A New Subclass of Meromorphic Starlike Functions Associated with Q-Hypergeometric Functions

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Abstract— The fractional calculus operator has used in various field of sciences, GFT and in the engineering, if we extend the ordinary fractional calculus in the q-theory we get fractional q-calculus operator. In this paper by making use of fractional q-calculus operator we have introduced a new subclass of Meromorphic starlike functions $\mathcal{N}_q(\lambda,\alpha,\beta)$ defined in the open disk and determined coefficient estimate, neighbourhood result, subordination results, extreme points and partial sums for the functions belonging to this class

Keywords- *fractional q-calculus operator, Meromorphic starlike functions.*

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I. Introduction

Let B be the class of analytic and univalent functions defined in the punchered open unit disk $U = \{z: |z| < 1\}$ is of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

normalized by $f(0) = 0 = f'(0) - 1$. The subclass S of $\mathcal{B}$ consisting of univalent functions in disk U of the form

$$g(z) = z - \sum_{n=2}^{\infty} b_n z^n, \quad b_n \geq 0 \tag{2}$$

We denote subclass of B by $S(\gamma)$ and $K(\gamma)$ consisting of all functions, which are starlike and convex of order γ introduced by Goodman [1], Ronning ([4]) and Silverman [3].

$$S(\gamma) = \left\{ f\epsilon\beta \,; Re\left(\frac{zf'(z)}{f(z)}\right) > \gamma \right\} \qquad \text{and} \qquad K(\gamma) = \left\{ f\epsilon\beta \,; Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \gamma \right\}$$

We also denote the functions f(z) belongs $\mathcal{B}$ that are convex in U as K.

Define new class of analytic functions in the punchered open unit disk various authors used fractional q- calculus. Recall some definitions of q- calculus operators of function $f(z)$.

The q-shifted, fractional is defined for a q ϵ $\mathbb{C}$ as a product of n factors by

$$(a\,;q)_n = \begin{cases} 1 & ,n = 0 \\ (1-a)(1-aq)\ldots\ldots\ldots..(1-aq^{n-1}) & ,n \in \mathbb{N} \end{cases} \tag{3}$$

and in terms of basic analogue of gamma function.

$$(q^a\,;q)_n = \frac{\Gamma_q(a+n)(1-q)^n}{\Gamma_q(a)} \;,n > 0 \tag{4}$$

The recurrence relation for q- gamma function is defined by Gasper and Rahman [2]

$$\Gamma_q(1+a) = \frac{(1-q^a)\Gamma_q(a)}{1-q} \tag{5}$$

and the q-binomial expansion is given by

$$(x-y)_v = x^v\left(\frac{-y}{x}\,;q\right) = x^v \prod_{n=0}^{\infty} \frac{1-\left(\frac{y}{x}\right)q^n}{1-\left(\frac{y}{x}\right)q^{v+n}} = x^v \phi_0\left[q^{-v}; -; q; \frac{yq^v}{x}\right] \tag{6}$$

The q-derivative and q-integral of function f defined by (1) is given by

$$D_{q,z}f(z) = \frac{f(z)-f(zq)}{z(1-q)} \;,(z \neq 0, q \neq 0) \tag{7}$$

$$\int_0^z f(t)d(t;q) = z(1-q)\sum_{k=0}^{\infty} q^k f(zq^k) \tag{8}$$

It is of interest to note that $\lim_{q\to 1^-} \frac{(q^a;q)_n}{(1-q)^n} = a_n = a(a+1)(a+2)\ldots\ldots\ldots\ldots(a+n-1)$ is the familiar pochhammer symbol. Recall the definitions of fractional q-derivative and fractional q-integral operators given by Kim and Srivastava [6].

