Batch Arrival Queuing Models with Periodic Review

R. Sivaraman
Ph.D. Research Scholar in Mathematics
Sri Satya Sai University of Technology and Medical Sciences
Bhopal, Madhya Pradesh
National Awardee for Popularizing Mathematics among masses
Chennai – 600 094
Email: rsivaraman1729@yahoo.co.in
Contact Number: + 91 7845014341

Dr. Sonal Bharti
Head, Department of Mathematics
Sri Satya Sai University of Technology and Medical Sciences
Bhopal, Madhya Pradesh
Email: sbsonalbharti6@gmail.com

Abstract: In this paper, we propose a periodic review policy for the M/M /c type of queue The embedded Markov chain technique is used for the analysis of this system. To determine the mean queue length of mean job waiting times and higher moments of these quantities the probability generating functions are calculated (for the queue length) A comparison is made between constant and state dependent lengths of the review period.

***Keywords**: Optimal length of the review period, Steady-state Probabilities, Probability generating functions (pgf's), Delay time distributions.*

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I. INTRODUCTION

Usually, the continuous monitoring policy is assumed in most queuing models Due to immediate - service and immediate departure policy, it utilizes the system facilities and reduces the waiting time of customers in many systems the inventory level is periodically reviewed because sometimes a continuous - monitoring policy is expensive and dispensable In periodic review policy the inventory level is observed at T, the equal review period length. If the inventory level is less than or equal to reorder level, it is essential to maintain the maximum level by placing an order,

Periodic review policy is widely used in manufacturing systems. Boozer and Srinivasan proposed the design of the tandem configuration for an automated guided vehicle (A G V) system A single A G V connects a number of workstations of a flexible manufacturing systems There are a number of identical machines in each workstation that can do different operations At the flexible manufacturing- system jobs arrive and join service randomly at a particular workstation. The length of service time is random in advance because of the flexibility of the machines. The single A 0 V visits these workstations. Whenever the A 0 V observes a machine is empty in the workstation, jobs are loaded. For departure jobs have to \a1t for the A 0 V Only at review instants, jobs mayjoin and leave the system. Boozer and Srinivasan discuss the efficacious advantages of single A G V - loop systems aver conventional system. They presented the minimum visit ratio for the A G V required for the stable F M S.

In production and inventory systems, queuing analysis has been frequently used. The assumption of continuous monitoring policy is considered in all articles Magazine proposed some optimal periodic review policies to minimize the cost for operating queuing systems. Dynamic programming is preferred far the solution of the problem. Wang considered both constant and state - dependent lengths of the review period and compared also recently the batch arrival queuing systems have been of importance because of their resemblance with the real life situation.

The purpose of this study is to obtain the optimal length of the review period T in the case of batch arrival queuing system. The system cost function consists of two types of costs job waiting cost and reviewing cost We have considered both constant and state - dependent review policies The basic distinction between these policies may he expressed as the system is revised at equal time interval according to a constant periodic review policy and in the state - dependent periodic review policy, the length of the next review period depends upon the current system state.

II. STEADY STATE PROBABILITY

Assume that the number of jobs U arrive randomly and uncompleted are R. in number during time interval T Let there be n jobs in operating manner in the beginning of the review period. The corresponding probabilities are expressed by d_n and b_{ni}.

Here jobs arrive as a Poisson process with parameter in groups of random size

$$Pr(x = m) = a_m$$

and the probability generating function (which vil1 be assumed to exist) of batch size.

$$A(z) = \sum_{m=1}^{\infty} a_m z^m$$

The arrivals form a compound Poisson processes. Then we have

$$d_n = Pr\{U = n\} = \sum_{k=0}^{n} \frac{e^{-\lambda T} . (\lambda T)^k}{k!} \{a_n\}^k. \quad (n \geq 0) \qquad (1)$$

where $\{a_n\}^k$ is the k fold convolution of $\{a_n\}$ with itself with

$$a_n^{(0)} \equiv \begin{cases} 1 & (n = 0) \\ 0 & (n > 0) \end{cases}$$

Now the probability of uncompleted jobs is

$$b_{ni} = Pr\{R_n = i\} \qquad i = 0,1, \ldots\ldots\ldots,n$$

$$n=0,1\ldots\ldots\ldots,c$$

$$[b_{00}=1 \text{ for } n=0]$$

A binomial distribution is followed by the random variable R_n because of the exponential distribution and the independent or operation times

