International Journal on Future Revolution in Computer Science & Communication Engineering ISSN: 2454-4248
Volume: 5 Issue: 1 08 – 11

# Euler Characteristic and Injective Modules

A. Pannerselvam
M.Sc., M.Phil.
Associate Professor, Department of Mathematics,
PRIST University,Vallam, Thanjavur

M. Parasakthi
M.Phil - Research Scholar, Department of Mathematics,
PRIST University, Vallam, Thanjavur

**ABSTRACT:** In this talk, we will work almost exclusively with the Euler characteristic and some of the consequences of its topological invariance. We discuss by following that motivated the study of the characteristic. This gives a relation between topological invariant of the surface and a quantity derived from its combinatorial description. Secondly, we obtain an inequality relating the number of normal triangles and normal quadrilaterals, that depends on the maximum number of tetrahedrons that share a vertex. In this thesis, we disscuss this and related injectivity conditions and show that there are many rings.

***Key words:*** *Euler characteristic, topological invariance, quadrilaterals, injective module.*

*****

## I. INTRODUCTION

The purpose of this paper is to introduce and investigate the concepts of Euler characteristics, which arise in algebraic topology as invariants describing topological spaces. Euler characteristics for finitely induce modules and topological invariant.

The Euler characteristic of a module is well-defined whenever the $0^{th}$ homology group is finite if and only if the applicable compact. All through this paper, all rings are associative rings with unity and all modules are unitary right modules.

Y. Utumi in a series of his papers on regular self injective rings observed three conditions on a ring which is satisfied if the ring is self injective. These conditions are currently known in the literature by conditions and subsequently extended to modules.

A module is called finite direct injective if every finitely generated submodule isomorphic to a direct summand is itself a direct summand. It is the generalization of direct injective modules.

A module in linear algebra, the most important structure is that of a vector space over a field. For commutative algebra it is therefore useful to consider the generalization of this concept to the case where the underlying space of scalars is a commutative ring R instead of a field. The resulting structure is called a module.

In fact, there is a another more subtle reason why modules are very powerful. They are unify many other structures that you already know.

Consequently general results on modules will have numerous consequences in the many different setups. So let us now start with definition of the modules. In principle, their theory that we will then quickly discuss in this chapter is entirely analogous to that of vector spaces.

In chapter 1, deals with some basic definitions are given.

In chapter 2, we discussed about Euler characteristic and the grothendieck group.

In chapter 3, illustrates the theorems and prepositions on injective modules.

In chapter 4, refreshes some lemmas and corollary on Bifunctors.

In chapter 5, obtained the prepositions and theorems on spectral sequences.

Finally, we end with conclusion and bibliography.

## II. BASIC DEFINITIONS

**DEFINITION: 1.1**

An algebra consists of a vector space V over a field F, together with a binary operation of multiplication on the set V of vectors, such that for all $a \in F$ and $\alpha, \beta, \gamma \in v$ the following conditions are satisfied.

1) $(a\alpha)\beta = a(\alpha\beta) = \alpha(a\beta)$
2) $(\alpha + \beta)\gamma = \alpha\gamma = \beta\gamma$
3) $\alpha(\beta + \gamma) = \alpha\beta+\alpha\gamma$

We shall somewhat incorrectly say of an algebra V over F. Also V is an associative algebra over F, if in addition to the preceding three conditions.

4) $(\alpha\beta)\gamma = \alpha(\beta\gamma)$ for all $\alpha, \beta, \gamma \in V$

**DEFINITION: 1.2**

The algebraic structure (G,.) is a group if the binary operation, satisfying the following conditions.

**1) Closure property:**

a b ∈ G for all a, b ∈ G

**2) Associativity:**

(a.b)c = a(b.c) for all a, b, c ∈ G

**3) Existence of identity:**

Then there exists an element e ∈ G such that,ea = a = ae for all a ∈ G. The element e is called the identity.

**4) Existence of inverse:**

If a ∈ G there exists an element b ∈ G such that ba = e = ab

The element b is called the inverse of a and we write $b = a^{-1}$

Thus $a^{-1}$ is an element of G such that $a^{-1}a = e = aa^{-1}$

**Example:**

The set of all integers I is a group with respect to the operation of addition of integers.

**DEFINITION: 1.3**

A subset H of a group G is called a subgroup if the subgroup has the following properties

**1) Closure:**

If a ∈ H, b ∈ H then a b ∈ H

**2) Identity:**

If I ∈ H.

