Fixed Point Theorem of Generalized Contradiction in Partially Ordered Cone Metric Spaces

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ABSTRACT: In this thesis we discuss the newly introduced concept of cone metric spaces, prove some fixed point theorems existence results of contractive mappings defined on such cone metric space and improve some well-known results in the normal case. The purpose of this paper is to establish the generalization of contractive type mappings on complete cone metric spaces. Also all the results in this paper are new. The main aim of this paper is to prove fixed point theorems is cone metric spaces which extend the Banach contraction mapping and others. This is achieved by introducing different kinds of Cauchy sequences in cone metric spaces.

Keywords: *Banach space, fixed point theorem, cone metric space, Cauchy sequence, contraction mapping.*

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I. INTRODUCTION

Let E be a real Banach Space. A nonempty convex closed subset $P \subset E$ is called cone in E if it satisfies:

1) P is closed, non-empty and $P \neq \{0\}$,
2) $a, b \epsilon \mathbb{R},\ a, b \geq 0$ and $x, y \epsilon P$ imply that $ax + by \epsilon P$,
3) $x \epsilon P$ and $-x \epsilon P$ imply that $x = 0$.

The space E can be partially ordered by the cone $P \subset E$; that is, $x \leq y$ if and only if $y - x \epsilon P$. Also we write $x \ll y$. if $y - x \epsilon P^0$, where P^0 denotes the interior of P.A cone P is called normal if there exists a constant $K > 0$ such that $0 \leq x \leq y$ implies $||x|| \leq K||y||$.

In the sequel, suppose that E is a real Banach space. P is a cone in E with nonempty interior $P^0 \neq 0$ and $\leq$ is the partial ordering with respect to P. A Cone metric spaces is Hausdorff and so has the property that any singleton is a closed subset of the space. In applications to computer science, especially to computer domains, a space induced by a distance function in which a singleton need not be closed is used. A partial cone metric space is such a space which might have a great application potential in computer science.

In Chapter I, refreshes some basic concepts of the metric space.
In Chapter II, deals with the concept of cone metric on poset.
In Chapter III, discuss about ciric's fixed point theorem in a cone metric space.
In Chapter IV, Illustrates the metrizability of cone metric spaces
In Chapter V, discusses the fixed point theorems in partial cone metric spaces.

PRELIMINARIES

Definition 1.1

Let X be a nonempty set. Assume that the mapping $d: X \times X \to E$ satisfies

i) $0 \leq d(x, y)$ for all $x, y \epsilon X$ and $d(x, y) = 0\ iff x = y$,
ii) $d(x, y) = d(y, x)$ for all $x, y \epsilon X$,
iii) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \epsilon X$.

Then d is the called a cone metric on X, and (X, d) is called **cone metric space**.

Definition 1.2

If $(X, \sqsubseteq)$ is a partially ordered set and$f: X \to X$, f is **monotone non-decreasing** if $x, y \epsilon X, x \sqsubseteq y \Rightarrow fx \sqsubseteq fy$.

Definition 1.3

The cone P is a called regular if every increasing sequence which is bounded form above is convergent.

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If $\{x_n\}$ is a sequence such that $x_1 \leq x_2 \leq \cdots \leq y$ for some yϵE, then there is xϵE such that $\lim_{n\to\infty} ||x_n - x|| = 0$.

Equivalently the cone P is regular if and only if every decreasing sequence which bounded from below is convergent. It has been mentioned that every regular cone is a normal.

Definition 1.4

P called **minihedral** cone if $sup\{x,y\}$ exists for all $x, y\epsilon E$, and **strongly minihedral** if every subset of E which is bounded above has a supremum.

If cone P be **strongly minihedral,** then every subset of P has infrimum.

CONE METRIC ON POSET

Theorem 2.1

Let $(X, \sqsubseteq)$ be a partially ordered set and suppose there exists a cone metric $d\epsilon X$ such that (X, d) is a complete cone metric space which the (ID) property holds.

Let $f: X \to X$ be a continuous and non-decreasing mapping such that

$$\psi\big(d(fx, fy)\big) \leq \psi\big(d(x, y)\big) - \varphi\big(d(x, y)\big)$$

for $x \sqsubseteq y$, where ψ and φ are altering distance functions. If there exists $x_0 \epsilon X$ with $x_0 \sqsubseteq fx_0$ then f has a unique fixed point.

Proof

If $x_0 = fx_0$ then the proof is finished.

Suppose that $x_0 \neq fx_0$.

