# Periodic Solutions Averaging Methods in Nonlinear Ordinary Differential Equations

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**ABSTRACT:** A nonstandard approach to averaging theory for ordinary differential equations and functional differential equations is developed. We define a notion of perturbation and we obtain averaging results under weaker conditions than the results in the literature. The classical averaging theorems approximate the solutions of the system by the solutions of the averaged system, for Lipschitz continuous vector fields, and when the solutions exist on the same interval as the solutions of the averaged system. We extend these results to perturbations of vector fields which are uniformly continuous in the spatial variable with respect to the time variable and without any restriction on the interval of existence of the solution.

***Keywords**: Ordinary Differential Equation, Functional differential equations, averaging, stroboscopic method, nonstandard analysis.*

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## INTRODUCTION

Differential equation is one of the most interesting and useful chapter in the study of mathematics. In the 17th century Isaac Newton of England and Gottfried Wilhelm Leibnitz of Germany invented the Calculus is the mathematical study of motion and change. Newton, one of the great mathematical and physicists of all time, applied the calculus to formulate his laws of motion and gravitation.

Calculus was invented for the purpose of solving problems that deal with continuously changing quantities. The problem of finding slop of the tangent line to the curve, is studied by the limiting process called differentiation and the equations are called differential equation.

Differential equations arise in many areas of science and technology, specifically whenever a deterministic relation involving some continuously varying quantities modeled by functions and their rates of changes in space and /or time is known its postulates. This is illustrated in classical mechanics where the motion of a body is described by its position and velocity as the time value varies.

Suppose that $x = \varphi(t)$ is a solution of the equation $\dot{x} = f(x)$ where

$x \in D \subset \Re^n$. Suppose that there exists a positive number T such that

$\varphi(t+T) = \varphi(t)$ for all $t \in \Re$ . Then $\varphi(t)$ is called a periodic solution of the equation with period T.

**DEFINTIONS WITH EXAMPLES**

**DEFINITION: 1.1**

**DIFFERENTIAL EQUATION**

An equation involving derivatives of one or more dependent variables with respect to one or more independent variables is called a **Differential Equation**.

**Ex:**

$$\frac{d^2y}{dx^2} + xy\frac{dy}{dx} = 0$$

**DEFINITION: 1.2**

**ORDINARY DIFFERENTIAL EQUATION**

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A differential equation involving ordinary derivatives of one or more dependent with respect to a single independent variable is called an **Ordinary Differential Equation.**

**Ex :**

$$\frac{dy}{dx} + x^2 y = xe^x + \sin x$$

**DEFINITION: 1.3**
**PARTIAL DIFFERENTIAL EQUATION**

A differential equation involving partial derivatives of one or more dependent variables with respect to more than one independent variable is called a **Partial Differential Equation.**

**Ex :**

$$\frac{\partial u}{\partial s} + \frac{\partial v}{\partial t} = 0$$

**DEFINITION: 1.4**
**ORDER OF A DIFFERENTIAL EQUATION**

The order of the highest ordered derivative involved in a differential equation is called the **Order of a Differential Equation.**

Here order= 2

**DEFINITION: 1.5**
**DEGREE OF A DIFFERENTIAL EQUATION**

The **Degree of a Differential Equation** is the degree of the highest derivative when the differential coefficients are cleared of fractions and radicals.

**Ex :** $[1 + (dy/dx)^2]^{3/2} = ad^2y/dx^2$

Here order=2

Degree=2

**PERIODIC SOLUTONS AVERAGING METHODS**
**AN ENERGY BALANCE METHOD FOR LIMIT CYCLES**

The non-linear character of isolated periodic oscillations makes their detection and construction difficult.

Here we discuss limit cycles and other periodic solutions in the phase plane $\dot{x}$=y,$\dot{y} = Y(x, y)$ which allows the mechanical interpretation and elsewhere, consider the family of equation.

$\dot{x} + \varepsilon h(x, \dot{x}) + x = 0$ (2.1) $\longrightarrow$

Phase plane we have,

$$\dot{X} = y,$$
$$\dot{y} = Y(x, y)$$
$$\ddot{X} = -\varepsilon h(x, \dot{x}) - x$$

$\dot{y} = -\varepsilon h(x, y) - x$ (2.2) $\longrightarrow$

Assume that $|\varepsilon| << 1$,

So that the nonlinearity is small, and that h(0,0)=0, so that origin is an equilibrium point. Suppose we have reaction to think that there is at least one periodic solution with space path surrounding the origin an appraisal of the phase plane in which energy loss and energy gain take place might give grounds for expecting a limit cycle.

