International Journal on Future Revolution in Computer Science & Communication Engineering
Volume: 4 Issue: 1
ISSN: 2454-4248
277 – 284

Some New Implication Operations Emerging From Fuzzy Logic

Manish K. Srivastava [1], S. Kumar[2] and R.S. Parihar

Sunrise University, Alwar, Rajsthan

[1] *tomanishibs@gmail.com*

[2] *sanjeevibs@yahoo.co.in*

Abstract: We choose, from fuzzy set theory, t-norms, t-conorms and fuzzy compliments which forms dual triplet that is *(i,u,c)* that satisfy the DeMorgan's law, these dual triplet are used in the construction of fuzzy implications in fuzzy logic. In this work introduction of fuzzy implication is given, which included definition of fuzzy implications and their properties and also distinct classes of fuzzy implication (S, R and QL-implications). Further also described previous work on fuzzy implication and supporting literature of construction of fuzzy implication are given. Finally main contribution of work is to design new fuzzy implication and their graphical representations.

Keywords: *Fuzzy Logic, membership function, implication operations.*

*****

1. INTRODUCTION

It seems intuitive that we should balance the degree of precision in a problem with the associated uncertainty in that problem. Hence, this work recognizes that uncertainty of various forms permeates all scientific endeavors and it exists as an integral feature of all abstractions, models, and solutions. It is the intent of this work to introduce methods to handle one of these forms of uncertainty in our technical problems, the form we have come to call fuzziness. When we start working with fuzziness the important part is fuzzy implications, therefore we worked on the fuzzy implications and other operators.

Fuzzy implication is the extension of the classical binary logic to fuzzy logic in a broad sense. The implication operator ($\rightarrow$) plays a significant role in the classical two-valued logic. Firstly, from the classical implication one can obtain all other basic logical connectives of the binary logic, *viz*, the binary operators ($\vee$) or (^) and the unary negation operator ($\neg$). Secondly, the implication operator holds the center stage in the inference mechanisms of any logic, like modus ponens, modus tollens, and hypothetical syllogism in classical logic.

The implication in classical binary logic works only on two truth values 0 and 1 while a fuzzy implication is a $[0; I]^2 \rightarrow [0; 1]$ mapping. So besides the boundary condition (1,1), the first step to work on fuzzy implications is naturally to determine which fundamental requirements, a fuzzy implication should fulfill. Most considerations are taken either from the point of view that a fuzzy implication is a generalization of the implication in classical binary logic, or from the point of view of fulfilling different requirements from specific applications, especially approximate reasoning.

In the earlier literature, different authors have proposed many individual definitions of fuzzy implications. Besides these individual definitions of fuzzy implications, Trillas et al.[16] proposed two classes of fuzzy implications generated from the fuzzy logic operators negation, conjunction and disjunction. They are strong implications (*S*-implications for short) and residuated implications (*R*-implications for short). *S*-implications are defined on the basis of; $p \rightarrow q = \neg p \vee q$ in classical binary logic, where p and q are two propositions.

R-implications are defined based on the fact that the implication is residuated with and in the classical binary logic. *S*- and *R*-implications are widely used in the early works about approximate reasoning (e.g., [7], [18]). Besides *S*- and *R*- implications, there is another class of fuzzy implications generated from the fuzzy logic operators negation, conjunction and disjunction coming from quantum logic. So they are called quantum logic implications (*QL*-implications for short). S-, R- and *QL*- implications are the most important classes of fuzzy implications which are widely studied in different aspects from the beginning until now. Examples of very recent works are: ([1], Baczy'nski 2006) and ([2], Baczy'nski 2007) work on the properties of S-implications generated from non-strong negations. Shi, (2008) work on the properties of a group of *QL*-implications. ([11], Mesiar 2004) and ([13], Pei 2002) work on the properties of *R*-implications generated from left-continuous *t*-norms. ([12], Morsi 2002) and ([19], Whalen 2007) work on how fuzzy rules are represented by *S*-, *R*- and *QL*-implications.

2. PRELIMINARIES:

Following are the fuzzy implications and their properties:

2.1 Definition

Fuzzy implication, I is the function of the form

$I:[0,1] \times [0,1] \rightarrow [0,1]$

In classical logic, I can be defined in several distinct forms and these are equivalent, but their extensions to fuzzy logic are not equivalent and result in distinct classes of fuzzy implication.