Since$x_0 \sqsubseteq fx_0$ and f is a non-decreasing function,

$x_0 \sqsubseteq fx_0 \sqsubseteq f^2x_0 \sqsubseteq f^3x_0$ ....

Put $x_{n+1} := fx_n = f^n x_0$ and $a_n := d(x_{n+1}, x_n)$.

Then for $n \geq 1$

$$\psi\big(d(x_{n+1}, x_n)\big) = \psi\big(d(fx_n, fx_{n-1})\big) \leq \psi\big(d(x_n, x_{n-1})\big) - \varphi(\,d(x_n, x_{n-1})),$$

Therefor $0 \leq \psi(a_n) \leq \psi(a_{n-1}) - \varphi(a_{n-1}) \leq \psi(a_{n-1})$ (1)

Since$x_n \sqsubseteq x_{n+1} \sqsubseteq x_{n+2}$ by the (ID) property

It follows that,

$$a_n \leq a_{n+1} \qquad (2)$$

or

$$a_{n+1} \leq a_n \qquad (3)$$

If (2) holds, since ψ is non-decreasing by (1)

It gives that

$$0 \leq \psi(a_n) \leq \psi(a_{n-1}) - \varphi(a_{n-1}) \leq \psi(a_n) - \varphi(a_{n-1}) \leq \psi(a_n) \qquad (4)$$

This implies that $\varphi(a_{n-1}) = 0$ and so $a_{n-1} = 0$ for $n \geq 1$

Thus $x_n = x_{n-1} = fx_{n-1}$ for $n \geq 1$ are fixed points of f.

If (3) holds, since ψ and φ are non-decreasing by, relation (1) and induction,

It implies that,

$$\begin{aligned}\varphi(a_{n+1}) \leq \varphi(a_n) \leq \psi(a_n) &\leq \psi(a_{n-1}) - \varphi(a_{n-1}) \\ &\leq \psi(a_{n-1}) - \varphi(a_n) \\ &\leq \psi(a_{n-2}) - \varphi(a_{n-2}) - \varphi(a_n). \\ &\leq \psi(a_{n-2}) - 2\varphi(a_n) \leq \cdots \\ &\leq \psi(a_0) - n\varphi(a_n)\end{aligned}$$

Then $0 \leq \varphi(a_n) \leq \frac{1}{1+n}\psi(a_0)$ for all n.

This implies that $\varphi(\lim_{n\to\infty} a_n)\ in\ P \cap -P$ and $\varphi(\lim_{n\to\infty} a_n) = 0$ and since φ is altering distance function.

Then $(\lim_{n\to\infty} a_n) = 0$, and

$$\lim_{n\to\infty} d(x_{n+1}, x_n) = 0. \qquad (5)$$

Now show that the sequence $\{x_n\}$ is cauchy.

Claim:

For every $c\ in\ E$ and $c \gg 0$ there exists N such that $d(x_{n+2}, x_n) \ll c$ for every $n \geq N$.

Choose $c \gg 0$, by (5) there exists N such that $d(x_{n+1}, x_n) \ll \frac{c}{2}$ for all $n \geq N$.

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It makes that,

$d(x_{n+2}, x_n) \leq d(x_{n+2}, x_{n+1}) + d(x_{n+1}, x_n) \ll c$ for every $n \geq N$.

Therefore,

For some N and $d(x_{n+2}, x_n) \ll c$ for every $n \geq N$.

Now by induction $d(x_{n+m}, x_n) \ll c$ for every $n \geq N$ and for all integer number $m \geq 1$.

The sequence $\{x_n\}$ is Cauchy and since (X, d) is complete, and thus there exists $x^* \, in \, X$ such that $x_n \to x^*$ and on the other hand f is continuous and $x_{n+1} = fx_n$.

Then $x^* = fx^*$.

For uniqueness let $x = fx$ and $y = fy$, and

$$\psi(d(x,y)) = \psi(d(fx,fy)) \leq \psi(d(x,y)) - \varphi(d(x,y))$$

The last inequality gives us $\varphi(d(x,y)) = 0$ and by property of the altering distance functions this implies $d(x,y) = 0$.

Therefore $x = y$.

In the next theorem,

The (Id) property is replaced by strongly minihedrallity of the cone.

Hence the proof.

CIRIC'S FIXED POINT THEOREM IN A CONE METRIC SPACE

Theorem (Fixed Point Thorem) 3.1

Let (X, d) be a complete cone metric space, P be a normal cone with normal constant $k(k \geq 1)$.

