A Study on Quasi – Groups Satisfying Partial Associative Law with Unique Right Unit

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Abstract: A Quasi- group is an algebraic structure resembling a group in the sense that division is possible. Quasi –groups differ from a group, that they are not necessarily be associative. In this paper we make a study of quasi- groups which satisfy Partial associative law and have a right unit. It is seen that these Quasi –groups have properties very similar to ordinary groups.

***Keyword**: Right unit Quasi-group*

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1. Introduction

The term Quasi –group was introduced by R.Moufang.Most of the results in the literature of Quasi –groups do depend upon special associative conditions. Also it is shown that the Quasi- group contains a set of minimal right unit sub-Quasi –group having no elements in common and at least one of them is contained in every sub-Quasi group of the Quasi-group. In this paper we make a study of those class of Quasi –Groups which satisfy an associative law 1 of the form

$a(bc) = (ab)c_1$, c_1 is independent of b and has a right unit.

2.Preliminaries:

Definition 2.1. A quasi-group G is a set together with an operation of multiplication such that

1) the set is closed under multiplication.
2) The equations ax = b and ya = b have unique solutions for x and y ,where a and b are any two (not necessarily distinct) elements of G

Condition (2) is sometimes referred to as quotient axiom. The quotient- axiom (2) implies both left and right cancellation laws .Since we are considering only finite quasi- groups, it is useful to note that every subset of G which is closed under multiplication satisfies the quotient- axiom and is therefore a sub-quasi group of G. From the quotient- axiom (2) it follows that every element a in G has a right unit e_a and a left unit e_a' defined by

$ae_a = e_a'a = a$

Definition 2.2. A quasi-group G with an identity element is called a loop.

We know that a finite set which is closed under an associative produt and in which both cancellation laws hold is a group. Hence a finite quasi –group differs from a group in that the associative law may fail to hold.

The following are few examples of a quasi - group.

1. The set of integers Z under the binary operation, subtraction (-) forms a quasi- group.
2. The non- zero rational numbers (or non- zero real numbers) with division forms a quasi- group.
3 .The set of non- zero elements of any division algebra forms a quasi- group.

3. Quasi- group satisfying partial associative laws

In a Quasi –Group we can define a sort of partial associative law as follows.

If a, b, c are any three elements of G, then

(1) $a(bc) = (ab)c_1$

wherec_1 is independent of b.

(1) can be called as associative law 1 .In what follows ,unless otherwise stated ,G denotes a finite quasi – group .In the following theorem we investigate the conditions under which all the right units of a quasi –group form a sub –quasi group.

<u>Theorem3.1</u>.

The set R of all right units of G form a sub-quasi group and $a \to e_a$ is a homomorphism of G on R, where e_a denotes the right unit of a.

<u>Proof</u> : -

Since G is a quasi –group, every element $a \in G$ has a right unit e_a and a left unit e'_a such that.

$ae_a = e'_a a = a.$

Assume that G satisfies the law

(1) $a(bc) = (ab)c_1$, for all a , b, $c \in G$ and c_1 is independent of b

since in (1) c_1 is independent of b, choose $b = e_a$

Then (1) becomes

(2) $a(e_a c) = (ae_a)c_1$
$= ac_1$

which implies $c_1 = e_a c$.

Now we denote by $f_a(c)$ the element defined by the equation

$e_a f_a(c) = c,$

Where c is any element of G and a is any fixed element.
But then,
We can define $f_a(e_a c) = e_a c$.

$$e_a f_a(e_a c) = e_a c.$$

Now by the left – cancellation law we have

$$f_a(e_a c) = c.$$

Therefore we can define the inverse function $f^{-1}{}_a o f f_a$ by

$$f^{-1}{}_a(c) = e_a c.$$

Then equation (1) becomes

$$(2)\quad \begin{cases} a(bc) = (ab)\, c_1 \\ \quad = (ab)(e_a c) \\ \quad = (ab)\, f^{-1}{}_a(c). \\ \text{and replacing c by } f_a(c) \text{ on both side we get} \\ (ab)\, c = a(b\, f_a(c)) \end{cases}$$

Now put $c = e_b$ in the first equation of (3)
Then we get,

$$a(be_b) = (ab)\, f^{-1}{}_a(e_b) \qquad [\because f^{-1}{}_a(c) = e_a c]$$

$$ab = (ab)(e_a e_b)$$

ie, $(ab)\, e_{ab} = (ab)(e_a e_b)$

$$\Rightarrow e_{ab} = e_a e_b$$

Thus the mapping $a \rightarrow e_a$ is a homomorphism of G on R.
Then to complete the proof of the theorem we have to prove that R is a sub-quasi –group of G.
Let $R = \{e_a / ae_a = a, \text{ for all } a \in G\}$