**3) Inverse:**

If a ∈ H then $a^{-1}$ ∈ H

**DEFINITION: 1.4**

A Permutation $\sigma \in S_n$ is a cycle if it has atmost one orbit containing more than one element.

**DEFINITION: 1.5**

Let (E, d) be a complex we let $z^i(E)$ = k e r $d^i$ and call $z^i(E)$ the module of i-cycles.

We frequently write $z^i$ instead of $z^i(E)$respectively.

## III. EULER CHARACTERISTIC AND THEGROTHENDIECK GROUP

**Theorem: 2.1**

Let F be a complex, which is of even length if it is closed. Assume that $\varphi(F^i)$ is defined for all i, $\varphi(F^i) = 0$ for almost all i, and $H^i(F) = 0$ for almost all i. Then φ(F) is defined, and

$$X\varphi(F) = \sum_i (-1)^i \varphi(F^i)$$

**Proof**:

Let $Z^i$ and $B^i$ be the groups of i-cycles and i-boundaries in b $F^i$ respectively.

We have an exact sequence

$$0 \longrightarrow Z^i \longrightarrow F^i \longrightarrow B^{i+1} \longrightarrow 0$$

Hence $\chi_\varphi(F)$ is defined, and

$$\varphi(F^i) = \varphi(Z^i) + \varphi(B^{i+1})$$

Taking the alternating sum, our conclusion follows at once.

A complex whose homology is trivial is called acyclic.

Hence the proof is complete.

**COROLLARY: 2.2**

Let F be an acyclic complex, such that $\varphi(F^i)$ is defined for all i, and equal to o for almost all i. If F is closed, we assume that F has length. Then

$$\varkappa\varphi(F) = 0$$

In many applications, an open complex F is such that $F^i = 0$ for almost all i.

One can then treat this complex as closed complex by defining an additional map going from a zero on the far left.

Thus in this case, the study of such an open complex is reduced to the study of closed complex.

## IV. INJECTIVE MODULES

**Theorem:3.1**

Every module is sub module of an injective module.

**Proof:**

ISSN: 2454-4248
08 – 11

The proof will be given by dualizing the situation, with some lemmas.

We first look at the situation in the category of abellan groups.

If M is an abelin group, let its dual group be M^=Hom(M, Q/Z).

If F is a Free abelian group, it is reasonable to exact, and in fact it is easily proved that its dual F^ is an injective module. Since injectivity is the dual notation of projectivetivity.

Furthermore,

M has a natural map into the double dual M^^.

Which is shown to be a monomorphism.

Now,

Represent M^ as a quotient of a free abelian group,

$$F \longrightarrow M\hat{} \longrightarrow 0$$

Dualizing this sequence yields a monomorphism

$$0 \longrightarrow M\hat{}\hat{} \longrightarrow F\hat{},$$

And since M is embedded naturally as a subgroup of M^^.

We get,

The desired embedding of M as a subgroup of F^.

This proof also works in general.

There are details to be filled in.

First we have to prove that,

The dual of a free module is injective.

Second,

We have to be careful when passing from the category of abelian groups to the category of modules over an arbitrary ring.

We now carry out the details.

We say that an abelian group T is divisible if for every integer m.

The homomorphism $m_T$:x⟶mx is surjective.

Hence the proof.

## V. BIFUNCTORS

**Lemma:4.1**

Let T be a bifunctor satisfying HOM 1,HOM 2 Let A ∈ and let MA⟶A⟶0. that is

$$\longrightarrow M_1 \longrightarrow M_0 \longrightarrow A \longrightarrow 0$$

be a T-exact resolution of A. Let $F^n(B) = H^n(T(M,B)$ for $B \in \mathscr{B}$. Then F is a δ-functor and $F^o(B) = T(A, B)$. If in addition T satisfies HOM 3, then $F^n(J) = 0$ for J injective and $n \geqq 1$.

**Proof:**

Given an exact sequence

$$0 \longrightarrow B' \longrightarrow B \longrightarrow B'' \longrightarrow 0$$

We get an exact sequence of complexes

$$0 \longrightarrow T(M, B')vT(M, B)vT(M, B'') \longrightarrow 0$$

When a cohomology sequence which makes A into a δ-functor.

For n = 0 we get $F^o(B) = T(A,B)$

Because X↣T(X,B) is contravariant and left exact for X∈.

If B is injective.

Then $F^n(B) = 0$ for $n \geqq 1$ by HOM 3

Because X↣T(X,B) is exact

This proves the lemma.