Suppose the mapping $T: X \to X$ satisfies the following constractive condition:

$$d(Tx,Ty) \leq A_1(x,y)d(x,y) + A_2(x,y)d(x,Tx) + A_3(x,y)d(y,Ty) + A_4(x,y)d(x,Ty) + A_4(x,y)d(y,Tx), \quad (9)$$

for all x, y in X, where $A_i: X \times X \to \mathcal{L}(E), i = 1,2,3,4$.

Further, assume that for all $x, y \, in \, X$,

$\exists \, \alpha \in [0,1/k) | \sum_{i=1}^{4} \| A_i(x,y) \| + \| A_4(x,y) \| \leq \alpha$ (10)

$\exists \, \beta \in [0,1) | \, \| s(x,y) \| \leq \beta$ (11)

$(A_1(x,y) + A_2(x,y))(P) \subseteq P$ (12)

$A_2(x,y)(P) \subseteq P$ (13)

$A_4(x,y)(P) \subseteq P$ (14)

$(I - A_3(x,y) - A_4(x,y))^{-1}(P) \subseteq P$ (15)

Here, $S: X \times X \to \mathcal{L}(E)$ *is given by:*

$$S(x,y) = (I - A_3(x,y) - A_4(x,y))^{-1}(A_1(x,y) + A_2(x,y) + A_4(x,y)),$$

for all $x, y \, in \, X$.

Then, T *has a unique fixed point*

Proof.

Let $x \in X$ be arbitrary and define the sequence $(x_n)_{n \in \mathbb{N}} \subset X$ by:

$x_0 = x, x_1 = Tx_0, \dots . x_n = Tx_{n-1} = T^n x_0, \dots .$

By (9) it gets that:

$$\begin{aligned} d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n) &\leq A_1(x_{n-1}, x_n)d(x_{n-1}, x_n) + A_2(x_{n-1}, x_n)d(x_{n-1}, x_n) \\ &\quad + A_3(x_{n-1}, x_n)d(x_n, x_{n+1}) + A_4(x_{n-1}, x_n)d(x_{n-1}, x_{n+1}) \\ &\quad + A_4(x_{n-1}, x_n)d(x_n, x_n) \\ &= (A_1(x_{n-1}, x_n) + A_2(x_{n-1}, x_n))d(x_{n-1}, x_n) \\ &\quad + A_3(x_{n-1}, x_n)d(x_n, x_{n+1}) + A_4(x_{n-1}, x_n)d(x_{n-1}, x_{n+1}). \end{aligned}$$

Using the triangular inequality, it given that:

$$d(x_{n-1}, x_{n+1}) \leq d(x_{n-1}, x_n) + d(x_n, x_{n+1}),$$

then

$$d(x_{n-1}, x_n) + d(x_n, x_{n+1}) - d(x_{n-1}, x_{n+1}) \, in \, P.$$

From (14), it follows that

$$A_4(x_{n-1}, x_n)[\, d(x_{n-1}, x_n) + d(x_n, x_{n+1}) - d(x_{n-1}, x_{n+1})] \, in \, P,$$

and

$$\begin{aligned} A_4(x_{n-1}, x_n)d(x_{n-1}, x_{n+1}) &\leq A_4(x_{n-1}, x_n)d(x_{n-1}, x_n) \\ &\quad + A_4(x_{n-1}, x_n)\, d(x_n, x_{n+1}). \end{aligned}$$

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Then it implies that:

$$d(x_n,x_{n+1})\leq(A_1(x_{n-1},x_n)+A_2(x_{n-1},x_n)+A_4(x_{n-1},x_n))d(x_{n-1},x_n)+(A_3(x_{n-1},x_n)+A_4(x_{n-1},x_n))d(x_n,x_{n+1}).$$

Then,

$$\big(I-A_3(x_{n-1},x_n)A_4(x_{n-1},x_n)\big)d(x_n,x_{n+1})\leq(A_1(x_{n-1},x_n)+A_2(x_{n-1},x_n)+A_4(x_{n-1},x_n))d(x_{n-1},x_n).$$

Using (15), $d(x_n,x_{n+1})\ \leq S(x_{n-1},x_n)(d(x_{n-1},x_n).$ (16)

It is not difficult to see that under hypotheses (12),(14) and (15),

$$S(x,y)(P)\ \subseteq P, for\ all\ \ x,y\ in\ X.$$

Using this remark, (16) and proceeding by iterations,

d(x_n,x_{n+1})≤ $S(x_{n-1},x_n)\ S(x_{n-2},x_{n-1})\ldots S(x_0,x_1)\ d(x_0,x_1),$

which implies by (11) that:

$\|$d(x_n,x_{n+1})$\|$≤ k $\|\ S\big(x_{n-1,}x_n\big)\ \|\|\|\ S(x_{n-2},x_{n-1})\ \|\ldots\ldots\|\ S(x_0,x_1)\ \|\|\|\ d(x_0,x_1)\ \|$ $\leq k\beta^n\ \|\ d(x_0,x_1)\ \|$.

