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A Study on Combinatories in Discrete Mathematics

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Abstract: - The next two chapters deal with Set Theory and some related topics from Discrete Mathematics. This chapter develops the basic theory of sets and then explores its connection with combinatorics (adding and multiplying; counting permutations and combinations), while Chapter 5 treats the basic notions of numerosity or cardinality for finite and infinite sets. Most mathematicians today accept Set Theory as an adequate theoretical foundation for all of mathematics, even as the gold standard for foundations.* We will not delve very deeply into this aspect of Set Theory or evaluate the validity of the claim, though we will make a few observations on it as we proceed. Toward the end of our treatment, we will focus on how and why Set Theory has been axiomatized. But even disregarding the foundational significance of Set Theory, its ideas and terminology have become indispensable for a large number of branches of mathematics as well as other disciplines, including parts of computer science. This alone makes it worth exploring in an introductory study of Discrete Mathematics.

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INTRODUCTION

Combinatory present approaches for solving counting and structural questions. It looks at how many ways a selection or arrangement can be chosen with a specific set of properties and determines if a selection or arrangement of objects exists that has a particular set of properties.

They also provide basic information on sets, proof techniques, and enumeration and graph theory-topics. The next few chapters explore enumerativeideas, including the pigeonhole principle and inclusion /exclusion. The text thencovers enumerative functions and the relations between them. It describes generating functions and recurrences. The authors also present introductions tocomputer algebra and group theory before considering structures of particularinterest in combinatory. Graphs, codes, Latin squares and experimental designs.

BASIC DEFINITION

DEFINITION: 1.1

A well-defined collection of object is called a **Set.**

EXAMPLE: 1.1.1

English alphabet is a set containing five elements, namely a, e, i, o, u.

DEFINITION: 1.2

The object in a set are called its member or elements.

DEFINITION: 1.3

The different arrangements which can be made out of a given number of things by taking some or all at a time are called a **Permutation.**

DEFINITION: 1.4

A group or a selection which can be formed by taking some or all of number of objects irrespective of the order of their arrangements is called a **Combination.**

DEFINITION: 1.5

The set of integers is the set of positive and negative whole numbers with zero. Therefore the set {….. −4, −3, −2, −1, 0, 1, 2, 3, 4, …….}

DEFINITION: 1.6

If x is real number the function that assigns the largest integer that is less than or equal to x is called the **Floor function** of x or **Simply the floor** of x and denoted by [x].

TYPES OF PERMUTATIONS COMBINATIONS PERMUTATIONS AND COMBINATIONS

DEFINITION: 2.1

An ordered arrangement of r elements of a set containing n distinct elements is called an r-Permutation of n elements and is denoted by P(n, r)or nPr, where r ≤n. An unordered selection of r elements of a set containingn distinct elements is called an r-combination of n elements and is denotedby C (n, r) or n_1 or $1^1 1$.

RESULT: 2.1.1

A permutation of objects involves ordering whereas a combination does not take ordering into account.

VALUES OF P(n, r) AND C (n,r) : 2.2

The first element of the permutation can be selected from a set having n elements in n ways. Having selected the first elements for the first positionof the permutation, the second elements can be selected in (n−1) ways, asthere are (n−1) elements left in the set.

Similarly, there are (n−2) ways of selecting the third element and so on. Finally there are

n −(r−1) = n−r +1 ways of selecting the r^{th} element. Consequently, by the product rule, there are,

n(n−1)(n−2) …… (n−r +1)

Ways of ordered arrangement of r elements of the given set,

Thus, P(n, r) = n(n−1)(n−2) …… (n−r +1)

$$p\,(n, r) = \frac{n!}{(n-r)!}$$

P (n, n) = n!

PRODUCT RULE: 2.3

If any activity can be performed in r successive steps and step 1 can be done in n ways, step 2 can be done in n ways, , step r can be done in n ways, then the activity can be done in (n , n , … . n) ways.

