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LONG TIME BEHAVIOR OF THE CAGINALP SYSTEM WITH SINGULAR POTENTIALS AND DYNAMIC BOUNDARY CONDITIONS

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(Dedicated to Michel Pierre on the occasion of his 60th birthday)

Abstract. This paper is devoted to the study of the well-posedness and the long time behavior of the Caginalp phase-field model with singular potentials and dynamic boundary conditions. Thanks to a suitable definition of solutions, coinciding with the strong ones under proper assumptions on the bulk and surface potentials, we are able to get dissipative estimates, leading to the existence of the global attractor with finite fractal dimension, as well as of an exponential attractor.

1. **Introduction.** The Caginalp system is a well-known model in phase transition, proposed in [1] to describe, in particular, melting-solidification phenomena in certain classes of materials. It consists of two parabolic equations in the state variables (w, u), the temperature and the order parameter, respectively. Here, we assume that the material undergoing phase transition is contained in a vessel, so that interactions with the walls need to be considered. For this purpose, physicists proposed, in the context of the Cahn-Hilliard equation, the so-called dynamic boundary conditions for the order parameter (in the sense that the kinetics, i.e., the time derivative of the order parameter, appears explicitly in the boundary conditions), see [7], [8] and [9].

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We consider in this paper, in a smooth and bounded domain $\Omega \subset \mathbb{R}^3$ with boundary $\partial\Omega = \Gamma$, the following initial and boundary value problem:

$$\begin{cases} \varepsilon\, \partial_t w - \Delta w = -\partial_t u, & t > 0, \quad x \in \Omega, \\ \partial_t u - \Delta u + f(u) - \lambda u = w, & t > 0, \quad x \in \Omega, \\ \partial_t \psi - \Delta_\Gamma \psi + g(\psi) + \partial_n u = 0, & t > 0, \quad x \in \Gamma, \\ \partial_n w|_\Gamma = 0, \quad u|_\Gamma = \psi, \\ w|_{t=0} = w_0, \quad u|_{t=0} = u_0, \quad \psi|_{t=0} = \psi_0, \end{cases} \tag{1}$$

where $\varepsilon \in (0,1)$, $\lambda \in \mathbb{R}$, Δ_Γ is the Laplace-Beltrami operator and ∂_n is the outward normal derivative. In particular, the second equation exhibits a nonlinear term; a thermodynamically relevant instance is the following (singular) logarithmic function:

$$f(s) - \lambda s = -2\kappa_0 s + \kappa_1 \ln \frac{1+s}{1-s}, \ s \in (-1,1), \ \kappa_0 > \kappa_1 > 0.$$

The mathematical study of the Cahn-Hilliard model with singular potentials F (here and below, the nonlinear term f is the derivative of the potential F) and/or dynamic boundary conditions has been developed in many papers, see, e.g., [12], [13], [16], [17] and [19]. The Caginalp model with singular potentials and dynamic boundary conditions has been considered for the first time in [4], assuming that the nonlinearity on the boundary has the right signs close to ± 1. With this requirement, the first and third authors were able to prove the well-posedness (and, in particular, the separation from the singularities), together with the dissipativity of the system, the existence of the global attractor and the convergence of solutions to steady states (this last issue when the nonlinear term on the boundary disappears and the potential is real analytic in $(-1,1)$). Unfortunately, the sign restrictions on g exclude, e.g., constant functions and, in [2], the authors tried to remove this assumption, with the result that even the well-posedness of the problem becomes a difficult task. Indeed, only the global existence (and uniqueness) of strong solutions (always separated from the singularities of f) was proved, provided that the singularities of the potential are strong enough, relying on a suitable elliptic problem, as well as on localization techniques, since reasonable super/sub-solutions were not available. This did not prevent the solutions from blowing up as the initial data approach the singularities, so that the asymptotic analysis was not possible, unless some technical assumptions related to the terms appearing in the dynamic boundary conditions are imposed (see [3]). In all these papers, strong solutions were considered, whereas, as we will see, the occurrence of both singular potentials and nonlinear dynamic boundary conditions gives rise to complicated dynamics.