2.2 Distinct Classes of Fuzzy Implications

i) S-implications

ii) R-implications

iii) QL-implications

2.2.1 *S*-Implications-

Fuzzy implications which are obtained by $I(a,b)=u(c(a),b)$ are usually referred to as S-implications. The following are examples of well-known S-implications, all of which are based on the standard fuzzy compliment and differ from one another by the chosen fuzzy unions:

Kleene-Dienes implication:

$I(a,b)=\max(1-a,b)$

2.2.2 *R*-Implications- This is characterized by,

$$I(a,b) = \sup\{x \in [0,1] \mid i(a,x) \le b\}$$

They are usually called R-implications, as they are closely connected with so called residuated semigroups.

2.2.3 *QL*-Implications- which are characterized by,

$I(a,b)=u(c(a),i(a,b))$,

2.3 Properties of Fuzzy Implications

- Monotonicity in first argument:

$$a \le b \text{ implies } I(1,x) \ge I(b,x)$$

- Monotonicity in second argument:

$$a \le b \text{ implies } I(x,a) \le I(x,b)$$

- Dominance of falsity: $I(0, a)$=l
- Neutrality: $I(1, b)=b$
- Identity: $I(a,a)$=l

Truth value of antecedent is equal to consequent.

- Exchange property:

ISSN: 2454-4248
277 – 284

$$I(a, I(b, x)) = I(b, I(a, x))$$

This is a generalization of equivalence of:

$$a \Rightarrow (b \Rightarrow x) \text{ and } b \Rightarrow (a \Rightarrow x)$$

- Boundary condition:
  $I(a,b)=1$ iff $a \leq b$
- Contraposition:
  $I(a,b)=I(c(b), c(a))$ for fuzzy compliment c.
- Continuity:
  I is a continuous function.

3. APPROXIMATE REASONING IN FUZZY RULE-BASED SYSTEMS

A fuzzy rule-based system can be applied to fuzzy control or to fuzzy decision making. There are four procedures of a fuzzy rule-based system: fuzzification, fuzzy rule base, approximate reasoning, and defuzzification. Fuzzy implications play significant roles in the approximate reasoning procedure in a fuzzy rule-based system. We give below an overview of the approximate reasoning and state the role fuzzy implications play in the approximate reasoning procedure.

3.1 Approximate Reasoning

The approximate reasoning procedure is based on the generalized modus ponens, generalized modus tollens, generalized fuzzy method of case etc., and realized through Zadeh's compositional rule of inference (1.3). Recall that the generalized modus ponens of a SISO (Single Input Single output) rule has the form:

if X is A then Y is B

where X and Y are linguistic variables on the universe of discourse U and V respectively.

A and A' are fuzzy sets on U, and B and B' are fuzzy sets on V.

The output fuzzy set B' is determined by Zadeh's compositional rule of inference.

The generalized modus tollens of an SISO rule has the form:

if X is A then Y is B

The output fuzzy set A' is determined by Zadeh's compositional rule of inference:

$$(\forall\, x \in U)\left[A'(x) = \sup_{y \in V} T\left(B'(y), R\left(A(x), B(y)\right)\right)\right]$$

In fuzzy logic, the classical binary negation, conjunction, disjunction and implication are extended to mappings that take values in the unit interval respectively. A fuzzy negation operator is normally modeled as a fuzzy negation. A fuzzy conjunction operator is normally modeled as a conjunction on the unit interval or (more usually) as a triangular norm (*t*-norm for short). A fuzzy disjunction operator is normally modelled as a triangular conorm (*t*-conorm for short). There are many approaches to model a fuzzy implication operator. It can be constructed from the other three fuzzy logic operators, or it can be constructed from some parameterized generating functions.

In many-valued logic we extend the classical binary negation to the unit interval as a [0; 1] →[0; 1] mapping as follows:

A mapping N: [0; 1] → [0; 1] is a fuzzy negation if it satisfies:

ISSN: 2454-4248
277 – 284

N1. Boundary conditions: $N(0) = 1$ and $N(l) = 0$,

N2. Monotonicity : $\left(\forall\, x, y \in [0,1]^2\right)\left(x \le y \Rightarrow N(x) \ge N(y)\right)$

Moreover, a fuzzy negation N is said to be strict if N is a continuous and strictly decreasing mapping.

A fuzzy negation N $\neg$ is said to be strong if $N(N(x))=x$, for all $x \in [0; 1]$.