For any positive integer p,

$$d(x_n,x_{n+p})\ \leq \textstyle\sum_{i=1}^{p} d(x_{n+i-1},x_{n+i}),$$

which implies that:

$$\|\ d(x_n,x_{n+p})\ \|\leq k\sum_{i=1}^{p}\|\ d(x_{n+i-1},x_{n+i})\ \|$$

$$\leq k^2\sum_{i=1}^{p}\beta^{n+i-1}\ \|\ d(x_0,x_1\ \|$$

$$\leq k^2\frac{\beta^n}{1-\beta}\ \|\ d(x_0,x_1)\ \| \qquad (17)$$

Since $\beta\in\ [0,1),\beta^n\ \to 0\ as\ n\to+\infty.$

So from (17) it follows that the sequence$(x_n)_{n\in N}$ is Cauchy. Since (X,d) is complete, there is a point $u\in X$ such that:

$$\lim_{n\to+\infty} d(Tx_n,u)=\lim_{n\to+\infty} d(x_n,u)\ \ =\lim_{n\to+\infty} d(Tx_n,x_{n+1})=0 \qquad (18)$$

Now, using the contractive condition (9),

$$d(Tu,Tx_n)\leq\ A_1(u,x_n)\mathrm{d}(u,x_n)+A_2(u,x_n)\ d(u,Tu)+A_3(u,x_n)d(x_n,x_{n+1})+A_4(u,x_n)d(x_n,x_{n+1})+A_4(u,x_n)d(x_n,Tu).$$

By the triangular inequality,

$$d(u,Tu)\leq d(u,x_{n+1})+d(x_{n+1},Tu)$$
$$d(x_n,Tu)\leq d(x_n,Tx_n)+d(Tx_n,Tu).$$

By (13) and (14),

$$A_2(u,x_n)d(u,Tu)\leq A_2(u,x_n)(d(u,x_{n+1})+d(x_{n+1},Tu))$$
$$A_4(u,x_n)d(x_n,Tu)\leq A_4(u,x_n)d(x_n,Tx_n)+A_4(u,x_n)d(Tx_n,Tu).$$

Then

$$d(Tu,Tx_n)\leq\ A_1(u,x_n)+d(u,x_n)+\ (A_2(u,x_n)+A_4(u,x_n))d(u,x_{n+1})+(A_2(u,x_n)+A_4(u,x_n))d(x_{n+1},Tu)\ +(\,A_3(u,x_n)+A_4(u,x_n))d(x_n,x_{n+1})$$

Using (10), this inequality implies that:

$$\|\ d(Tu,Tx_n)\ \|\leq\frac{k\alpha}{1-k\alpha}(\|\ d(u,x_n)\ \|+\|\ d(u,x_{n+1})\ \|+\|\ d(x_n,x_{n+1})\ \|).$$

From (18), it follows immediately that:

$$\lim_{n\to+\infty} d(Tu,Tx_n)=0. \qquad (19)$$

Then, (18), (19) and the uniqueness of the limit imply that $u=Tu, then\ u$ is a fixed point of T . and T has least one fixed point $u\in X.$

Now, if $v\in X$ is another fixed point of T , by (9),

$$d(u,v)=d(Tu,Tv)\leq\ A_1(u,v)d(u,v)+2A_4(u,v)d(u,v),$$

Which implies that:

$$\|\ d(u,v)\ \|\leq k(\|\ A_1(u,v)\ \|+2\ \|\ A_4(u,v)\ \|)\ \|\ d(u,v)\ \|\leq k\alpha\ \|\ d(u,v)\ \|,$$

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$$(1-k\alpha)\parallel d(u,v)\parallel \leq 0.$$

Since $0 \leq \alpha < \frac{1}{k}$, we get $d(u,v) = 0$, i.e., $u = v$. So the proof of the theorem is completed.

METRIZABILITY OF CONE METRIC SPACES

Theorem 4.1

For every cone metric $D: X \times X \to E$ there exists metric $d: X \times X \to \mathbb{R}^+$ which is equivalent to D on X.

Proof:

Define $d(x,y) = \inf\{\parallel u \parallel : D(x,y) \leq u\}$. We shall to prove that d is an equivalent metric to D. If $d(x,y) = 0$ then there exists u_n such that$\parallel u_n \parallel \to 0$ and $D(x,y) \leq u_n$.