Definition 1. Let the function f(z) be the analytic in a simply connected region of the z-plane containing the origin . The fractional q-integral of f of order μ $(\mu > 0)$ is defined by

$$J_{q,z}^{\mu} f(z) = D_{q,s}^{-\mu} f(z) = \frac{1}{\Gamma_q(\mu)} \int_0^z (z-qt)_{\mu-1} f(t)d(t;q) \tag{9}$$

Where $(z-tq)_{\mu-1}$ can be expressed as q- bionomial given by (6) and the series $\phi_0\ [\mu;-;q;z]$ is a single valued when $|\arg(z)|<\pi$, $|z|<1$, therefore the function $(z-tq)_{\mu-1}$ in (9) is single valued when $\left|arg\left(\frac{-tq^{\mu}}{z}\right)\right|<\pi$, $|tq^{\mu}|<1$ and $||arg(z)|<\pi$.

Definition 2 . The fractional q- derivative operator of orderμ $(0\le\mu<1)$ is defined by function f(z) by

$$D_{q,z}^{\mu} f(z) = D_{q,z} J_{q,z}^{\mu} f(z) = \frac{1}{\Gamma_q(1-\mu)} D_{q,z} \int_0^z (z-qt)_{-\mu} f(t)d(t;q) \tag{10}$$

Where the function f(z) is constrained , and the multiplicity of function $(z-qt)_{-\mu}$ is removed as in Definition 1.

Definition 3. Under the hypothesis of Definition 2 , the fractional derivative of order μ is defined by

$$D_{q,z}^{\mu} f(z) = D_{q,z}^{m} J_{q,z}^{m-\mu} f(z),\ (m-1\le\mu<m;m\,\epsilon\,\mathbb{N}). \tag{11}$$

By using known extensions involving q-differ-integral operator ,we define the Linear opartor

$$H_{q,z}^{\mu}f(z):\ \mathrm{I}\to\mathrm{I}$$

$$H_{q,z}^{\mu}f(z) = \frac{\Gamma_q(2-\mu)}{\Gamma_q(2)} z^{\mu-1} D_{q,z}^{\mu} f(z) = z - \sum_{n=2}^{\infty} T_q(n,\mu)\, a_n\, z^n \tag{12}$$

Where

$$T_q(n,\mu) = \frac{\Gamma_q(2-\mu)\Gamma_q(n+1)}{\Gamma_q(2)\Gamma_q(n+1-\mu)} \tag{13}$$

Where we can easily check that $T_q(n,\mu)$ is a decreasing function of n for $-\infty<\mu<2, 0<\mu<1$.

In this paper we define a following subclass of starlike functions of order γ based on q-fractional operator .

For $\mu<2$, $0\le\alpha<1$, $\beta\ge0$ and $0\le\lambda<1$, we let $\mathcal{N}_q(\lambda,\alpha,\beta)$ be a subclass of $\mathcal{B}$ consisting of functions of the form (2) and satisfying

$$Re\left\{\frac{z\left(H_{q,z}^{\mu}f(z)\right)''+(1-\lambda)\left(H_{q,z}^{\mu}f(z)\right)'}{(1-\lambda)\left(H_{q,z}^{\mu}f(z)\right)'+\lambda z\left(H_{q,z}^{\mu}f(z)\right)''}\right\} \ge \beta\left\{\frac{z\left(H_{q,z}^{\mu}f(z)\right)''+(1-\lambda)\left(H_{q,z}^{\mu}f(z)\right)'}{(1-\lambda)\left(H_{q,z}^{\mu}f(z)\right)'+\lambda z\left(H_{q,z}^{\mu}f(z)\right)''}-1\right\}+\alpha \tag{14}$$

Where $H_{q,z}^{\mu}f(z)$ given by (12)

II. Coefficient Estimate

To obtain main results we recall the following lemmas

Lemma 1. If α is a real number and w is complex number , then

$$\mathrm{R}(w)\ \ge\alpha \Leftrightarrow |w+(1-\alpha)| - |w+(1+\alpha)| \ge 0 \tag{15}$$

Lemma 2. If w is a complex number and α, β are real numbers , then

$$\mathrm{R}(w)\ \ge \beta\,|w-1|+\alpha\ \Leftrightarrow \mathrm{R}\left\{w\left(1+\beta\,e^{i\theta}\right)-\beta\,e^{i\theta}\right\}\ge\alpha, -\pi\le\theta\le\pi \tag{16}$$

Theorem 1.The function f(z) defined by (2) is in the class $\mathcal{N}_q(\lambda,\alpha,\beta)$ if and only if

$$\sum_{n=2}^{\infty} n[(1-\lambda)(1-\alpha)+(n-1)(1+\beta-\lambda\alpha-\lambda\beta)]\ T_q(n,\mu)\, a_n \le (1-\lambda)(1-\alpha) \tag{17}$$

Where $\mu<2, 0\le\lambda<1, \beta\ge0$ and $0\le\alpha<1$.