If $e_a, e_b \in R$ then

$e_a e_b = e_{ab} \in R$

Next we show that $e_a x = e_b$ and $ye_a = e_b$ have unique solution for all a, b in G. since G is a quasi – group, $ax = b$ and $ya = b$ have unique solutions.
∴ There exists elements c and d such that $ac = b$ and $da = b$
Then,
$e_a e_c = e_{ac} = e_b$ and
$e_d e_a = e_{da} = e_b$
$\Rightarrow e_c, e_d \in R$.
Hence e_c is the solution of $e_a x = e_b$ and e_d is the solution of $ye_a = e_b$. Hence R is a sub-quasi-group of G.

Theorem 3.2.

Any finite quasi-group G satisfying law I, contains a set of minimal right unit sub-quasi-groups, no two of which have elements in common and at least one of which is contained in every sub-quasi-group of G.

Proof:-

From theorem 3.1 it follows that the homomorphism $a \rightarrow e_a$ maps G on R. Then R could be mapped onto its right unit quasi-group R_1, R_1 to its right unit quasi-group R_2 and so on. Since G is a finite quasi-group we finally reach a sub-quasi-group R_t which is mapped onto itself.

Since every sub-quasi-group of R_t must contain its own right units, it follows from theorem 3.1 that the mapping $a \rightarrow e_a$ is an automorphism of R_t and also of every sub-quasi group of R_t.

Let $E_1, E_2, \ldots\ldots, E_r$ be the set of all minimal sub-quasi groups of R_t where 'minimal' means that each E_i does not contain any other sub-quasi-group of R_t

Suppose $a \in E_i \cap E_j$.
Then $a \in E_i$ and $a \in E_j$
Since $a \rightarrow e_a$ is an automorphism , for,
$a \in E_i$, $e_a \in E_i$
Similarly $a \in E_j \Rightarrow e_a \in E_j$
$\therefore e_a \in E_i \cap E_j$

In any sub-quasi-group of R_t, the process $a \rightarrow e_a$ must terminate at some stage, since it is finite. That is, at some stage an element will be its own right unit. Applying this reasoning to E_i and E_j we see that e_a, the right unit of a become its own right unit and will belong to

$E_i \cap E_j$

ie, $E_i \cap E_j$ is a sub-quasi-group of R_t.
But $E_i \cap E_j \subset E_i$ (and E_j) which contradicts the minimality of E_i.

$$\therefore E_i \cap E_j = \emptyset$$

Since every sub-quasi-group of G contains a sub-quasi of R, and therefore of $R_1, R_2, \ldots\ldots, R_t$; it must contain at least one of these E_i. This completes the proof of the theorem .

Definition 3.3

If one of the minimal right unit sub-quasi group of a quasi-group G consists of a single element e, then e will become its own right (and left) unit. In this case e will be called the principal unit.

4. QUASI – GROUPS WITH UNIQUE RIGHT UNIT:-

In this section we consider the quasi-groups which satisfy associative law I and has a unique right unit e.

$$ae = a \text{ for all } a \in G.$$

With this hypothesis equation (2) of theorem 3.1 becomes

$$(4)\quad \begin{cases} a(bc) = (ab)\, c^s \\ (ab)\, c = a(bc^{s-1}) \end{cases}$$

respectively where $c^s = ec$ and $ec^{s-1} = c$
putting $a = e$ in the first equation of (4) we get

$$(bc)^s = b^s c^s.$$

$$\therefore s \text{ is an automorphism of G}$$

This idea is seen to be helpful in proving the following result:-

Theorem: 4.1 The set H, of all elements which commute with e, is a group, the largest group contained in G.

Proof :-

$$\text{Let } H = \{x \in G / xe = ex\}$$

we have to show that H is closed with respect to multiplication.

Let x, y ∈ H

Then xe = ex and ye = ey

ie $x = x^s$ and $y = y^s$

Now $e(xy) = (xy)^s$

$= x^s y^s$

(∵s is an automorphism)

$= xy$

$= (xy)e$.

ie, xy∈ H.

Al so since ex = x= xe for all x ∈ H, e is the identify in H. Since H is finite and closed under multiplication,

H is a sub-quasi-group of G.

Let a∈ H and let a_{-1}, a^{-1} denote the left and right inverses of 'a' respectively.