And $u_n \to 0$ and consequently for all $c \gg 0$ there exists $N \in \mathbb{N}$ such that $u_n \ll c$ for all $n \geq N$.

Thus for all $c \gg 0, 0 \leq D(x,y) \ll c$.

Namely $x = y$. If $x = y$ then $D(x,y) = 0$ which implies that $d(x,y) \leq\parallel u \parallel$ for all $0 \leq u$.

Put $u = 0$ it implies $d(x,y) \leq \parallel 0 \parallel = 0$, on the other hand $0 \leq d(x,y)$,

Therefore $d(x,y) = 0$.

It is clear that $d(x,y) = d(y,x)$.

To prove triangle inequality, for $x,y,z \in X$,

$$\forall \epsilon > 0\ \exists u_1\ \parallel u_1 \parallel < d(x,z) + \epsilon, D(x,z) \leq u_1,$$
$$\forall \epsilon > 0\ \exists u_2\ \parallel u_2 \parallel < d(z,y) + \epsilon, D(z,y) \leq u_2.$$

But $D(x,y) \leq D(x,z) + D(z,y) \leq u_1 + u_2$,

Therefore

$$d(x,y) \leq \parallel u_1 + u_2 \parallel \leq \parallel u_1 \parallel + \parallel u_2 \parallel \leq d(x,z) + d(z,y) + 2\epsilon.$$

Since $\epsilon > 0$ was arbitrary so $d(x,y) \leq d(x,z) + d(z,y)$.

Claim:

For all $\{x_n\} \subseteq X\ and\ x \in X, x_n \to x$ in (X,d) if and only if $x_n \to x$ in (X,D).

$\forall n,m \in N\ \exists u_{nm}$ such that $\parallel u_{nm} \parallel < d(x_n,x) + \frac{1}{m},\ D(x_n,x) u_{nm}$.

$$Put\ v_n := u_{nn}\ then\ \parallel v_n \parallel < d(x_n,x) + \frac{1}{n}$$

and $D(x_n,x) \leq v_n$. Now if $x_n \to x$ in (X,d) then $d(x_n,x) \to 0$ and $v_n \to 0$. Therefore for all $c \gg 0$ there exists $N \in \mathbb{N}$ such that $v_n \ll c$ for all $n \geq \mathbb{N}$.

This implies that $D(x_n,x) \ll c$for all $n \geq \mathbb{N}$. Namely $x_n \to x$ in(X,D).

Conversely,

For every real $\epsilon > 0$, choose $c \in E$ with $c \gg 0$

And $\parallel c \parallel < \epsilon$.

Then there exists $N \in \mathbb{N}$ such that $D(x_n,x) \ll c$ for all $n \geq N$.

This means that for all $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $d(x_n, x) \leq\parallel c \parallel < \varepsilon$ for all $n \geq N$.

Therefore $d(x_n,x) \to 0$ as $n \to \infty$ so$x_n \to x\ in\ (X,d)$.

FIXED POINT THEOREMS IN PARTIAL CONE METRICSPACES

Theorem5.1

Let (X,p) be a partial cone metric space and P be a normal cone with normal constant K, then (X,p) is T_0.

Proof.

Suppose $p: X \times X \to E$is a partial cone metric, and

suppose$x,y \in X$ with $x \neq y$, from (p1) and (p2)

$p(x,x) < p(x,y)$ or $p(y,y) < p(x,y)$.

Suppose $p(x,x) < p(x,y)$ and $0 < p(x,y) - p(x,x)$,

$$0 < \parallel p(x,y) - p(x,x) \parallel = \delta_x.$$

For $\delta_x > 0$, choose $c_x \in intP$with $\parallel c_x \parallel < \delta_x$.

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Then $x \in B_p(x, c_x)$and$y \notin B_p(x, c_x)$.

Consequently (X, p) partial cone metric space isT_0.

II. CONCLUSION

The present work contains not only an improvement and a generalization of the concept of a partial metric, as it has been presented in a more general setting, a partial cone metric space which is more general than the partial metric space. But also an investigation of some fixed point theorems one of which is also new for a partial metric space.

So that one may expect it to be more useful tool in the field of topology in modeling various problems occurring in many areas of science, computer science, information theory, and biological science.

On the other hand, a concept of fuzzy partial cone metric is investigated fixed points theorems for fuzzy functions.

However, due to the change in settings, the definitions and methods of proofs will not always be analogous to those of the present results.

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