When $\varepsilon = 0$,

$\ddot{x} + x = 0$ (2.3) $\longrightarrow$

Called the linearized equation.

Auxiliary equation

$$m^2 + 1 = 0$$

M=$\pm i$

Complementary function

$= \text{A} \cos \beta\, x + \sin \beta\, x$

$= A \cos t + B \sin t$

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$=a\cos\alpha\cos t + a\sin\alpha\sin t$

$=a\cos t + \alpha$

It's general solutions is $x(t) = a\cos(t+\alpha)$

Where a and $\alpha$ arbitrary constants. So for as the phase diagram is concerned, we may restrict a and $\alpha$ to the cases.

a$> 0, a = 0$

since different values of a simply correspond to different time origins, the phase paths and representative points remain unchanged. The family of phase paths for (2.3) is given parametrically

$x= a\cos\theta$

$y= -a\sin\theta$

which is the family of circles $x^2 + y^2 = a^2$.

The period of all these motions is equal to $2\pi$.

For small enough $\varepsilon$we expect that any limit cycle or any periodic motion,

Of (2.1) will be close to one of the circular motions (2.4) and will

Approach it as $\varepsilon \to 0$.

Therefore the value of

$$x(t) \approx a\cos t$$

$$y(t) \approx -\sin t \; and T \approx 2\pi \qquad (2.4) \longrightarrow$$

on the limit cycle, where T is its period.

[with$\varepsilon$ h in place of h and g(x) = x] the change in energy.

$\varepsilon(t) = \frac{1}{2}x^2(t) + \frac{1}{2}y^2(t)$

Over one period $0 \le t \le T$, is given by

$$\varepsilon(t) - \varepsilon\int_0^t h\big(x(t), y(t)\big)y(t)dt$$

Since the path is closed, $\varepsilon$ returns to its original value after one circuit.

Therefore ,$\int_0^t h(t), y(t)y(t)dt = 0$ on the limit cycle.

This relation is exact. Now insert the approximations (2.4) into the integral. We obtain the energy balance equation for the amplitude $a> 0$of the periodic motion.

$\varepsilon(2\pi) - \varepsilon(0) = \varepsilon a\int_0^{2\pi} h(a\cos t, -a\sin t)\sin t dt = 0 \qquad (2.5) \longrightarrow$

(or)

$\int_o^{2\pi} h(a\cos t, -a\sin t)\ \sin t dt=0 \qquad (2.6) \longrightarrow$

[since after getting rid of the factor ($-\varepsilon a$)]

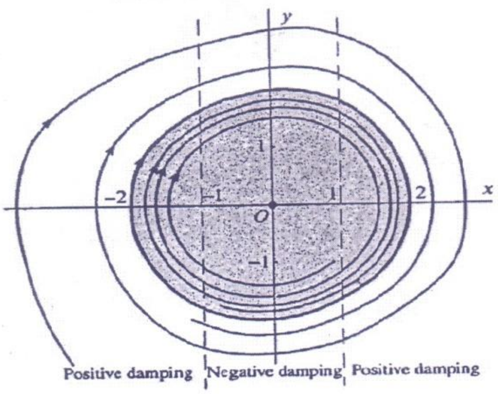
This is an equation which, in principle can be solved for the unknown amplitude a of a limit cycle.

**EXAMPLE: 2.1.**

Find the approximate amplitude of the limit cycle of the van der pol equation

$\ddot{x} + \varepsilon(x^2 - 1)\dot{x} + x = 0$, when$\varepsilon$is small $\qquad (2.7) \longrightarrow$

**FIGURE: 1**




Phase diagram for the vander pol equation $\ddot{x}+\varepsilon(x^2 - 1)\dot{x} + x = 0$ with $\varepsilon = 0.1$

the limit cycle is the outer rim of the shaded region.

*h(x,y)=*$(x^2-1)y$

assuming that $x= a\cos t$, the energy balance equation

$$\int_0^{2\pi}((a^2\cos^2 t-1)\sin t)\sin t dt=0$$

$$a^2\int_0^{2\pi}\cos^2 t\sin^2 t - \int_0^{2\pi}\sin^2 t=0$$

$$a^2\int_0^{2\pi}(\cos t\sin t)^2 - \int_0^{2\pi}\sin^2 t dt=0$$

$$a^2\int_0^{2\pi}(\frac{\sin 2t}{2})^2 dt-\int_0^{2\pi}\sin^2 t dt=0$$

$x=2t$, $\frac{dx}{dt}=2$,dt= $\frac{dx}{2}$

$$\frac{a^2}{4}\int_0^{4\pi}\sin^2 x\frac{dx}{2}-\int_0^{2\pi}\sin^2 t dt=0$$

$$\frac{a^2}{8}\int_0^{4\pi}\sin^2 x dx-\int_0^{2\pi}\Delta\sin^2 t dt=0$$

$$\frac{a^2}{8}\int_0^{4\pi}(\frac{(1-\cos 2x)}{2})\,dx - \int_0^{2\pi}(\frac{(1-\cos 2t)}{2})\,dt=0$$