A way to overcome such difficulties consists in using duality techniques, as in [14], where a problem similar to ours, with dynamic boundary conditions for the temperature as well, has been addressed (see also [20] for a Cahn-Hilliard model). Again, the existence of attractors is proved under sign or growth restrictions on the nonlinear terms.

An analogous outcome, in the context of the Cahn-Hilliard equation, was deeply analyzed by Miranville and Zelik in [19], exhibiting the possible appearance of strong singularities close to the boundary, in particular, when the aforementioned sign restrictions are not satisfied, and showing that, already in the 1-D case, solutions in the sense of distributions may not exist, due to the jumps of the normal derivative close to the boundary. Thus, the authors modified the notion of a solution, by

introducing a variational inequality (we can note that the aforementioned duality techniques are somehow similar to this approach, but at an abstract level), and proved that such a solution is the usual one when it does not reach the singular values on the boundary, a fact prevented by a fast growing nonlinear term f or by the sign conditions on g.

Our aim in this paper is to extend this approach to the Caginalp system which, as the numerical simulations at the end of the paper point out, in presence of a logarithmic potential and without the sign requirements on g, exhibits solutions stopping at a certain time when they reach one of the singular values ± 1 on the boundary. On the contrary, there exist global solutions, provided that g has proper signs at ± 1. In other words, it is again relevant to adopt the variational definition of a solution, which allows, in particular, to prove the uniform in time estimates needed for dissipativity, without any additional assumption on f and g. Having gained the lacking ingredient, we are now in a position to accomplish our asymptotic analysis and prove the existence of finite-dimensional attractors.

This paper is organized as follows. Section 2 is devoted to our assumptions and notation. In Section 3, we introduce regularized problems in which the singular nonlinearity is approximated by regular functions, obtaining uniform a priori estimates on the corresponding solutions. Then, a variational formulation of (1) is given, for which well-posedness and regularity estimates are proved in Section 4. In fact, under the sufficient conditions stated in Section 5, we prove that a variational solution coincides with a solution in the usual (distribution) sense. Then, the existence of finite-dimensional (global and exponential) attractors is shown in Section 6. In Section 7, we give numerical simulations which suggest the possible nonexistence of classical solutions. Finally, in Appendix, we recall, for the reader's convenience, some results of [19] which are used in our proofs.

2. Setting of the problem.

We set $\tilde f(u) := f(u) - \lambda u$ and rewrite (1) in the form

$$\begin{cases} \varepsilon\, \partial_t w - \Delta w = -\partial_t u, & t > 0, \quad x \in \Omega, \\ \partial_t u - \Delta u + \tilde f(u) = w, & t > 0, \quad x \in \Omega, \\ \partial_t \psi - \Delta_\Gamma \psi + g(\psi) + \partial_n u = 0, & t > 0, \quad x \in \Gamma, \\ \partial_n w|_\Gamma = 0, \quad u|_\Gamma = \psi, \\ w|_{t=0} = w_0, \quad u|_{t=0} = u_0, \quad \psi|_{t=0} = \psi_0, \end{cases} \tag{2}$$

where f is a singular function satisfying

$$\begin{cases} 1. \quad f \in C^2(-1,1), \\ 2. \quad f(0) = 0, \quad \lim_{s\to\pm 1} f(s) = \pm\infty, \\ 3. \quad f'(s) \geq 0, \quad \lim_{s\to\pm 1} f'(s) = +\infty, \\ 4. \quad f''(s)\,\mathrm{sgn}\, s \geq 0. \end{cases} \tag{3}$$

As a consequence, the following properties hold for $\tilde f$:

$$\tilde f'(s) \geq -\lambda \ \text{ and } \ -\tilde c \leq \tilde F(s) \leq \tilde f(s)s + \tilde C, \quad \forall s \in (-1,1), \tag{4}$$

where $\tilde F(s) = \int_0^s \tilde f(r)dr$ and $\tilde c$, $\tilde C$ are strictly positive constants.