Proof. If $f\in\mathcal{N}_q(\lambda,\alpha,\beta)$ then by (14), we have

$$Re\left\{\frac{z\left(H_{q,z}^{\mu}f(z)\right)''+(1-\lambda)\left(H_{q,z}^{\mu}f(z)\right)'}{(1-\lambda)\left(H_{q,z}^{\mu}f(z)\right)'+\lambda z\left(H_{q,z}^{\mu}f(z)\right)''}\right\} \ge \beta\left\{\frac{z\left(H_{q,z}^{\mu}f(z)\right)''+(1-\lambda)\left(H_{q,z}^{\mu}f(z)\right)'}{(1-\lambda)\left(H_{q,z}^{\mu}f(z)\right)'+\lambda z\left(H_{q,z}^{\mu}f(z)\right)''}-1\right\}+\alpha$$

Using Lemma (2), we have

$$Re\left\{\frac{z\left(H_{q,z}^{\mu}f(z)\right)''+(1-\lambda)\left(H_{q,z}^{\mu}f(z)\right)'}{(1-\lambda)\left(H_{q,z}^{\mu}f(z)\right)'+\lambda z\left(H_{q,z}^{\mu}f(z)\right)''}\left(1+\beta e^{i\theta}\right)-\beta e^{i\theta}\right\}\ge\alpha,\quad -\pi\le\theta\le\pi \tag{18}$$

Or equivalently

$$Re\left\{\frac{\left[z\left(H_{q,z}^{\mu}f(z)\right)''+(1-\lambda)\left(H_{q,z}^{\mu}f(z)\right)'\right](1+\beta e^{i\theta})-\left[(1-\lambda)\left(H_{q,z}^{\mu}f(z)\right)'+\lambda z\left(H_{q,z}^{\mu}f(z)\right)''\right]\beta e^{i\theta}}{(1-\lambda)\left(H_{q,z}^{\mu}f(z)\right)'+\lambda z\left(H_{q,z}^{\mu}f(z)\right)''}\right\}\ge\alpha$$

Let $A(z)=\left[z\left(H_{q,z}^{\mu}f(z)\right)''+(1-\lambda)\left(H_{q,z}^{\mu}f(z)\right)'\right](1+\beta e^{i\theta})-\left[(1-\lambda)\left(H_{q,z}^{\mu}f(z)\right)'+\lambda z\left(H_{q,z}^{\mu}f(z)\right)''\right]\beta e^{i\theta}$

and $B(z)=(1-\lambda)\left(H_{q,z}^{\mu}f(z)\right)'+\lambda z\left(H_{q,z}^{\mu}f(z)\right)''$

by Lemma (1), (18) is equivalent to

$$|A(z)+(1-\alpha)B(z)|\ge|A(z)+(1+\alpha)B(z)| \text{ for } 0\le\alpha<1$$

By substituting the values of A(z) and B(z), we get

$$\sum_{n=2}^{\infty} n\,[(1-\lambda)(1-\alpha)+(n-1)(1+\beta-\alpha\lambda-\lambda\beta)]T_q(n,\mu)\,a_n \le (1-\lambda)(1-\alpha)$$

Conversely, suppose that (17) holds. Then we must show

$$Re\left\{\frac{\left[z\left(H_{q,z}^{\mu}f(z)\right)''+(1-\lambda)\left(H_{q,z}^{\mu}f(z)\right)'\right](1+\beta e^{i\theta})-\left[(1-\lambda)\left(H_{q,z}^{\mu}f(z)\right)'+\lambda z\left(H_{q,z}^{\mu}f(z)\right)''\right]\beta e^{i\theta}}{(1-\lambda)\left(H_{q,z}^{\mu}f(z)\right)'+\lambda z\left(H_{q,z}^{\mu}f(z)\right)''}\right\}\ge\alpha$$

