Some Sequences of Fuzzy Numbers Associated With a Modulus Function

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Abstract. In this article we introduce fuzzy sequence space $m_F(f, \phi, p)$, $0 < p < 1$, defined by a modulus function. We study its different properties like solidity, symmetricity, completeness etc.
Keywords: *Modulus function, solid space, symmetric space.*
AMS Classification No. 40A05 ; 46A45.

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I. Introduction

Let P_s denote the class of all subsets of N, the set of natural numbers, those do not contain more than s elements. Throughout $\{\phi_n\}$ represents a non-decreasing sequence of real numbers such that $n\,\phi_{n+1} \le (n+1)\,\phi_n$, for all $n \in N$.

The class of these sequences $\{\phi_n\}$ is denoted by Φ.

The sequence space $m(\phi)$ introduced by Sargent [15] is defined as

$$m(\phi) = \{(x_k) \in w : \sup_{s\ge 1, \sigma\in P_s} \frac{1}{\phi_s}\sum_{k\in\sigma} |x_k| < \infty\},$$

which becomes a Banach space, normed by

$$\|x\|_{m(\phi)} = \sup_{s\ge 1, \sigma\in P_s} \frac{1}{\phi_s}\sum_{k\in\sigma} |x_k|.$$

The notion of modulus function was introduced by Nakano [11]. Later on different sequence spaces were defined by using modulus function and their different properties were investigated by Ruckle [14], Maddox [8], Bilgin [4] and many others.

Let D denote the set of all closed and bounded intervals $X = [a_1, a_2]$ on R, the real line. For $X, Y \in D$ we define

$$d(X, Y) = \max(|a_1 - b_1|, |a_2 - b_2|),$$

where $X = [a_1, a_2]$ and $Y = [b_1, b_2]$. It is known that (D, d) is a complete metric space.

A fuzzy real number X is a fuzzy set on R, *i.e.* a mapping $X : R \to I\ (=[0,1])$ associating each real number t with its grade of membership $X(t)$.

A fuzzy real number X is called *convex* if $X(t) \ge X(s) \wedge X(r) = \min\{X(s), X(t)\}$, where $s < t < r$.

If there exists $t_0 \in R$ such that $X(t_0) = 1$, then the fuzzy real number X is called *normal.*

A fuzzy real number X is said to be *upper-semi continuous* if, for each $\varepsilon > 0$, $X^{-1}([0, a + \varepsilon))$, for all $a \in I$ is open in the usual topology of R.

The set of all upper-semi continuous, normal, convex fuzzy real numbers is denoted by $R(I)$ and throughout the article, by a fuzzy real number we mean that the number belongs to $R(I)$.

The α - *level* set $[X]^\alpha$ of the fuzzy real number X, for $0 < \alpha \le 1$, defined as $[X]^\alpha = \{t \in R : X(t) \ge \alpha\}$. If $\alpha = 0$, then it is the closure of the strong 0-cut.

The set R of all real numbers can be embedded in $R(I)$. For $r \in R$, $\bar{r} \in R(I)$ is defined by

$$\bar{r}(t) = \begin{cases} 1, & \text{for } t = r, \\ 0, & \text{for } t \ne r \end{cases}$$

The *absolute* value, $|X|$ of $X \in R(I)$ is defined by (see for instance Kaleva and

ISSN: 2454-4248
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Seikkala [6])

$$|X|(t) = \max\{X(t), X(-t)\}, \text{ if } t \geq 0, \\ = 0, \quad \text{if } t < 0.$$

A fuzzy real number X is called *non-negative* if $X(t) = 0$, for all $t < 0$. The set of all non-negative fuzzy real numbers is denoted by $R^*(I)$.

Let $\bar{d} : R(I) \times R(I) \to R$ be defined by

$$\bar{d}(X, Y) = \sup_{0 \leq \alpha \leq 1} d\left([X]^\alpha, [Y]^\alpha\right).$$

Then $\bar{d}$ defines a metric on $R(I)$.

The additive identity and multiplicative identity in $R(I)$ are denoted by $\bar{0}$ and $\bar{1}$ respectively.

The sequence space $m(\phi)$ was introduced by Sargent [15], who studied its different properties and obtained its relations with the spaces ℓ^p and ℓ^∞. Later on the notion was further investigated and linked with summability theory by Tripathy [16], Tripathy and Sen [18] and many others.

Spaces of sequences of fuzzy numbers were studied by Matloka [9], Nuray and Savas [13] and many others.

Throughout the article w^F and $(\ell_\infty)_F$ denote the spaces of *all* and *bounded* sequences of fuzzy numbers, respectively.

II. Definition and Preliminaries

Definition. A sequence space E is said to be *symmetric* if $(X_n) \in E$ implies $(X_{\pi(n)}) \in E$, where π is a permutations of N.

Definition. A sequence space E is said to be *convergence free* if $(Y_k) \in E$, whenever $(X_k) \in E$ *and* $X_k = \bar{0}$ implies $Y_k = \bar{0}$.

Definition. A function $f : [0, \infty) \to [0, \infty)$ is called a *modulus* if

(*a*) $f(x) = 0$ if and only if $x = 0$

(*b*) $f(x + y) \leq f(x) + f(y)$, for $x \geq 0$, $y \geq 0$.