$$\frac{a^2}{16}\int_0^{4\pi}(1-\cos 2x)\,dx - \frac{1}{2}\int_0^{2\pi}(1-\cos 2t)dt=0$$

$$\frac{a^2}{16}[x-\left(\frac{\sin 2x}{2}\right)]_0^{4\pi} - \frac{1}{2}[t-(\frac{\sin^2 t}{2})]_0^{2\pi}=0$$

$$\frac{a^2}{16}[4\pi]-\frac{1}{2}[2\pi]=0$$

$$\left(\frac{a^2}{4}\right)\pi-\pi=0$$

$$\pi\left(\left(\frac{a^2}{4}\right)-1\right)=0$$

$$(\frac{a^2}{4})-1=0$$

This leads to the equation $(\frac{1}{4})\,a^2-1=0$ with the periodic solution a=2

Shows the limit cycle for $\varepsilon=0.1$

As $\varepsilon$ becomes larger, the shape of the limit cycle becomes significantly different.

From a circle although the amplitude remains close to 2.

This is shown in fig. (2) for the case $\varepsilon=0.5$

The corresponding time solution is shown in fig(2) the period is slightly greater than $2\pi$.

We should expect that unclosed paths near enough to the limit cycle spiraling gradually,

$$x\approx a(t)\cos t$$

$$y\approx -a(t)\sin t$$

Where a(t) is nearly constant over a limit interval if $0\leq t\leq 2\pi$

The approximation equation (2.6) is given by

$$g(a)=\varepsilon a\int_0^{2\pi}h(acost,-asint)sintdt \qquad (2.8) \longrightarrow$$

Let a $\approx a_0(>0)$ on the limit cycle.

Then by equation (2.6).

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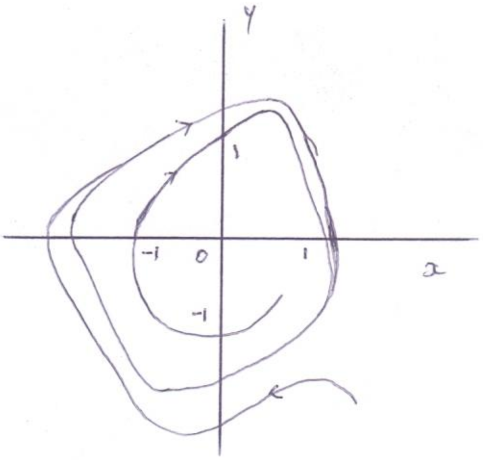


$g(a_0) = 0$

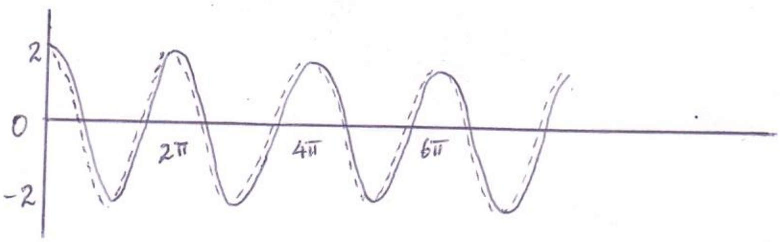


**FIGURE: 2**

a) Phase diagram
b) Time solution

the limit cycle of the vander pol equation $\ddot{x}+\varepsilon(x^2 - 1)\dot{x} + x = 0$ with $\varepsilon = 0.5$.
if the limit cycle is stable, then along nearby interir segments (a$< a_0$)
energy is gained, and along exterior segments $(a > a_0)$ energy is lost.

This is to say, for some value of $\delta > 0$,

$$g(a) > 0, when a_0 - \delta\rho < a < a_0 \longrightarrow (2.9)$$

$$g(a) < 0 \; when \; a_0 < a < a_0 + \delta$$

Similarly if the signs of the inequalities are both reversed, the limit cycle is unstable.

The existence and stability of a limit cycle of amplitude $a_0$ are determined by the conditions. Stable if

$$g(a_0) = 0$$

Stable if $g'(a_0) < 0$

Unstable if $g'(a_0) > 0 \longrightarrow (2.10)$

**NOTE:**

The signs of these inequalities are reversed when the sign of ε is reversed.