$$b_{ni} = \binom{n}{i} r^i (1-r)^{n-i} \qquad i = 0,1, \ldots\ldots\ldots,n$$

n=0,1………,c (2)

where

$$\binom{n}{i} = n! \ / \ 1! \times (n-i)\,!$$

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and

$$r = e^{-\mu t}$$

may be defined as the probability of the operation time which is longer than T.
Suppose the total number of arriving jobs and uncompleted jobs in the duration T are denoted by V and defined as

$$V_n = U + R_n$$

and it's probability

$$\hat{d}_{ni} = \Pr\{U + R_n = i\}$$
$$= \sum_{k=0}^{\min(i,\ n)} b_{nk}\, d_{i-k}$$

$$i = 0,1, \ldots\ldots, n$$

$$n = 0,1 \ldots\ldots\ldots, c \quad (3)$$

After releasing the completed jobs, Suppose X_k are the number of jobs at the k^{th} review period. Then we may define $\{X_k, k \geq 0\}$ is as a Markov chain.
The steady state probability is

$$P_n = \lim_{k \to \infty} \Pr\{X_n = n\}$$

III. PROBABILITY GENERATING FUNCTIONS

Now, we define the probability generating functions (p g f s)

$$D(z) = \sum_{n=0}^{\infty} d_n z^n$$

$$B_n(z) = \sum_{j=0}^{\infty} b_{ni} z^i$$

$$\hat{D}_n(z) = \sum_{i=0}^{\infty} d_{ni} z^i$$

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$$P(z)=\sum_{n=0}^{\infty} p_n z^n$$

We have the following results from equations (1) - (3)

$$D(z) = e^{\lambda T(A(z)-1)} \dots (4)$$

$$B_n(z) = (1-r+rz)^n \dots (5)$$

$$DD^{\wedge}{}_n(z) = D(z)B_n(z) \qquad \dots (6)$$

We have following stochastic balance equations after releasing the completed jobs at review instants -

$$p_0 = \hat{d}_{00}p_0 + \hat{d}_{10}p_1 + \hat{d}_{20}p_2 + \dots\dots + \hat{d}_{c0}p_c \qquad \dots(7)$$

$$p_1 = \hat{d}_{01}p_0 + \hat{d}_{11}p_1 + \hat{d}_{21}p_2 + \dots\dots + \hat{d}_{c1}p_c + \hat{d}_{c0}p_{c-1} \qquad \dots(8)$$

....
....

$$p_n = \hat{d}_{0n}p_0 + \hat{d}_{1n}p_1 + \hat{d}_{2n}p_2 + \dots\dots + \hat{d}_{c0}p_{c+n} \qquad \dots(9)$$

Multiplying above equations with appropriate powers of z and then summing up, we have

$$P(z)=\sum_{i=0}^{c-1} \hat{D}_i(z)p_i + \sum_{i=c}^{\infty} z^{i-c}\hat{D}_c(z)p_i \qquad \dots(10)$$

$$[z^c - \hat{D}_c(z)]P(z) = \sum_{i=0}^{c-1} [z^c \hat{D}_i(z) - z^i \hat{D}_c(z)]p_i$$

……(11)

$$P(z) = \frac{D(z) = \sum_{i=0}^{c-1} [z^c B_i(z) - z^i B_c(z)]p_i}{[z^c - \hat{D}_c(z)]} \qquad (12)$$

Estimating $\lim_{z-l} P(z) = 1$ via L. Hospital's rule, we obtain

$$\sum_{i=0}^{c-1} (c-i)(1-r)p_i = c(1-r) - \lambda E[X]T \qquad (13)$$

[E [X] is the expectation of batch size x]

$$\sum_{i=0}^{c-1} (c-i)(1-r)p_i$$

Here must be positive for steady state probabilities.

i.e. $c(1-r)-\lambda\ E[X]\ T>0$ $[r=e^{-\mu T}]$

or

$$\lambda E[X] T < c(1-r)$$

We may find out c unknown probabilities [p_i, i=0, 1, …. c -1] in equation (12) with the help of Rouche' s theorem According to the theorem there will be exactly c roots inside and on the unit circle |z| < 1 of the denominator z^c -D_c ^ (z).