(*c*) f is increasing.

(*d*) f is continuous from the right at 0.

Hence f is continuous everywhere in $[0, \infty)$.

We define the following sequence space

$$m_F(f, \phi, p) = \left\{ (X_k) \in w^F : \sup_{s \geq 1, \sigma \in P_s} \frac{1}{\phi_s} \sum_{k \in \sigma} [f(\bar{d}(X_k, \bar{0})]^p < \infty \right\}$$

III. Main Results

Theorem 3.1. *The set* $m_F(f, \phi, p)$ *is a complete linear metric space, with respect to the metric g defined by*

$$g(X, Y) = \sup_{s \geq 1, \sigma \in P_s} \frac{1}{\phi_s} \sum_{k \in \sigma} [f(\bar{d}(X_k, Y_k)]^p$$

Proof. Since the linearity of $m_F(f, \phi, p)$ with respect to the co-ordinate wise addition and scalar multiplication is trivial, we omit the details.

Theorem 3.2. *Let* f *be a modulus function. Then,*

$$m_F(f, \phi, p) \subseteq m_F(f, \psi, p) \text{ if and only if } \sup_{s \in N} \frac{\phi_s}{\psi_s} < \infty.$$

for the sequences (ϕ_s) *and* (ψ_s) *of real numbers.*

Proof. Let $\sup_{s\geq 1} \frac{\phi_s}{\psi_s} = K \ (< \infty)$, then $\phi_s \leq K\,\Psi_s$ for all $s \in N$.

Then the inclusion $m_F(f, \phi, p) \subseteq m_F(f, \psi, p)$ follows from the following inequality:

$$\frac{1}{K\psi_s}\sum_{k\in\sigma}[f(\bar{d}(X_k, \bar{0}))]^p \leq \frac{1}{\phi_s}\sum_{k\in\sigma}[f(\bar{d}(X_k, \bar{0}))]^p .$$

Conversely let $m_F(f, \phi, p) \subseteq m_F(f, \psi, p)$ and $\sup_{s\geq 1} \eta_s = \infty$, where $\eta_s = \frac{\phi_s}{\psi_s}$.

Then there exists a subsequence $<\eta_{s_i}>$ of $<\eta_s>$ such that $\lim_{i\to\infty} \eta_{s_i} = \infty$.

Let $(X_k) \in m_F(f, \phi, p)$.

$$\text{Now} \ \sup_{s\geq 1, \sigma\in P_s} \frac{1}{\Psi_{s_i}}\sum_{k\in\sigma}[f(\bar{d}(X_k, \bar{0}))]^p \geq \sup_{s\geq 1, \sigma\in P_s} \frac{\eta_{s_i}}{\phi_{s_i}}\sum_{k\in\sigma}[f(\bar{d}(X_k, \bar{0}))]^p$$

$$\geq \left(\sup_{i\geq 1}\eta_{s_i}\right)\left(\sup_{s\geq 1, \sigma\in P_s} \frac{1}{\phi_{s_i}}\sum_{k\in\sigma}[f(\bar{d}(X_k, \bar{0}))]^p\right)$$

$$= \infty .$$

Thus $(X_k) \notin m_F(f, \psi, p)$ as such we arrive at a contradiction.

Corollary 3.1. *Let* $0 < p < 1$, *then* $m_F(f, \phi, p) = m_F(f, \Psi, p)$ *if and only if* $\sup_{s\geq 1} \eta_s < \infty$ *and* $\sup_{s\geq 1} \eta_s^{-1} < \infty$ *where* $\eta_s = \frac{\phi_s}{\psi_s}$.

The following result is obvious in view of the definition of the space.

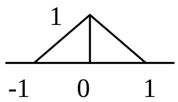
Proposition 3.3. *The space* $m_F(f, \phi, p)$ *is symmetric*.

Property 3.4. *The space* $m_F(f, \phi, p)$ *is not convergence free*.

Proof. The proof follows from the following example.

Example 3.1. Let $f(x) = x$, $\phi_n = n$ for all $n \in \mathbb{N}$. Let the sequence (X_k) be defined as,

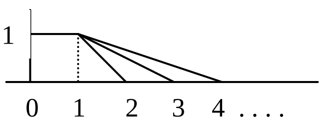
$$\text{For } k > 2, \qquad X_k(t) = \begin{cases} t+1, & \text{for } -1 < t < 0 \\ -t+1, & \text{for } 0 < t < 1 \\ 0, & \text{otherwise.} \end{cases}$$



and $X_k = \bar{0}$, otherwise.

Let the sequence (Y_k) be defined as,

$$\text{For } k > 1, \qquad Y_k(t) = \begin{cases} 1, & \text{for } 0 < t < 1, \\ (1-k)^{-1}t + k(k-1)^{-1}, & \text{for } 1 < t < k, \\ 0, & \text{otherwise.} \end{cases}$$



and $Y_k = \bar{0}$, otherwise.

Then $(X_k) \in m_F(f, \phi, p)$, but $(Y_k) \notin m_F(f, \phi, p)$.

Hence $m_F(f, \phi, p)$ is not convergence free.

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