3.2 CONJUNCTIONS

We already know the truth table of the classical binary conjunction ^. In many-valued logic we extend the classical binary conjunction to the unit interval as a $[0; I]^2 \rightarrow [0; 1]$ mapping as follows:

A mapping C: $[0; I]^2 \rightarrow [0; 1]$ is a conjunction on the unit interval if it satisfies:

C_l. Boundary conditions: $C(0; 0) = C(0; 1) = C(l; 0) = 0$ and $C(l; 1) = 1$,

C_2. Monotonic: $\left(\forall\, x, y, z \in [0,1]^3\right)\left(x \le y \Rightarrow C(x,z) \le C(y,z)\right) and\ C(z,x) C(z,y)$

3.3 FUZZY IMPLICATIONS GENERATED FROM OTHER FUZZY LOGIC OPERATORS

Fuzzy implications generated from additive generating functions Yager [97] introduced a class of fuzzy implications generated from additive generating functions, and analysed their roles in approximate reasoning. They are fuzzy implications generated from *f*-generators.

A generator *f* is a continuous $[0, 1] \rightarrow [0, l]$ mapping which is strictly decreasing and $f(l) = 0$. Moreover the pseudo-inverse of *f*, $f^{(0\text{-}1)}$ is defined as

$$f^{(-1)}_{(x)} = \begin{cases} f^{-1}(x), if\ x \le f(0) \\ 0, \quad otherwise \end{cases}$$

3.4 Definition A f-generated implication *If* is defined as

$$\left(\forall\, x, y \in [0,1]^2\right)\left(I_f(x,y) = f^{(-1)} x f(y)\right)$$

Remark: Observe that if the generator *f* is defined as $f(x) = -\log x$, then the *f*-generated implication is the widely-known Yager implication *Iy* (see in [94]):

$$\left(\forall\, x, y \in [0,1]^2\right)\left(I_y(x,y) = y^x\right.$$

3.5 Theorem - A function I: $[0,1]^2 \rightarrow [0,1]$ satisfies properties 1-9 of fuzzy implication for a particular fuzzy compliment *c iff* there exists a strict increasing continuous function

f: $[0,1] \rightarrow [0, \infty)$ such that $f(0) = 0$.

$$I(a,b) = f^{-1}\left(f(1) - f(a) + f(b)\right) \ for\ all\ a, b \in [0,1]$$

 ISSN: 2454-4248
277 – 284

$$C(a) = f^{-1}(f(1) - f(a)) \text{ } for \text{ } all \text{ } a \in [0,1]$$

Consider the function $f(a) = \ln(l+a)$, $a \in [0,1]$.

Pseudo-inverse of function is:

$$f^{(-1)}(a) = \begin{cases} e^a - 1 & 0 \le a \le \ln 2 \\ 1 & otherwise, \end{cases}$$

Fuzzy compliment generated by f is:

$$c(a) = \frac{1-a}{1+a} \qquad \forall a \in [0,1],$$

then the fuzzy implication is,

$$I(a,b) = f^{-1}(f(1) - f(a) + f(b)) \qquad \text{...(1)}$$

$$f(1) = \ln(1+1) = \ln 2$$

$$f(a) = \ln(1+a)$$

$$f(b) = \ln(1+b)$$

$$I(a,b) = f^{(-1)}(\ln[(2+2b)/(1+a)])$$

$$I(a,b) = \min\left(1, \frac{1-a+2b}{1+a}\right), \forall a,b \in [0,1] \qquad \text{...(2)}$$

4. Contribution towards Design of Implications

S-implication: *S* implication in classical logic logic is defined as

$$i(a,b) = \bar{a} \vee b$$

We interpret the disjunction and negation as a fuzzy union (t-conorm) and a fuzzy compliment, respectively. This results in defining i in fuzzy logic by the formula:

$$1(a,b) = u(c(a),b)$$

I. Choosing $u(a,b) = \left\{\left\{1 + \left[\left(\frac{1}{a} - 1\right)^{\lambda} + \left(\frac{1}{b} - 1\right)^{\lambda}\right]^{\frac{-1}{\lambda}}\right\}\right\}^{-1}$

$$i(a,b) = u(1-a,b)$$

$$i(a,b)=\left\{1+\left\{\left(\frac{1}{1-a}-1\right)^{\lambda}\right\}^{\left(\frac{-1}{\lambda}\right)}\right\}^{-1}$$

$$i(a,b)=\left\{1+\left\{\left(\frac{1-1+a}{1-a}\right)^{\lambda}+\left(\frac{1}{b}-1\right)^{\lambda}\right\}^{\left(\frac{-1}{\lambda}\right)}\right\}^{-1}$$

$$i(a,b)=\left\{1+\left\{\left(\frac{a}{1-a}\right)^{\lambda}+\left(\frac{1}{b}-1\right)^{\lambda}\right\}^{\left(\frac{-1}{\lambda}\right)}\right\}^{-1}$$