Upon using the values of z on the positive real axis where $0 \le 2 = r < 1$, the above inequality reduces to

$$Re\left\{\frac{[-\sum_{n=2}^{\infty} n(n-1)T_q(n,\mu)\, a_n\, r^{n-1} + (1-\lambda)(1-\sum_{n=2}^{\infty} nT_q(n,\mu)\, a_n\, r^{n-1}](1+\beta e^{i\theta})}{(1-\lambda)(1-\sum_{n=2}^{\infty} nT_q(n,\mu)\, a_n\, r^{n-1}) - \lambda\sum_{n=2}^{\infty} n(n-1)T_q(n,\mu)\, a_n\, r^{n-1}} - \frac{[(1-\lambda)(1-\sum_{n=2}^{\infty} nT_q(n,\mu)\, a_n\, r^{n-1}) - \lambda\sum_{n=2}^{\infty} n(n-1)T_q(n,\mu)\, a_n\, r^{n-1}](\alpha+\beta e^{i\theta})}{[(1-\lambda)(1-\sum_{n=2}^{\infty} nT_q(n,\mu)\, a_n\, r^{n-1}) - \lambda\sum_{n=2}^{\infty} n(n-1)T_q(n,\mu)\, a_n\, r^{n-1}]}\right\} \ge 0$$

Since $Re(e^{i\theta}) \ge -|e^{i\theta}| = -1$,the inequality is correct for all $z \in U$,lettering $r \to 1$ yeilds

$$Re\left\{\frac{(1-\lambda)(1-\alpha) - \sum_{n=2}^{\infty} T_q(n,\mu)\, a_n\, [n(n-1)+(1-\lambda)n - \alpha(1-\lambda)n - \alpha\lambda n\,(n-1) + \beta n\,(n-1)(1-\lambda)}{(1-\lambda) - \sum_{n=2}^{\infty} T_q(n,\mu)\, a_n\, [n - \lambda n + \lambda n\,(n-1)}\right\} \ge 0$$

And so by the Mean value theorem, we have

$$Re\{(1-\lambda)(1-\alpha) - \sum_{n=2}^{\infty} T_q(n,\mu)\, a_n\, n[(n-1) + (1-\lambda) - \alpha(1-\lambda) - \alpha\lambda(n-1) + \beta n(n-1)(1-\lambda)]\} \ge 0$$

We get desired conclusion.

Corollary1. If$f(z) \in \mathcal{N}_q(\lambda, \alpha, \beta)$,then

$$a_n \le \frac{(1-\lambda)(1-\alpha)}{n[(1-\lambda)(1-\alpha)+(n-1)(1+\beta-\alpha\lambda-\lambda\beta)]T_q(n,\mu)}, (n \ge 2) \tag{19}$$

Where $0 \le \alpha < 1$, $0 \le \lambda < 1$, $\beta \ge 0$ and $\mu < 2$ .The result is sharp for the function

$$f(z) = z - \frac{(1-\lambda)(1-\alpha)}{n[(1-\lambda)(1-\alpha)+(n-1)(1+\beta-\alpha\lambda-\lambda\beta)]T_q(n,\mu)} z^n, (n \ge 2) \tag{20}$$

III. RADII OF CLOSE-TO CONVEXITY AND CONVEXITY AND STARLIKENESS

Theorem2. Let the function f(z) defined by (2)be in the class $\mathcal{N}_q(\lambda, \alpha, \beta)$ then f(z) is close-to-convex of order $\varphi(0 \le \varphi < 1$ in $|z| < r_1$,where

$$r_1 = \inf_{n \ge 2} \left\{\frac{(1-\varphi)[(1-\lambda)(1-\alpha)+(n-1)(1+\beta-\lambda\beta-\alpha\lambda)]T_q(n,\mu)}{(1-\lambda)(1-\alpha)}\right\}^{\frac{1}{n-1}} \tag{21}$$