Only one root is equivalent to and the remaining c - 1 root exists inside unit circle. Let z k (k = 0, 1… c -l) are these roots and z_0 = 1. We describe that the numerator vanishes at these roots z_k, k = 1, 2,........, c -1 as P(z) is analytic inside the unit circle |z| < 1 and that the numerator has a polynomial withdegree at least c.

Consequently, the (c -1)linear equations are

$$\sum_{i=0}^{c-1} z_k{}^c B_i(z_k) - z_k{}^i B_c(z_k)]p_i = \quad , \qquad k = 1,2, \dots \dots \dots, c-1 \qquad \dots (15)$$

P_i can be find out with the help of equation (13) and (15). Chaudhary et. al. and Zhao proposed some fixed properties of these roots in a more general characteristic equation.

By differentiating equation (12) at limit z = 1, the mean queue length is

$$L = \frac{\lambda E[X]T(2c - \lambda E[X]T) - c(c-1)(1-r^2) \div (1-r^2)\sum_{i=0}^{c-1}[c(c-1-i(i-1)]P_i}{2(c - cr - \lambda E[X]T)}$$

In case of c=1:

$$D^\wedge{}_0(z) = D(z)B_0(z) = D(z)$$

$$D^\wedge{}_1(z) = D(z)B_1(z) = D(z)$$

$$P(z) = \frac{D(z) - (z-1)(1-r)}{z - D^\wedge(z)} . p_0$$

Now applying the condition $\lim_{z\to 1} P(1) = 1, we have$

$$P_0 = \frac{1-r-\lambda E[X]T}{1-r} \qquad \dots(18)$$

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Thus $P(z) = \frac{D(z)(1-r-\lambda E[X]T)(z-1)}{z-D^\wedge(z)}$ ... (19)

The mean queue length

$$L = P'(1) = \frac{\lambda E[X]T(2-\lambda E[X]T)}{2(1-r-\lambda E[X]T)} \dots (20)$$

IV. DELAY TIME DISTRIBUTION

The total time that a job passes in the system is termed as delay time (waiting time before and after the operation and operation time).

Assuming that f (t) is the probability density function of the delay time and F(s) is the Laplace Transform (L T) of the delay time, we have

$$F(s) = \int_0^\infty e^{-st} f(t)\, dt$$

Now we define Q (z), the p g f of the queue length after a job's departure and P (z) is the p g r of the queue length, which is observed at every review instant.

Suppose N jobs arrive between the departure of jobs during a busy period and probability is c_n.

$$c_n = \Pr\{N = n\}$$

Thus the p g f of N,

$$C(z)=E[z^N]$$

$$= \frac{(1-r)D(z)}{1-rD(z)} \qquad (21)$$

(Consider a condition that service of job is finished or not at a review instant.)

Assume that g_n is the probability or the remaining n customers in the queue after releasing the first customer of a busy period. n = 0, 1. 2,.... . At least one customer must be present there during a review period for a busy period to begin (probability (1 - a_0). Then (n + 1) customers have to join the system in the duration or an idle period and the time that the first job is finished

Consequently ,

$$g_n = \sum_{i=0}^{n} \frac{a_{i-1}}{1-a_0} c_{n-1}$$

Thus the p g f,

$$G(z) = \sum_{n=0}^{\infty} z^n g_n = \frac{D(z) - a_0}{z(1-a_0)} . C(z)$$

Let, after releasing a job, there are n jobs in the system and its probability is q Therefore the stochastic balance equation is

$$q_n = g_n q_0 + \sum_{i=1}^{n+1} c_{n-i+1} \dots q_i, \qquad n \geq 0 \text{ (A 1)}$$

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From this equation, we have

$$= \frac{[D(z)-1]D(z)(1-r-\lambda E[X]T)}{[z-D(z)(1+rz-r)-\lambda E[X]T} \qquad \text{(A 2)}$$