The r- permutations of the set can be obtained by the first forming the C(n,

r) r-combinations of the set and then arranging (ordering) the elements in each r-combinations, which can be done in P(r, r)

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ways. Thus

$$P(n,r) = C(n,r).\, P(r,r)$$

Therefore $(n, r) = \frac{p(n,r)}{p(r,r)} = \frac{n!(n-r)!}{r!/(r-r)!}$

$$= \frac{n!}{r!/(r-r)!}$$

$$C(n, n) = 1$$

RESULT: 2.3.1

Since the number of ways of selecting out r elements from a set of n elements is the same as the number of ways of leaving (n-r) elements in theset, it follows that

$C(n, r) = C(n, n-r)$

$$C(n, n-r) = \frac{n!}{(n-r)!(n-(n-r)\}!}$$

$$= \frac{n!}{(n-r)!\,r!}$$

$$= C(n, r).$$

PASCAL'S IDENTITY: 2.4

If n and r are positive integers. Where n $\geq$r,

$$\text{Then } \binom{n}{r-1} + \binom{n}{r} = \binom{n+1}{r}$$

Proof:

Let S be a set containing (n+1) elements, one of which is 'a',Let S' = S – {a}.

The number of subset of S containing r elements is1^{n} 1.

A subset of S with r elements either contains 'a' together with (r-1)

elements of S' or contains r elements of S' which do not include 'a'.

The number of subsets of (r-1) elements of S' = $\binom{n}{r-1}$

There for the number of subsets of r elements of S that contain 'a' = $\binom{n}{r-1}$.

Also the number of subset of r elements of S that do no contain 'a' =

that of S' = $\binom{n}{r}$.

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Consequently, $\binom{n}{r-1} + \binom{n}{r} = \binom{n+1}{r}$

RESULT: 2.4.1

This result can also be proved by using the values of $\binom{n}{r-1}$, $\binom{n}{r}$ and $\binom{n+1}{r}$.

COROLLARY: 2.4.2

$$\mathrm{C}(\mathrm{n}+1, \mathrm{r}+1) = \sum_{i-1}^{n} C(i, r)$$

Proof:

By Pascal's identity we get,

$$\binom{n}{r-1} + \binom{n}{r} = \binom{n+1}{r}$$

C(n, r−1)+ C(n, r) = C (n+1,r)

Changing n to i and r to r+1, we get

C (i, r) +C (i, r+1) = C (i+1, r+1)

That is C (i, r) = C (i+1, r+1) −C (i, r+1) -------------------- (1)

Putting i=r, r+1, …… n in (1) and adding, we get,

$$\sum_{\mathrm{i}-1}^{\mathrm{n}} \mathrm{C}(\mathrm{i}, \mathrm{r}) = \mathrm{C}(\mathrm{n}+1, \mathrm{r}+1) - \mathrm{C}(\mathrm{r}, \mathrm{r}+1)$$

= C(n+1, r+1). [since C(r, r+1)= 0]

VANDERMONDE'S IDENTITY: 2.5

If m, n, r are non- negative integer where r≤ m orn, then

$$\mathrm{C}(\mathrm{m}+\mathrm{n}, \mathrm{r}) = \sum_{\mathrm{i}-1}^{\mathrm{n}} \mathrm{C}(\mathrm{m}, \mathrm{i}-\mathrm{r}). \mathrm{C}(\mathrm{n}, \mathrm{i})$$

Proof:

Let m and n be the number of elements in sets 1 and 2 respectively.

Then the total number of ways of selecting r elements from the union of sets1 and 2.

= C(m+ n, r)

The r elements can also be selected by selecting i elements from set 2 and (n-i) elements from set 1, where i=0, 1, 2, ……. r. This selection can bedone in C(m, r-i)C(n, i) ways by the product rule.

The (r+1) selections corresponding to i=0, 1, 2, …..r are disjoint.