The result is sharp ,with the extremal function f(z) given by (20)

Proof. r_1 is given by (21).Indeed we find from (2) that

$$|f'(z) - 1| \le \sum_{n=2}^{\infty} na_n |z|^{n-1}$$

Thus $|f'(z) - 1| \le 1 - \varphi$ if $\sum_{n=2}^{\infty}\left(\frac{n}{1-\varphi}\right) a_n |z|^{n-1} \le 1$

But by the theorem 1, we have

$$\sum_{n=2}^{\infty} \frac{n[(1-\lambda)(1-\alpha)+(n-1)(1+\beta-\lambda\beta-\lambda\alpha)]T_q(n,\mu)}{(1-\lambda)(1-\alpha)} a_n \le 1$$

Hence (22) will be true if

$$\frac{n|z|^{n-1}}{1-\varphi} \le \frac{n[(1-\lambda)(1-\alpha)+(n-1)(1+\beta-\lambda\beta-\lambda\alpha)]T_q(n,\mu)}{(1-\lambda)(1-\alpha)}$$

Equivalently if

$$|z| \le \left\{\frac{(1-\varphi)[(1-\lambda)(1-\alpha)+(n-1)(1+\beta-\lambda\beta-\alpha\lambda)]T_q(n,\mu)}{(1-\lambda)(1-\alpha)}\right\}^{\frac{1}{n-1}}, \ (n \ge 2) \tag{24}$$

The theorem follows from (24).

Theorem3. Let f(z) defined by (2)be in the class $\mathcal{N}_q(\lambda, \alpha, \beta)$. Then f(z) is convex of order $\varphi(0 \le \varphi < 1$ in $|z| < r_2$, where

$$r_2 = \inf_{n \ge 2} \left\{\frac{(1-\varphi)[(1-\lambda)(1-\alpha)+(n-1)(1+\beta-\lambda\beta-\alpha\lambda)]T_q(n,\mu)}{(n-\varphi)(1-\lambda)(1-\alpha)}\right\}^{\frac{1}{n-1}} \tag{25}$$

The result is sharp,with extremal function f(z) given by (20).

Proof. We must show that

$$\left|\frac{zf''(z)}{f'(z)}\right| < 1 - \varphi \text{ for } \quad |z| < r_2 \tag{26}$$

Substituting the series expansionsof $f''(z)$ and $f'(z)$ in the left hand of (25), we have

$$\left|\frac{-\sum_{n=2}^{\infty} n(n-1)a_n z^{n-1}}{1-\sum_{n=2}^{\infty} na_n z^{n-1}}\right| \le \frac{\sum_{n=2}^{\infty} n(n-1)a_n |z|^{n-1}}{1-\sum_{n=2}^{\infty} na_n |z|^{n-1}}$$

The last expansion above is bounded by $(1-\varphi)$ if

$$\sum_{n=2}^{\infty} \frac{n(n-\varphi)}{(1-\varphi)} a_n |z|^{n-1} \le 1 \tag{27}$$

In view of (26),it follows that(27) is true if

$$\frac{n(n-\varphi)}{(1-\varphi)} |z|^{n-1} < \frac{n[(1-\lambda)(1-\alpha)+(n-1)(1+\beta-\lambda\beta-\alpha\lambda)]T_q(n,\mu)}{(1-\lambda)(1-\alpha)}$$

Or

$$|z| < \left\{\frac{(1-\varphi)[(1-\lambda)(1-\alpha)+(n-1)(1+\beta-\lambda\beta-\alpha\lambda)]T_q(n,\mu)}{(1-\varphi)(1-\lambda)(1-\alpha)}\right\}^{\frac{1}{n-1}}, (n \ge 2) \tag{28}$$

Theorem (22) follows easily from (28).