The nonlinear function $g \in C^2([-1,1])$ can be extended, without loss of generality, to the whole real line by writing

$$g(s) = s + g_0(s), \quad \forall\, s \in \mathbb{R}, \quad \text{where} \quad \|g_0\|_{C^2(\mathbb{R})} := C_0 < +\infty. \tag{5}$$

In the whole paper, $\|\cdot\|_\Gamma$ and $\langle\cdot,\cdot\rangle_\Gamma$ stand for the norm and the scalar product in $L^2(\Gamma)$ (or $[L^2(\Gamma)]^3$, depending on the context), respectively; moreover, we denote by $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$ the norm and the scalar product or the duality pairing in $L^2(\Omega)$ (or, again, $[L^2(\Omega)]^3$ or $[L^2(\Omega)]^6$). Finally, we introduce the spaces

$$\mathcal{L}^2 := L^2(\Omega)\times L^2(\Gamma)\times L^2(\Omega) \quad \text{and} \quad \mathcal{H}^1 := H^1(\Omega)\times H^1(\Gamma)\times H^1(\Omega),$$

both endowed with their standard norms. We then set, concerning the temperature,

$$H^2_N(\Omega) := \{w\in H^2(\Omega): \quad \partial_n w|_\Gamma = 0\}. \tag{6}$$

Besides, for further convenience, given two normed function spaces X in Ω and Y on Γ, we set, whenever it makes sense,

$$X\otimes Y := \{u\in X: \quad u|_\Gamma\in Y\}$$

and we endow this space with the norm

$$\|u\|^2_{X\otimes Y} = \|u\|^2_X + \|u|_\Gamma\|^2_Y.$$

The problem is characterized by the conservation law

$$\langle u(t)+\varepsilon\, w(t)\rangle = \langle u_0+\varepsilon\, w_0\rangle := I_0, \quad \forall t\ge 0, \tag{7}$$

where $\langle v\rangle = \frac{1}{|\Omega|}\int_\Omega v dx$. Thus, we will often use the obvious inequalities

$$\langle w\rangle\langle u\rangle \le \frac{1}{2\varepsilon}\langle u+\varepsilon w\rangle^2 \le \frac{1}{2\varepsilon|\Omega|}\|u+\varepsilon w\|^2 \tag{8}$$

and

$$0\le \|w-\langle w\rangle\|^2 = \|w\|^2 - |\Omega|\langle w\rangle^2 \le \|w\|^2. \tag{9}$$

Besides, we introduce the spaces

$$\tilde H^1(\Omega) = \{v\in H^1(\Omega):\ \langle v\rangle = 0\} \text{ and } [\tilde H^1(\Omega)]' = \{v^*\in [H^1(\Omega)]^*:\ \langle v^*, 1\rangle = 0\} \tag{10}$$

and the cartesian products

$$\tilde{\mathcal{H}}^1 = H^1(\Omega)\times H^1(\Gamma)\times \tilde H^1(\Omega) \quad \text{and} \quad (\tilde{\mathcal{H}}^1)' = [H^1(\Omega)]^*\times [H^1(\Gamma)]^*\times [\tilde H^1(\Omega)]'.$$

The symbol $\langle\cdot,\cdot\rangle$ in $[\tilde H^1(\Omega)]'$ stands for the duality pairing between $H^1(\Omega)$ and $[H^1(\Omega)]^*$. Notice that $[\tilde H^1(\Omega)]'$ is not the dual space of $\tilde H^1(\Omega)$. In particular, we denote by c_Ω the positive constant such that

$$\|v-\langle v\rangle\|^2_{[\tilde H^1(\Omega)]'} \le c_\Omega\|v\|^2, \quad v\in L^2(\Omega). \tag{11}$$

We further use $-\Delta$ as the realization of the Laplacian with homogeneous Neumann boundary conditions acting on the space of the $L^2(\Omega)$-functions with null average (it is thus a positive invertible operator with compact inverse $(-\Delta)^{-1}$).

In view of assumption (3), we a priori assume that $\|u(t)\|_{L^\infty(\overline\Omega)} < 1$, for a.e. $t\ge 0$. Besides, we observe the equivalence between the $H^1(\Omega)\times H^1(\Gamma)-$norm and another norm arising naturally in computations

$$\frac{1}{\kappa}[\|u\|^2_{H^1(\Omega)} + \|u|_\Gamma\|^2_{H^1(\Gamma)}] \le \|\nabla u\|^2 + \|u|_\Gamma\|^2_{H^1(\Gamma)} \le \kappa[\|u\|^2_{H^1(\Omega)} + \|u|_\Gamma\|^2_{H^1(\Gamma)}], \tag{12}$$

for any $u\in H^1(\Omega)\times H^1(\Gamma)$, for some $\kappa>1$.