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Hence, by the sum rule.

$$C(m+n,r)=\sum_{i-0}^{n} C(m,i-r).C(n,i) \; or \; \sum_{i-0}^{n} C(m,i).C(n,r-i)$$

SUM RULE: 2.5.1

If ‘r’ activities can be performed in n , n , … . n ways and if they are disjoint,cannotbeperformedsimultaneously,then anyoneoftheractivities

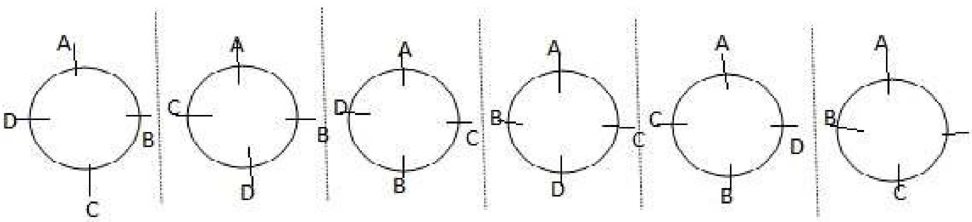
can be performed in (n + n + ⋯ + n) ways.

CIRCULAR PERMUTATION: 2.6

The permutations discussed so far can be termed as linear permutation, as on the objects were assumed to be arranged in a line. If theobjects are arranged in a circle or any closed curve is called circular permutation and the number of circular permutations will be different fromthe number of linear permutations.

EXAMPLE: 2.7

An arrange 4 elements A, B, C, D in a circle as follows: Fix one of the elements, A is the top point of the circle. The other 3 elements B, C, D are permuted in all possible ways, resulting ways, resulting in 6 = 3! Different circular permutations are



RESULT: 2.8

Circular arrangement are considered the same when one can be obtained fromtheotherbyrotation,therelativepositions and notthe actual positions of the objects alone count for different circular permutations. Let the number of different circular arrangements of 4 elements = (4−1)! = 6.

Similarly, the number of different circular arrangement of n objects

= (n−1)!

If no distinction is made between clockwise and counterclockwise circular arrangements. For example, if the circular arrangements in the firstand the last figures are assumed as the same, then the number of different

circular arrangements = (n − 1)! .

PIGEONHOLE PRINCIPLE: 2.9

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Though this principle stated is deceptively simple, it is sometimes useful in counting methods. The deception often lies in recognizing the problems.

STATEMENT: 2.10

If n pigeons are accommodated in m pigeon-holes and n >m then at least one pigeonhole will contain two or more pigeons. Equivalently, if n objects are put in m boxes and n>m, then at least one box will contain twoor more objects.

Proof:

Let the n pigeons be labeled P, P, … … P_n and the m pigeonholes be labeled H , H , … … H_1. If P, P, … … P_1 are assigned to H , H , … … H_1 respectively.

Left with the (n−m) pigeons P_1 , P_1 , … … P_n. If these left over pigeons are assigned to the m pigeonholes again in any random manner, at least one pigeonhole will contain two or more pigeons.

PRINCIPLE OF INCLUSION – EXCLUSION:

STATEMENT:

If A and B are finite subsets of a finite universal set ∪, then

$|A \cup B| = |A| + |B| - |A \cap B|$, where $|A|$ denotes the cardinality of the number of elements in the set A.

This principle can be extended to a finite number of finite sets A , A , … . . A_n as follows:

$|A \cup A \cup \ldots . . \cup A_n|$

$$= \sum_{i} |Ai| - \sum_{i<j} |Ai - \cap Aj| + \sum_{i<j<k} |Ai - \cap Aj \cap Ak|$$

$$- \cdots \ldots . + (-1)^n \quad |A \cup A \cup \ldots . . \cup A_n|$$

Where the first sum is over all I, the second sum is over all pairs I, j with i>j the third sum is over all triples I, j, k with i<j<k and so on.

Proof:

Let A\B = {a , a , … . . a_r }

B\A = {b , b , … . . b_s}

A∩B = {x , x , … . . x_t}

Where A\B is the set of those elements A which are not in B.