Theorem 4. Let f(z) defined by (2)be in the class $\mathcal{N}_q(\lambda,\alpha,\beta)$. Then f(z) is starlike of order $\varphi(0 \le \varphi < 1)$ in $|z| < r_3$, where

$$r_3 = \inf_{n\ge 2}\left\{\frac{(1-\varphi)n[(1-\lambda)(1-\alpha)+(n-1)(1+\beta-\lambda\beta-\lambda\alpha)]T_q(n,\mu)}{(n-\varphi)(1-\lambda)(1-\alpha)}\right\}^{\frac{1}{n-1}} \quad (29)$$

The result is sharp, with external function f(z) given by (20).

Proof. It is sufficient to show that

$$\left|\frac{zf'(z)}{f'(z)} - 1\right| \le 1-\varphi \text{ for } |z| < r_3$$

We have

$$\left|\frac{zf'(z)}{f'(z)} - 1\right| \le \frac{\sum_{n=2}^{\infty}(n-1)a_n|z|^{n-1}}{1-\sum_{n=2}^{\infty}a_n|z|^{n-1}}.$$

Thus

$$\left|\frac{zf'(z)}{f'(z)} - 1\right| \le 1-\varphi$$

If

$$\sum_{n=2}^{\infty}\frac{n(n-\varphi)}{(1-\varphi)}a_n|z|^{n-1} \le 1 \quad (30)$$

Hence (30) will be true if

$$\frac{(n-\varphi)}{(1-\varphi)}|z|^{n-1} \le \frac{n[(1-\lambda)(1-\alpha)+(n-1)(1+\beta-\lambda\beta-\alpha\lambda)]T_q(n,\mu)}{(1-\lambda)(1-\alpha)},$$

Or equivalently

$$|z| \le \left\{\frac{(1-\varphi)n[(1-\lambda)(1-\alpha)+(n-1)(1+\beta-\lambda\beta-\alpha\lambda)]T_q(n,\mu)}{(n-\varphi)(1-\lambda)(1-\alpha)}\right\}^{\frac{1}{n-1}}, (n \ge 2) \quad (31)$$

Theorem follows easily from (31).

IV. Closure Theorems

Theorem 5. Let

$$f_i(z) = z - \sum_{n=2}^{\infty}a_{n,i}z^n \in \mathcal{N}_q(\lambda,\alpha,\beta) \quad \text{where } i \in \{1,2,\dots\dots\dots,l) \text{ and } 0 < C_i < 1$$

Such that

$$\sum_{i=1}^{l}C_i = 1$$

Then the function f(z) defined by

$$F(z) = \sum_{i=1}^{l}C_i f_i(z) = 1$$

Is also in the class $\mathcal{N}_q(\lambda,\alpha,\beta)$.

Proof. For every $i \in \{1,2,\dots\dots\dots,l)$,we obtain

$$\sum_{n=2}^{\infty}\frac{n[(1-\lambda)(1-\alpha)+(n-1)(1+\beta-\lambda\beta-\alpha\lambda)]T_q(n,\mu)}{(1-\lambda)(1-\alpha)}a_{n,i} \le 1$$

Since

$$F(z) = \sum_{i=1}^{l}C_i f_i(z) = \sum_{i=1}^{l}C_i\left[z - \sum_{n=2}^{\infty}a_{n,i}z^n\right] = z - \sum_{n=2}^{\infty}\left[\sum_{i=1}^{l}C_i a_{n,i}\right]z^n.$$

Therefore

$$\sum_{n=2}^{\infty}\frac{n[(1-\lambda)(1-\alpha)+(n-1)(1+\beta-\lambda\beta-\alpha\lambda)]T_q(n,\mu)}{(1-\lambda)(1-\alpha)}\left[\sum_{i=1}^{l}C_i a_{n,i}\right]$$
$$= \sum_{i=1}^{l}C_i\left[\frac{n[(1-\lambda)(1-\alpha)+(n-1)(1+\beta-\lambda\beta-\alpha\lambda)]T_q(n,\mu)}{(1-\lambda)(1-\alpha)}a_{n,i}\right]$$
$$\le \sum_{i=1}^{l}C_i = 1$$

Hence $F(z) \in \mathcal{N}_q(\lambda,\alpha,\beta)$

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