For the sake of simplicity, throughout the whole paper, c denotes any positive constant, allowed to vary in the same line and independent of the initial data, but possibly influenced by the average I_0, ε and the other structural parameters (λ, K, etc.). Further dependencies will be specified on occurrence.

3. **A priori estimates for approximating regular problems.** Since the interaction between the singular potential and the dynamic boundary condition may give rise to singular solutions, we first focus on approximating problems for which the former difficulty is weakened. Following [19], we introduce a family of regular approximating functions: given any $N \in \mathbb{N}$, we set

$$f_N(s) = \begin{cases} f(-1+1/N) + f'(-1+1/N)(s+1-1/N), & -1 < s < -1+1/N \\ f(s), & |s| \le 1-1/N \\ f(1-1/N) + f'(1-1/N)(s-1+1/N), & 1-1/N < s < 1. \end{cases}$$

Then, we call F_N the primitive $F_N(s) = \int_0^s f_N(r)dr$ and, having set $\tilde{f}_N(s) = f_N(s) - \lambda s$, we define $\tilde{F}_N$ analogously, with $\tilde{f}_N$ in place of f_N. For the reader's convenience, we recall a particular instance of [19, formulae (2.14) and (2.17), with $c = 0$], namely, there exist $\alpha > 0$ and $c > 0$ such that, for N large enough, there hold

$$\begin{cases} \tilde{f}_N(s)s \ge \alpha f_N(s)s - c \ge \dfrac{\alpha}{2}|f_N(s)| - c, & \forall s \in \mathbb{R}, \\ 2F_N(s) + c \ge \tilde{F}_N(s) \ge \dfrac{1}{2}F_N(s) - c, & \forall s \in \mathbb{R}, \end{cases} \tag{13}$$

together with $f_N(s)s \ge F_N(s) \ge 0$. Our first aim is to study the family of problems

$$\begin{cases} \varepsilon\, \partial_t w - \Delta w = -\partial_t u, & t > 0, \quad x \in \Omega, \\ \partial_t u - \Delta u + \tilde{f}_N(u) = w, & t > 0, \quad x \in \Omega, \\ \partial_t \psi - \Delta_\Gamma \psi + g(\psi) + \partial_n u = 0, & t > 0, \quad x \in \Gamma, \\ \partial_n w|_\Gamma = 0, \quad u|_\Gamma = \psi, & \\ w|_{t=0} = w_0, \quad u|_{t=0} = u_0, \quad \psi|_{t=0} = \psi_0. & \end{cases} \tag{14}$$

The well-posedness of these problems is already well established (see, e.g., [11]) and will not be considered in this paper. Unfortunately, we cannot find a uniform (with respect to N) $H^2(\Omega)$-estimate on u_N. Nevertheless, we can control some Hölder norm of u_N in Ω, together with the H^2-norm in some interior domain and the L^2-boundary norm of the gradient of the tangential derivative, $\nabla D_\tau u_N$ (we recall that $D_\tau u = \nabla u - (\partial_n u)n$ is the tangential part of the gradient ∇u and $n(x)$ stands also for some smooth extension of the unit normal vector field at the boundary inside Ω). As it will be clear below, this is enough for our purpose.

For the sake of simplicity, we drop the subscript N in u_N, ψ_N, w_N. In particular, we recall that the positive constants c appearing below are independent of N and the initial data.