Then the equation ax = e and ya = e have unique solutions in H.

ie, there exists a^{-1}, a_{-1}∈ H such that

$a a^{-1} = a_{-1} a = e$

Now $a a^{-1} = e \Rightarrow a_{-1}(aa^{-1}) = a_{-1} e$

$\Rightarrow a_{-1}(a a^{-1}) = a_{-1}$

$\Rightarrow (a_{-1} a)(a^{-1})^s = a_{-1}$

$\Rightarrow e(a^{-1})^s = a_{-1}$

(5) $\Rightarrow (a^{-1})^{s2} = a_{-1}$

Now to show that $a^{-1} = a_{-1}$

$(a^{-1})^{s2} = [(a^{-1})^s\ ]^s$

$= (a^{-1})^s$ [∵ $(a^{-1})^s = a^{-1}$)]

(6) $= a^{-1}$

From (5) and (6) we get

$a^{-1} = a_{-1}$

Hence the inverse exists in H.

Let x, y, z ∈H.

Now $x(yz) = (xy)z^s$

$= (xy)z$. [∵$z^s = z$)

∴ Associative law holds in H.

Next we prove that H is the largest sub-group,

Contained in G. Let K be a sub-group of G such that H⊂ K.

Let x ∈ K ⇒xe = ex

⇒x∈ H

⇒K∈ H,

∴ K = H.

Hence the theorem.

If H is a sub quasi- group of a quasi-group G and ifaH = Ha holds for all a in G, then G/H is a group.

Let G be any Abelian quasi-group with unique right unit e, and let ϕ(a) denote any power of a. Then to any such power ϕthere corresponds two sub-quasi-groups of G. The first, which we shall denote by G_ϕ, consists of all elements x of G such that

$\phi(x) = e$,

and the second, $G^{(\phi)}$, consists of all elements of the form ϕ(x), where x runs through all elements of G.

Theorem : 4.2

The quotient quasi-group G/G_ϕ is isomorphic to $G^{(\phi)}$ where G_ϕ, consists of all elements x of G such that.

$\phi(x) = e$

and $G^{(\phi)}$, all elements of the from ϕ(x), where x runs through all elements of G.

Proof :-

we know that G_ϕ denote the set of all elements x of G such that

$\phi(x) = e$

and $G^{(\phi)}$ consists of all elements of the from ϕ(x), where x runs through all elements of G.

we have to show that

$G/G_\phi \cong G^{(\phi)}$.

Define a function $\psi : G/G_\phi \rightarrow G^{(\phi)}$.

First to show that ψ is well – defined.

Suppose $aG_\phi = bG_\phi$.

Then $b \in aG_\phi$.

∴ b = ax for some $x \in G_\phi$.

$\phi(b) = \phi(ax)$

$= \phi(a)\phi(x)$

[∵G is an abelian quasi – group.]

$= \phi(a)e$ [∵ x G_ϕ]

$= \phi(a)$.

$\therefore \phi(a) = \phi(b)$

$\because \phi(a) = \phi(b)$; $(aG_\phi)\psi = (bG_\phi)\psi$

Hence ψ is well – defined.

To show that ψ is a homomorphism.

For,

$[(aG_\phi)(bG_\phi)]\psi = [(ab)G_\phi]\psi$

$= \phi(ab)$

$= (ab)^m$

$= a^m b^m$

$= \phi(a)\phi(b)$

$= (aG_\phi)\psi(bG_\phi)\psi$.

$[(aG_\phi)(bG_\phi)]\psi = (aG_\phi)\psi(bG_\phi)\psi$.

∴ψ is a homomorphism.

Next we have to show that ψ is one-one and onto.

For that it is enough to show that $\phi(a) = \phi(b)$, if and only if b lies in $aG\phi$

Let b lies in $aG\phi$. To prove $\phi(a) = \phi(b)$.

Since b lies in $aG\phi$, b = ag where $g \in G\phi$.

$\therefore \phi(b) = \phi(ag)$

$= \phi(a)\phi(g)$

$= \phi(a)e$ [∵ $g \in G_\phi$]

$= \phi(a)$.

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$\therefore \phi(a) = \phi(b)$.

Conversely suppose $\phi(a) = \phi(b)$. To show that b lies in aG_ϕ.

For,

$$\phi(a_{-1} b) = \phi(a_{-1})\phi(b)$$
$$= \phi(a_{-1})\phi(a) \quad [\because \phi(a) = \phi(b)]$$
$$= \phi(a_{-1} a)$$
$$= e.$$
$$\therefore a_{-1} b \text{ belongs to } G_\phi.$$

So b belongs to aG_ϕ. Thus we get a one-one correspondence from G/G_ϕ onto $G^{(\phi)}$. So the correspondence

$$aG_\phi \leftrightarrow \phi(a)$$

is an isomorphism between G/G_ϕ and $G^{(\phi)}$

Hence the result.

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