## VI. SPECTRAL SEQUENCES

**Proposition: 5.1**

Let F be a filtered differential object. Then there exist a spectral sequence $\{E_r\}$ with:

$$E_0^p = F^p/F^{p+1} ; E_1^p = H(G_r^p F); E_\infty^p = Gr^p H(F)$$

**Proof** :

ISSN: 2454-4248
08 – 11

Define

$$Z_r^p = \{x \in F^p \text{such that } d\,x \in F^{p+r}\}$$

$$E_r^p = Z_r^p / [dZ_{r-1}^{p-(r-1)} + Z_{r-1}^{p+1}]$$

The definition of $E_r^p$ makes sense, since $Z_r^p$ is immediately verified to contain $dZ_{r-1}^{p-(r-1)} + Z_{r-1}^{p+1}$

Furthermore, d maps $Z_r^p$ into $Z_r^{p+r}$ ; and hence includes a homomorphism

$$d_r: E_r^p\, E_r^{p+r} \rightarrow$$

We shall now compute the homology and show that it is what we want.

First, for the cycles: An element $x \in Z_r^p$ represents a cycle of degree p in $E_r$ if and only if $dx \in dZ_r^{p+1}$

In other words,

$dx = dy + z$, with $y \in Z_{r-1}^{p+1}$ and $Z \in Z_{r-1}^{p+r+1}$

write, $x = y+u$, so $du = z$

Then, $u \in F^n$ and $du \in F^{p+r+1}$, that is $u \in Z_{r+1}^p$

If follows that,

P-cycles of $E_r = (Z_{r+1}^p + Z_{r-1}^{p+1}) / (dZ_{r-1}^{p-r+1} + Z_{r-1}^{p+1})$

On the other hand,

The P-boundaries in $E_r$ are represented by elements of $dZ_r^{p-r}$

Which contains $dZ_{r-1}^{p-r+1}$

Hence,

P-boundaries of $E_r = (dZ_r^{p-r} + Z_{r-1}^{p+1}) / (dZ_r^{p-r} + Z_{r-1}^{p+1})$

Therefore,

$$H^p(E_r) = (Z_{r+1}^p + Z_{r-1}^{p+1}) / (Z_r^{p-r} + Z_{r-1}^{p+1})$$

$$= Z_{r+1}^p / (Z_{r+1}^p n (Z_r^{p-r} + Z_{r-1}^{p+1})$$

Since,

$Z_{r+1}^p \supset dZ_r^{p-r}$ and $Z_{r+1}^p \cap Z_{r-1}^{p+1} = Z_{r-1}^p$

It follows that,

$H^p(E_r) = Z_{r+1}^p / (dZ_r^{p-r} + Z_{r-1}^{p+1}) = E_{r+1}^p$

Thus proving the property of a spectral sequence.

## VII. CONCLUSION

Clearly, the Euler characteristic is invariance. It is very important in the classification of surfaces, as we learned in class. There are also many more ways to use it with simplified complexes. But that important much more detail than I notice was required for this paper.

But I hope that at the very least this paper shed some light on the importance and uses of Euler characteristic as well as the different ways to calculate it. We hope this introduction has sparked an interest in injective modules and will motivate the reader to study them further.

## REFERENCES


[1]. Cartan, H. Eilenberg. S: Homological algebra, Princeton: University Press 1956.
[2]. Cohn, P.M: On the free products of associative rings. Math. Z.71, 380-389, 1959.
[3]. Goldman, O: Rings and modules of quotients. J. Algebra, 10-47, 1969.
[4]. Levy, L: Torsion-free and divisible modules over non-integral-domains. Canadian J. Math.15, 132-151, 1963.
[5]. Moddox, B.H: Absolutely pure modules. Proc. Amer. Math. Soc.18, 155-158, 1967.
[6]. Matlis, E: Injective modules over Noetherian rings. Pacific J. Math.8, 511-528, 1958.
[7]. Megibbe , C: Absolutely pure modules. Proc. Amer. Math. Soc.26, 561-566, 1970.
[8]. Nakayamma, T: Note on uniserial and generalized uniserial rings. Imp. Acad. Tokyo.16, 285-289, 1940.
[9]. Aluffi. P, Generalized Euler characteristics, graph hypersurfaces and Feynman periods. Accessed june, 2015.
[10]. R. Bott& L. Tu. Differential Forms in Algebraic Topology, Springer-Verlag, New York, N.Y., 1982.
[11]. A. Hatcher, Algebraic Topology, Cambridge University Press, 2002.