The main result of this section is the

Theorem 3.1. *We assume that f and g satisfy the above assumptions. Then, for N large enough, any sufficiently regular solution $z(t) = (u(t), \psi(t), w(t))$ to (14) is such that $u \in C^\alpha(\Omega) \otimes H^2(\Gamma)$, for some $\alpha < 1/4$, $F_N(u) \in L^1(\Gamma)$, $\nabla D_\tau u \in [L^2(\Omega)]^6$, $u \in H^2(\Omega_\delta)$, where $\Omega_\delta = \{x \in \Omega : \quad \mathrm{dist}(x, \Gamma) > \delta\}$, for every $\delta > 0$, and there*

holds

$$\begin{aligned}
&\|u(t)\|^2_{H^1(\Omega)} + \|u(t)\|^2_{C^\alpha(\Omega)} + \|\nabla D_\tau u(t)\|^2 \\
&\quad + \|u(t)\|^2_{H^2(\Omega_\delta)} + \|\psi(t)\|^2_{H^2(\Gamma)} + \|w(t)\|^2_{H^2(\Omega)} \\
&\quad + \|f_N(u(t))\|_{L^1(\Omega)} + \|F_N(u(t))\|_{L^1(\Gamma)} + \|\partial_t z(t)\|^2_{\mathcal{L}^2} + \int_t^{t+1} \|\partial_t z(s)\|^2_{\mathcal{H}^1} ds \\
&\leq c\left(\|u_0\|^2 + \|\psi_0\|^2_\Gamma + \|w_0\|^2_{H^2(\Omega)} + \|\partial_t u(0)\|^2 + \|\partial_t \psi(0)\|^2_\Gamma\right) e^{-\nu t} + c,
\end{aligned} \tag{15}$$

where the positive constant ν is independent of N and the initial data. Moreover, the following smoothing property holds:

$$\begin{aligned}
&\|u(t)\|^2_{H^1(\Omega)} + \|u(t)\|^2_{C^\alpha(\Omega)} + \|\nabla D_\tau u(t)\|^2 \\
&\quad + \|u(t)\|^2_{H^2(\Omega_\delta)} + \|\psi(t)\|^2_{H^2(\Gamma)} + \|w(t)\|^2_{H^2(\Omega)} \\
&\quad + \|f_N(u(t))\|_{L^1(\Omega)} + \|F_N(u(t))\|_{L^1(\Gamma)} + \|\partial_t z(t)\|^2_{\mathcal{L}^2} + \int_t^{t+1} \|\partial_t z(s)\|^2_{\mathcal{H}^1} ds \\
&\leq c\,\frac{t^2+1}{t^2}\,(1 + \|z_0\|^2_{\mathcal{L}^2}), \quad t > 0.
\end{aligned} \tag{16}$$

Proof. The estimates on w and $\partial_t z$ are obtained in Lemma 3.6 below. In order to control (u, ψ), we follow the rationale of [19], that is, we consider the nonlinear elliptic problem

$$\begin{cases} -\Delta u + f_N(u) + u = u + \lambda u + w - \partial_t u \ := \tilde{h}_1, & x \in \Omega, \\ -\Delta_\Gamma \psi + \psi + \partial_n u = -\partial_t \psi - g_0(\psi) \ := \tilde{h}_2, & x \in \Gamma. \end{cases} \tag{17}$$

Then, we apply Theorem 8.1 (namely, [19, Theorem 6.1] which is written in Appendix for the reader's convenience): this is possible, provided that $\tilde{h}_1(t) \in L^2(\Omega)$ and $\tilde{h}_2(t) \in L^2(\Gamma)$ and that their norms can be (uniformly in N) controlled by the initial data. Actually, by definition of $\tilde{h}_1$ and $\tilde{h}_2$, from the assumptions on g, it is straightforward to get

$$\|\tilde{h}_1(t)\|^2 + \|\tilde{h}_2(t)\|^2_\Gamma \leq c(1 + \|z(t)\|^2_{\mathcal{L}^2} + \|\partial_t u(t)\|^2 + \|\partial_t \psi(t)\|^2_\Gamma).$$

Hence, we are led to establish proper estimates on $\|z(t)\|_{\mathcal{L}^2}$, $\|\partial_t u(t)\|$ and $\|\partial_t \psi(t)\|_\Gamma$, a task that we accomplish in the next lemmas. □

We are thus left to prove several technical lemmas which are needed in the proof of the above Theorem 3.1.