The relation between P(z) and Q(z) is

$$Q(z) = \frac{P(z).(D(z)-1)}{\lambda E[X]T\ (z-1)}$$

And the mean queue length after releasing a job is

$$L^{\sim} = L + \frac{\lambda E[X]T}{2} \qquad (22)$$

For c=0, equation (10) is

$$P(z) = \sum_{i=0}^{\infty} D_i{}^{\wedge}(z)\, p_i$$

$$= e^{\lambda T(A(z)-1)}.P(1-r+rz)$$

Substituting the value of P(1 - r + r z) and using the condition $\lim_{n\to\infty} r^n = 0$, we have

$$\text{P(z)} = e^{\lambda T(A(z)-1)/(l-r)} \qquad (23)$$

Comparing with the p g f of a Poisson distribution, we may observe that the queue length follows Poisson distribution at the review period with mean

$$L = \frac{\lambda E[X]T}{1-r}$$

we may find our P (z) at z = 0 for the probability of the emptiness of the system

$$p_0 = P(0) = e^{\lambda T(A(z)-1)/(l-r)}$$

For the approximation of $M^{[X]}$ /M /c queues (with periodic review) with larger values of c, these results are useful.

Now, we have deliberated a state - dependent review period (i. e. the review period depends upon the queue length). Suppose. i customers are in the system at the present review instant and T_i , (i =0, 1, 2...) is the next review period length. Constants T_i are determined already & rely upon the cost structure

The p g f of queue length is

$$P(z) = \sum_{i=0}^{\infty} z_i D_i{}^{\wedge}(z) p_i$$

Where

$$D_i{}^{\wedge}(z) = e^{\lambda Ti(A(z)-1)}.[1-r_i+r_i z]^i$$

With $\qquad r_i = e^{-\mu Ti}$

For general values of T_i, the existence of analytic solution of P (z) is not possible. So letting the length of next review period length is dependable only on the number of working machines at the current review instant.

Suppose that

$T_i = T_c$, $i \geq c$ where c is the number of machine in the system and T_c is a constant.

Now, after releasing the jobs, the stochastic process (X_k , $k \geq 0$ } is a discrete - time semi - Markov chain where X_k is the number of jobs at the k^{th} review instant T_n is the mean time for the Markov chain being in state n. Assume that π_n be the time for the Markov chain existing in state n for long -run proportion.

The relation between π_n and p_n is

$$\pi_n = \frac{p_n T_n}{\bar{T}} \tag{25}$$

where

$$\bar{T} = \sum_{i=0}^{\infty} p_i T_i \tag{26}$$

is the average review time.

[Semi-Markov Theory]

The long-run average of jobs in the system is

$$L = \sum_{n=0}^{\infty} n\pi_n \tag{27}$$

V. CONSTANT REVIEW PERIOD

Jobs are characterized into two groups during one review period - jobs arrive in the duration of the review period and the jobs already present in the system before this period.

The average number of jobs = $L + \frac{\lambda E[X]T}{2}$

Therefore, the cost function per unit of time is given as

$$f(T) = C_1 L + \frac{\lambda E[X]T}{2} + \frac{C_2}{T}$$

(job waiting cost plus reviewing cost)

where C_1 is the cost per unit of time for a waiting job and C_2 is per review cost for $M^{[X]}$ /M /1 queue with periodic review.

$$f(T) = C_1 \left[\frac{\lambda E[X]T(2 - \lambda E[X]T)}{2(1 - r - \lambda E[X]T)} + \frac{\lambda E[X]T}{L} \right] + \frac{C_2}{T}$$

And the optical value T can be obtained by

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$$\frac{df}{dt} = 0$$

$$\frac{df(T)}{dt} = C_1 \lambda E[X] \left[\frac{1 - r - \mu T_r}{(1 - r - \lambda E[X]T)^2} - \frac{1}{2} \right] + \frac{C_2}{T^2} = 0$$

In the case of $M^{[X]}$ / M /∞ queue (with periodic review), the cost function is

$$f(T) = C_1 \left[\frac{\lambda E[X]T}{1 - r} + \frac{\lambda E[X]T}{2} \right] + \frac{C_2}{T}$$

& for optimal value of T. the equation is

$$\frac{df(T)}{dt} = C_1 \lambda E[X] \left[\frac{1 - r - \mu T_r}{(1 - r)^2} - \frac{1}{2} \right] + \frac{C_2}{T^2} = 0$$

The value of T from this equation, provide the review period T in the case of batch arrival queuing system.

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