for λ =1

$$i(a,b)=\left\{1+\left[\frac{ab+(1-a)(1-b)}{(1-a)b}\right]^{-1}\right\}^{-1}$$

$$i(a,b)=\left\{1+\left[\frac{(1-a)b}{2ab-a-b+1}\right]\right\}^{-1}$$

$$i(a,b)=\left\{\frac{2ab-a-b+1}{ab-a+1}\right\}^{-1}$$

II. **Choosing:** $u(a,b)=\dfrac{a+b+(r-2)}{r+(r-1)ab}$

$$i(a,b)=u(1-a,b)$$

$$i(a,b)=\frac{(1-a)+b+(r-2)(1-a)b}{r+(r-1)(1-a)b}$$

$$i(a,b)=\frac{1-a+b+(r-2)(b-ab)}{r+(r-1)(b-ab)}$$

for r =1

$$i(a,b)=\frac{1-a+b+(1-2)(b-ab)}{1+(1-1)(b-ab)}$$

$$i(a,b)=\frac{1-a+b+(-1)(b-ab)}{1+0}$$

$$i(a,b)=1-a+b-b+ab$$

$$i(a,b)=1-a+ab$$

***QL*-Implications:** *QL*-implication in fuzzy logic is defined as:

$$i(a,b)=u(c(a),i(a,b))$$

We choose *t*-norm and *t*-conorm as,

$$i(a,b)=\max\left(0,(1+\lambda)(a+b-1)-\lambda ab\right)$$

 ISSN: 2454-4248
277 – 284

$$u(a,b) = \min(1, a+b+\lambda ab)$$

then the *QL* implication is,

$$1(a,b) = u\left(1-a, \max\left(0, (1+\lambda)(a+b-1) - \lambda ab\right)\right)$$

$$i(a,b) = \min\left[1, (1-a) + \max\left(0, (1+\lambda)(a+b-1) - \lambda ab\right) + \lambda(1-a)\max\left(0, (1+\lambda)(a+b-1) - \lambda ab\right)\right]$$

$$i(a,b) = \min\left[1, (1-a) + \lambda(1-a)\max\left(0, (1+\lambda)(a+b-1) - \lambda ab\right) + \max\left(0, (1+\lambda)(a+b-1) - \lambda ab\right)\right]$$

By taking (1-*a*) common then we get,

$$i(a,b) = \min\left\{1, (1-a)\left[1+\lambda\max\left(0, (1+\lambda)(a+b-1) - \lambda ab\right)\right] + \max\left(0, (1+\lambda)(a+b-1) - \lambda ab\right)\right\}$$

$$i(a,b) = \min\left\{1, (1-a)\left[1+\max\left(0, \lambda(1+\lambda)(a+b-1) - \lambda ab\right)\right] + \max\left(0, (1+\lambda)(a+b-1) - \lambda ab\right)\right\}$$

$$i(a,b) = \min\left\{1, (1-a)\left[\max\left(0+1, 1+\lambda(1+\lambda)(a+b-1) - \lambda ab\right)\right] + \max\left(0, (1+\lambda)(a+b-1) - \lambda ab\right)\right\}$$

$$i(a,b) = \min\left\{1, (1-a)\left[\max\left(1, 1+\lambda(1+\lambda)(a+b-1) - \lambda ab\right)\right] + \max\left(0, (1+\lambda)(a+b-1) - \lambda ab\right)\right\}$$

As implication can take maximum value *i* so,

$$\max\left(1, 1+\lambda(1+\lambda)(a+b-1) - \lambda ab\right) = 1$$

Now, the equation (1) become -,

$$i(a,b) = \min\left\{1, (1-a)^{*} 1 + \max\left(0, (1+\lambda)(a+b-1) - \lambda ab\right)\right\}$$

$$i(a,b) = \min\left\{1, 1-a + \max\left(0, (1+\lambda)(a+b-1) - \lambda ab\right)\right\}$$

$$i(a,b) = \min\left\{1, 1+\max\left(0, (1+\lambda)(a+b-1) - \lambda ab\right) - a\right\}$$

$$i(a,b) = \min\left\{1, \max\left(1, 1+(1+\lambda)(a+b-1) - \lambda ab\right) - a\right\}$$

$$i(a,b) = \min\{1, 1, -a\}$$

R-Implications: Residual implications is defined as:

$$i(a,b) = \sup\{x \in [0,1] \mid i(a,x) \le b\}$$

I. We choose the Schweizer and Sklarl fuzzy intersection:

$$i(a,b) = \left\{\max\left(0, a^p + x^p - 1\right)\right\}^{\frac{1}{p}}$$

then we obtained,

$$i(a,b) = \sup\left\{x \in [0,1] \mid \left\{\max\left(0, a^p + x^p - 1\right)^{\frac{1}{p}} \le b\right\}\right.$$