Lemma 3.2. *There exists $\nu > 0$ small enough such that, for any $t \geq 0$ and N large enough, we have*

$$\|z(t)\|^2_{\mathcal{L}^2} + c\int_0^t e^{-\nu(t-s)}[\|z(s)\|^2_{\mathcal{H}^1} + \langle F_N(u(s)), 1\rangle] ds \leq c(1 + \|z_0\|^2_{\mathcal{L}^2} e^{-\nu t}), \tag{18}$$

for some positive constants c which are independent of N, but monotone increasing with respect to $|I_0|$, $1/\varepsilon$ and $1/\nu$, where $I_0 = \langle u_0 + \varepsilon w_0\rangle$. Moreover, for any $t > 0$, there holds

$$\int_t^{t+1} [\|z(s)\|^2_{\mathcal{H}^1} + \langle F_N(u(s)), 1\rangle] ds \leq c(1 + \|z_0\|^2_{\mathcal{L}^2} e^{-\nu t}). \tag{19}$$

Proof. Multiplying the second equation of (14) by u in $L^2(\Omega)$ and exploiting decomposition (5), we have

$$\frac{\mathrm{d}}{\mathrm{dt}}(\|u\|^2+\|\psi\|_\Gamma^2)+2\|\nabla u\|^2+2\|\psi\|^2_{H^1(\Gamma)}+2\langle\tilde f_N(u),u\rangle+2\langle g_0(\psi),\psi\rangle_\Gamma=2\langle w,u\rangle.$$

Next, we rewrite the first equation of (14) as

$$\partial_t(u+\varepsilon w-I_0)-\Delta(w-\langle w\rangle)=0,$$

where $I_0=\langle u_0+\varepsilon w_0\rangle=\langle u(t)+\varepsilon w(t)\rangle$, $\forall t>0$. Then, multiplying the above equation by $(-\Delta)^{-1}(u+\varepsilon w-I_0)$, we get

$$\frac{\mathrm{d}}{\mathrm{dt}}\|u+\varepsilon w-I_0\|^2_{[\tilde H^1(\Omega)]'}+2\varepsilon\|w-\langle w\rangle\|^2+2\langle w,u\rangle-2|\Omega|\langle w\rangle\langle u\rangle=0,$$

that is, since $\langle u\rangle=I_0-\varepsilon\langle w\rangle$,

$$\frac{\mathrm{d}}{\mathrm{dt}}\|u+\varepsilon w-I_0\|^2_{[\tilde H^1(\Omega)]'}+2\varepsilon\|w-\langle w\rangle\|^2+2\varepsilon|\Omega|\langle w\rangle^2=-2\langle w,u\rangle+2|\Omega|I_0\langle w\rangle.$$

This, by (9), reduces to

$$\frac{\mathrm{d}}{\mathrm{dt}}\|u+\varepsilon w-I_0\|^2_{[\tilde H^1(\Omega)]'}+2\varepsilon\|w\|^2=-2\langle w,u\rangle+2|\Omega|I_0\langle w\rangle.$$

We also take the product of the above equation by $\varepsilon(u+\varepsilon w)$, getting

$$\frac{\mathrm{d}}{\mathrm{dt}}(\varepsilon\|u+\varepsilon w\|^2)+2\varepsilon^2\|\nabla w\|^2=-2\varepsilon\langle\nabla w,\nabla u\rangle.$$

Adding the three equalities, introducing the functional

$$Y(t)=\|u(t)\|^2+\|\psi(t)\|_\Gamma^2+\|u(t)+\varepsilon w(t)-I_0\|^2_{[\tilde H^1(\Omega)]'}+\varepsilon\|u(t)+\varepsilon w(t)\|^2,\quad(20)$$

we find

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{dt}}Y+2\|\nabla u\|^2+2\|\psi\|^2_{H^1(\Gamma)}+2\langle\tilde f_N(u),u\rangle+2\varepsilon\|w\|^2+2\varepsilon^2\|\nabla w\|^2\qquad(21)\\&=-2\langle g_0(\psi),\psi\rangle_\Gamma+2|\Omega|I_0\langle w\rangle-2\varepsilon\langle\nabla w,\nabla u\rangle.\end{aligned}$$