(The case $g(a_0)$ is $> 0$ or $< 0$ on both sides also implies instability, but this is not shown by testing the sign of $g'(x_0)$ because its value is zero in this is not shown by testing the sign of $g'(x_0)$ because its value is zero in this cases)

**AMPLITUED AND FREQUENCY ESTIMATES POLAR CO-ORDINATE**

The equation,

$$\ddot{X} + \varepsilon h(x, \dot{X}) + x = 0$$

and the equivalent system

$$\ddot{x} = y \qquad (3.1) \longrightarrow$$

$$\dot{y} = -\varepsilon h(x, y) - x$$

and suppose that it has at least one periodic time solution, corresponding to a closed path.

Let any phase path be represented parametrical by time dependent polar co-ordinates $a(t)$,$\theta(t)$

the polar co-ordinates,

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$$\dot{a} = -\varepsilon h \sin\theta \qquad (3.2)$$

$$d\theta = -1 - \varepsilon a^{-1} h \cos\theta \qquad (3.3)$$

And the differential equation for the phase path

$$\left(\frac{da}{d\theta}\right) = \frac{\varepsilon h \sin\theta}{1+\varepsilon a - 1h \cos\theta} \qquad (3.4)$$

Where for brevity, $h$ stands for $hacos\theta, asin\theta$

This equation hold generally, whether $\varepsilon$ is small or not

**FIGURE(A):**

Shows a closed path

Which may be a limit cycle or one the of the curves constituting a center.

Let its limit period be T.

Then $a(t), \theta(t)$ and therefore h all have time period T,

meaning that along the closed path $a(t_0 + T) = a(t_0)$ for every to and so for other variables.

Regarded as functions of the angular co-ordinates $\theta$, all the variables

**FIGURE:(B**)

When t increases, $\theta$ decreases.

The direction along the path for t increasing is clockwise, following

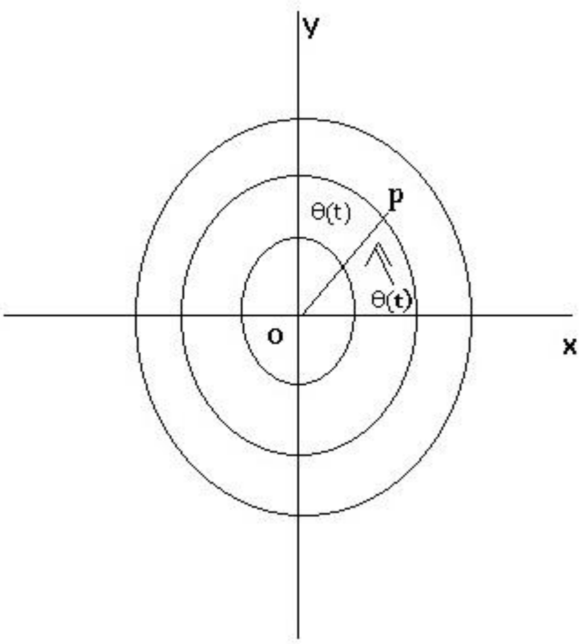
the general rule for the phase plane with $y = \dot{X}$.

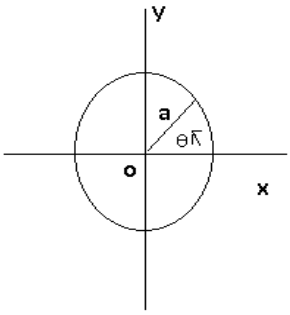
The positive rule for the polar co-ordinates $\theta$ is counter clockwise.

A typical cycle of a closed path is therefore described by the end conditions.

$a = a_0, \quad \theta = 2\pi, at\ t = 0 \qquad (3.5)$

$a = a_0 \quad \theta = 0, at\ t = T. \qquad (3.6)$




a) A limit cycle and its approximating phase diagram for $\varepsilon = 0$

b) Showing relation between $\theta$ and t.

Now suppose that $\varepsilon$ is small. $|\varepsilon| << 1$.

As described in the previous section we expect that the closed path will consist of a small distortion or perturbation of one of the circular paths of the linearized system

$$\ddot{X} + x = 0$$

(or) $$\dot{X} = y,$$

$$\dot{y} = -x$$

Which are indicated as broken lines in figure(a).

We shall give more precision to this idea, and also provide estimates of the period t and the circular frequency

$$\omega = \frac{2\pi}{T}$$

by approximation to equation (3.2)(3.3)(3.4) for small values of $\varepsilon$ by expanding the right –hand side of (3.4) in powers of $\varepsilon$ we obtain.