$$\Rightarrow \left\{\max\left(0, a^P + x^P - 1\right)\right\}^{\frac{1}{p}} \le b \quad \Rightarrow \left\{\max\left(0, a^p + x^p - 1\right)\right\} \le b^p$$

$$\Rightarrow a^p + x^p - 1 \le b^p \quad \Rightarrow x^p \le b^p - a^p + 1$$

$$\Rightarrow x \le \left(b^p - a^p + 1\right)^{\frac{1}{p}}$$

for *p*=1,

ISSN: 2454-4248
277 – 284

$$i(a,b)=\left(b^p-a^p+1\right)^{\frac{1}{p}}, \quad \text{when } \frac{1}{2}<a,b<1$$

III. Choosing $i(a,b)=\dfrac{ab}{r+(1-r)(a+b-ab)}$

$$i(a,b)=\sup\left\{x\in[0,1] \mid i(a,x)\le b\right\}$$

$$\frac{ax}{r+(1-r)(a+x-ax)}\le b$$

$$ax\le b\left(r+(1-r)(a+x-ax)\right)$$

$$ax-bx+bxr+abx-abrx\le br+ab-abr$$

$$x\left(a-b+br+ab-abr\right)\le br+ab-abr$$

$$x\le\frac{b(r+a-ar)}{a-b(1-r)(1-a)}$$

for *r=1*,

$$x\le\frac{b}{a}$$

$$i(a,b)=\frac{b}{a}$$

5. Conclusion

An investigation about fuzzy implication and their properties is discussed here, and developed the *S*-implications-implication and *QL*-implication. Here also provides their mathematical form and geometrical representation. In future research, new fuzzy implications can also be developed by using inverse and contrapositive condition.

REFERENCES:

[1] Jaczynski, M., and Jayaram, B. On the characterizations of *(S,N)*- implications generated from continuous negations. Proceedings of the 11th Conf. Information Processing and Management of Uncertainty in Knowledge-Based Systems, Paris, France 2006, 436-443.
[2] Jaczynski, M., and Jayaram, B. "On the characterizations of *(S, N)*-implications". Fuzzy Sets and Systems 158 (2007), 1713-1727.
[3] Baldwin, J. "A new approach to approximate reasoning using a fuzzy logic". Fuzzy Sets and Systems 2, 309-325.
[4] Jayaram B. and Radko M. "On special fuzzy implications", Fuzzy Sets and Systems 160 (2009), 2063-2085.
[5] Dubois, D., and Prade, H. "The generalized modus ponens under sup Dmin composition, a theoretical study". Approximate Reasoning in Expert Systems Elsevier Science Publisher, North-Holland (1985), 217-232.
[6] George J. Klir, and Bo Youn, "Fuzzy Sets and Fuzzy Logic, Theory and Applications", 1st edition, PHI Learning Pvt. Ltd.
[7] Jayaram, B. "On the distributivity of implication operators over *t* and *s* norms". IEEE Iran. Fuzzy Systems 12,194-198.
[8] Morsi, N., and Fahmy, A. "On generalized modus ponens with multiple rules and a residuated implication". Fuzzy Sets and Systems 129 (2002), 267-274J.
[9] Pei, D. "*R*-implication: characteristics and applications". Fuzzy Sets and Systems 131 (2002), 297-302.
[10] Shi, Y., Van Gasse, B., Ruan, D., and Kerre, E. "On the first place Mititonicity in *QL* implications". Fuzzy Sets and Systems 159 (2008), 2988-3013.
[11] Trillas, E., and Valverde, L. "On implication and indistinguishability in the setting of fuzzy logic". Management Decision Support Systems using Fuzzy Set and Possibility Theory Verlag TUV, 196-212.
[12] Trillas, E., and Valverde, L. "On some functionally expressible implications for fuzzy set theory". Proc. 3rd International Seminar of Fuzzy Set Theory Johannes Kepler Univ. Linz (1981), 173-190.
[13] Whalen, T. “Interpolating between fuzzy rules using improper S-implications”. Int J. Approximate Reasoning 45 (2007), 30-42.
[14] Yager, R. “On the implication operator in fuzzy logic”. Information Sciences, 31 (1983), 141-164.
[15] Yager, R. “On global requirements for implication operators in fuzzy modus ponens”. Fuzzy Sets and Systems 106 (1999), 3-10.