Thus, since g_0 is a globally bounded function, by (9), we have, concerning the right-hand side of the above differential equation,

$$\begin{aligned}&-2\langle g_0(\psi),\psi\rangle_\Gamma+2|\Omega|I_0\langle w\rangle-2\varepsilon\langle\nabla w,\nabla u\rangle\\&\le 2C_0\sqrt{|\Gamma|}\|\psi\|_\Gamma+\varepsilon\|w\|^2+|\Omega|\frac{I_0^2}{\varepsilon}+\varepsilon^2\|\nabla w\|^2+\|\nabla u\|^2\\&\le\|\nabla u\|^2+\|\psi\|_\Gamma^2+\varepsilon\|w\|^2+\varepsilon^2\|\nabla w\|^2+c,\end{aligned}$$

where C_0 is the constant in (5) and now c also depends on $|I_0|$. Collecting the above estimates and taking N large enough for (13) to hold, we end up with

$$\frac{\mathrm{d}}{\mathrm{dt}}Y+\|\nabla u\|^2+\|\psi\|^2_{H^1(\Gamma)}+\varepsilon\|w\|^2+\varepsilon^2\|\nabla w\|^2+2\alpha\langle F_N(u),1\rangle\ \le\ c.$$

Notice that (9) and (11) entail

$$0\le\|u(t)+\varepsilon w(t)-I_0\|^2_{[\tilde H^1(\Omega)]'}\le c_\Omega\|u(t)+\varepsilon w(t)-I_0\|^2_{L^2(\Omega)}\le c_\Omega\|u(t)+\varepsilon w(t)\|^2_{L^2(\Omega)},$$

which allows to see that, for some $c>1$, there holds

$$\frac1c\|z(t)\|^2_{\mathcal L^2}\le Y(t)\le c\|z(t)\|^2_{\mathcal L^2}.\qquad(22)$$

Hence, for $\nu > 0$ small enough, by (12) we have

$$\frac{\mathrm{d}}{\mathrm{dt}} Y(t) + \nu Y(t) + \nu[\|z(t)\|^2_{\mathcal{H}^1} + \langle F_N(u(t)), 1\rangle] \leq c. \tag{23}$$

An application of Gronwall's lemma, together with an integration over $(0, t)$, finishes the proof of (18). Finally, an integration of (23) over $(t, t+1)$ $(t > 0)$ provides, thanks to (18) and (22),

$$\int_t^{t+1} [\|z(s)\|^2_{\mathcal{H}^1} + \langle F_N(u(s)), 1\rangle] ds \leq c(1 + Y(t)) \leq c(1 + \|z_0\|^2_{\mathcal{L}^2} e^{-\nu t}).$$

□

Lemma 3.3. *For $t \geq 1$, N large enough and $\nu > 0$ small enough, we have*

$$\begin{aligned} &\|z(t)\|^2_{\mathcal{H}^1} + \langle F_N(u(t)), 1\rangle + \int_1^t e^{-\nu(t-s)} \|\partial_t z(s)\|^2_{\mathcal{L}^2} ds \\ &\leq c(\|z(1)\|^2_{\mathcal{H}^1} + \langle F_N(u(1)), 1\rangle) e^{-\nu(t-1)} + c. \end{aligned}$$

Proof. Taking the product in $L^2(\Omega)$ of the first equation of (14) first by w,

$$\frac{\mathrm{d}}{\mathrm{dt}} (\varepsilon \|w\|^2) + 2\|\nabla w\|^2 = -2\langle \partial_t u, w\rangle,$$

and then by $\partial_t w$,

$$\frac{\mathrm{d}}{\mathrm{dt}} \|\nabla w\|^2 + 2\varepsilon \|\partial_t w\|^2 = -2\langle \partial_t u, \partial_t w\rangle, \tag{24}$$

and adding this second equation multiplied by ε to the first one, we obtain

$$\frac{\mathrm{d}}{\mathrm{dt}} (\varepsilon \|w\|^2_{H^1(\Omega)}) + 2\|\nabla w\|^2 + 2\varepsilon^2 \|\partial_t w\|^2 = -2\varepsilon \langle \partial_t u, \partial_t w\rangle - 2\langle \partial_t u, w\rangle.$$

The sum of this equality with the product in $L^2(\Omega)$ of the second equation of (14) by $\partial_t u$, namely,