$\frac{da}{d\theta} = \varepsilon h sin\theta + o(\varepsilon^2)$ (3.7)

Integrate with respect to $\theta$, over the range $\theta = 2\pi$ decreasing to $\theta = 0$

$$a(\theta) - a(2\pi) = o(\varepsilon),$$

(or) $a(\theta) = a_0 + o(\varepsilon)$ (3.8)

Since $a(2\pi) = a_0$ from (3.5)

The deviation from a circular path of radius $a_0$ is therefore small, of order of magnitude $\varepsilon$.

Integrate (3.6) over the full range of the cycle (3.5) from

$$\theta = 2\pi \ to \ \theta = 0$$

We obtain

$$a_0 - a_0 = 0$$

$$0 = \varepsilon \int_0^{2\pi} h(acos\theta, asin\theta)sin\theta d\theta + o(\varepsilon^2)$$

$$= -\int_0^{2\pi} h(acos\theta, asin\theta)sin\theta d\theta + o(\varepsilon)$$

After cancelling the factor $\varepsilon$ and reversing the direction of the integral.

Now substitute for a from (3.5) and (3.6) expand $hacos\theta, asin\theta$

Under the integral sign.

We have

$$\int_0^{2\pi} h(a_0 \, cos\theta, a_0 sin\theta)sin\theta d\theta = o(\varepsilon)$$

Since the integral on the left does not depend on $\varepsilon$, a necessary condition for

The phase path to be closed is that

$\int_0^{2\pi} h(a_0 \, cos\theta, a_0 sin\theta)sin\theta d\theta = 0$ (3.9)

This serves as an equation for the approximation amplitude the expression (2.7) for $\theta$ with respect to t,

$\theta = 2\pi - t + o(\varepsilon)$ (3.10)

When this substitude into (3.9) and the leading term in the expansion of h

Retained, we have to obtain the period of the cycle, T,

$$T = \int_0^{2\pi} \frac{d\theta}{1 + \varepsilon a^{-1}} h(acos\theta, asin\theta)cos\theta$$

Substitute $a = a_0 + o(\varepsilon)$,retain the leading terms then expand

$$= \frac{1}{(1 + \varepsilon a_0^{-1} hcos\theta)}$$

We have

$$T = \int_0^{2\pi} (1 - \varepsilon a_0^{-1} \, h(a_0 cos\theta, a_0 sin\theta)cos\theta + o(\varepsilon^2)d\theta \quad (3.11)$$

$$\approx 2\pi - \frac{\varepsilon}{a_0} \int_0^{2\pi} h(a_0 \, cos\theta, a_0 sin\theta)cos\theta d\theta$$

The error being of order $\varepsilon^2$

The circular frequency of the periodic oscillation is

$$\frac{2\pi}{T} \approx 1 + \frac{\varepsilon}{2\pi a_0} \int_0^{2\pi} h(a_0 \, cos\theta, a_0 sin\theta)cos\theta d\theta \quad (3.12)$$

With error of order $\varepsilon^2$.

**EXAMPLE: 3.1**

Obtain the frequency of the limit cycle of the Vander pol equation

$$\ddot{X} + \varepsilon(x^2 - 1)\dot{X} + X = 0$$

Correct to order $\varepsilon$.

Since $h(x, \dot{X}) = (x^2 - 1)\dot{X}$ and the amplitude $a_0 = 2 \ to$ order $\varepsilon$

$$\omega = 1 + \frac{\varepsilon}{4\pi} \int_0^{2\pi} (cos\theta)^2 - 1)(2sin\theta)cos\theta d\theta$$

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$$= 1 + \frac{\varepsilon}{4\pi}\int_0^{2\pi}(4cos^2\theta - 1)(2sin\theta)cos\theta d\theta$$
$$= 1 + \text{zero}$$

## CONCLUSION

A perturbated problem whose solution can be approximated on the whole problem domain, whether space or time by a single asymptotic expansion has a regular perturbation. Most often in applications, an acceptable approximation to a regular perturbated problem is found by simply replacing the small parameter $\varepsilon$ by zero everywhere in the problem statement. This corresponds to taking only the first term of the expansion yielding an approximation that converges perhaps slowly to the true solution as decreases.

Singular perturbation theory is a rich and on going area of exploration or mathematicians physics and other researches. The method used to tackle problem in this field are many. The more basic of these include the method asymptotic expansion and WKB approximation for spatial problems and in time, The Poincare-Linstedt method, The method of multiple scales and periodic averaging.

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