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Then A = {a ,a , ... a , x , x , ... x_1} and

B = {b ,b , ... b_1, x , x , ... x_1 }

Hence, A∪B = {a ,a , ... a , x , x , ... x_1, b , b , ... b_1} Now |A| +|B|−|A∩B| = (r+t) +(r+t) −t

=r+s+t

=|A∪B| ---------------------(1)

Let us now extent the result to 3 finite sets A, B, C.

$$|A\cup B\cup C|=|A\cup(B\cup C)|$$

$$=|A|+|B\cup C|-|A\cap(B\cup C)|$$

$$=|A|+|B|+|C|-|B\cap C|-\{(A\cap B)\cup(A\cap C)\} \quad \text{[by (1)]}$$

$$=|A|+|B|+|C||-|B\cap C| \quad |-\{(A\cap B)\cup \quad (A\cap C)\}-|(A\cap B)\cap(A\cap C)|\}$$

[by (1)]

$$=|A|+|B|+|C|-|A\cap B| \quad -|B\cap C|--|C\cap A|+|A\cap B\cap C|$$

$$|A\cup B\cup C|=|A|+|B|+|C|-|A\cap B| \quad -|B\cap C|--|C\cap A|+|A\cap B\cap C|$$

PROBLEM:

a) Assuming that repetitions are not permitted, how many four digitnumbers can be formed form the six digits 1,2,3,,5,6,,8?

b) How many of these numbers are less than 4000?

c) How many of the numbers in part (a) are even?

d) How many of the numbers in part (a) are odd?

SOLUTION:

a) The 4-digit number can be considered to be formed by filling up 4 blank spaces with the available 6 digits. Hence, the number of 4-digitsnumbers.

= the number of 4-permutatuions of 6 numbers

= P(6,4)

"P(n,r)=n(n−1)(n−2)…..(n−r+1)"

= P(6, 4)

= 6×5×4×3

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$= 360$

b) If a 4-digit number is to be less than 4000, the first digit must be 1, 2, or 3. Hence the first space can be filled up in 3 ways. Corresponding to any one of these 3 ways, the remaining 3 spaces can be filed up with the remaining 5 digits in P (5, 3) ways.

Hence, the required number

$= 3 \times P(5, 3)$

"$P(n,r) = n(n-1)(n-2)\ldots..(n-r+1)$"

$3 \times P(5, 3) = 3 \times 5 \times 4 \times 3$

$=180$

c) If the 4-digit number is to be even, the last digit must be 2 or 8. Hence, the last space can be filled up in 2 ways. Corresponding to any one of these 2 ways, the remaining 3 spaces can be filled up with the remaining 5 digits in P(5, 3) ways.

Hence the required number of even number

$=2 \times P(5, 3)$

"$P(n,r) = n(n-1)(n-2)\ldots..(n-r+1)$"

$2 \times P(5, 3) = 2 \times 5 \times 4 \times 3$

$= 120$

d) Similarly the required number of odd numbers

$=4 \times P(5, 3)$

$=$ "$P(n,r) = n(n-1)(n-2)\ldots..(n-r+1)$"

$4 \times P(5, 3) = 4 \times 5 \times 4 \times 3$

$= 240$

CONCLUSION

A large number of mathematics departments now offer a course in combinatorial mathematics covering graph theory and enumeration. The major question for an instructor in teaching this course is the balance between problem-solving and theory. Typically the course's primary focus is on problem-solving with some associated theory. Graph theoretic and enumerative problem solving skills are critical for computer science and operations research as well as much of discrete probability. Since some combinatorial problem-solving topics are part of the Common Core high school curriculum, an applied focus is appropriate for prospective high school teachers. However, based on the philosophy of the mathematics major and preferences of an instructor, a valuable course for students can be designed that puts the primary focus on theory. Graph theory, in particular, is a good setting for strengthening proof skills because of the visual nature of the subject and its proofs. That said, the focus of the following discussion is on a problem-solving approach to the course.