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{dt}} (\|\nabla u\|^2 + \|\psi\|^2_{H^1(\Gamma)} + 2\langle \tilde{F}_N(u), 1\rangle + 2\langle G_0(\psi), 1\rangle_\Gamma) \\ &\quad + 2\|\partial_t u\|^2 + 2\|\partial_t \psi\|^2_\Gamma = 2\langle w, \partial_t u\rangle, \end{aligned}$$

furnishes

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{dt}} (\|\nabla u\|^2 + \|\psi\|^2_{H^1(\Gamma)} + \varepsilon \|w\|^2_{H^1(\Omega)} + 2\langle \tilde{F}_N(u), 1\rangle + 2\langle G_0(\psi), 1\rangle_\Gamma) \\ &\quad + 2\|\nabla w\|^2 + 2\|\partial_t u\|^2 + 2\|\partial_t \psi\|^2_\Gamma + 2\varepsilon^2 \|\partial_t w\|^2 = -2\varepsilon \langle \partial_t w, \partial_t u\rangle. \end{aligned} \tag{25}$$

Summing (21) and (25) allows to see that the functional

$$E(t) = Y(t) + \|\nabla u(t)\|^2 + \|\psi(t)\|^2_{H^1(\Gamma)} + \varepsilon \|w(t)\|^2_{H^1(\Omega)} + 2\langle \tilde{F}_N(u(t)), 1\rangle + 2\langle G_0(\psi(t)), 1\rangle_\Gamma$$

being $Y(t)$ defined by (20), satisfies

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{dt}} E + 2\|\nabla u\|^2 + 2\|\psi\|^2_{H^1(\Gamma)} + 2(1+\varepsilon^2)\|\nabla w\|^2 + 2\varepsilon \|w\|^2 + 2\langle \tilde{f}_N(u), u\rangle + 2\langle g_0(\psi), \psi\rangle_\Gamma \\ &+ 2\|\partial_t u\|^2 + 2\|\partial_t \psi\|^2_\Gamma + 2\varepsilon^2 \|\partial_t w\|^2 = -2\varepsilon \langle \partial_t w, \partial_t u\rangle + 2|\Omega| I_0 \langle w\rangle - 2\varepsilon \langle \nabla w, \nabla u\rangle. \end{aligned}$$

Due to our assumptions, $F_N(s) = \int_0^s f_N(\tau) d\tau$ satisfies $f_N(s)s \geq F_N(s)$, $\forall s \in \mathbb{R}$, which, in view of (13), implies

$$2\langle \tilde{f}_N(u), u\rangle \geq 2\alpha \langle F_N(u), 1\rangle - c.$$

Therefore, we obtain

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{dt}}E + 2\|\nabla u\|^2 + 2\|\psi\|^2_{H^1(\Gamma)} + 2(1+\varepsilon^2)\|\nabla w\|^2 + 2\varepsilon\|w\|^2 + 2\alpha\langle F_N(u),1\rangle\\&+ 2\|\partial_t u\|^2 + 2\|\partial_t\psi\|^2_\Gamma + 2\varepsilon^2\|\partial_t w\|^2\\&\le -2\varepsilon\langle\partial_t w,\partial_t u\rangle + 2|\Omega|I_0\langle w\rangle - 2\varepsilon\langle\nabla w,\nabla u\rangle + 2C_0\|\psi\|_\Gamma + c\\&\le \|\nabla u\|^2 + \|\psi\|^2_\Gamma + \varepsilon^2\|\nabla w\|^2 + \varepsilon\|w\|^2 + \|\partial_t u\|^2 + \varepsilon^2\|\partial_t w\|^2 + c.\end{aligned}$$

This furnishes

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{dt}}E + \|\nabla u\|^2 + \|\psi\|^2_{H^1(\Gamma)} + 2\|\nabla w\|^2 + \varepsilon\|w\|^2 + 2\alpha\langle F_N(u),1\rangle\\&+ \|\partial_t u\|^2 + \|\partial_t\psi\|^2_\Gamma + \varepsilon^2\|\partial_t w\|^2 \le c.\end{aligned}$$

Since, using again (5), (13) and (22),

$$\begin{cases}E(t) \ge \dfrac{1}{c}(\|z(t)\|^2_{\mathcal{H}^1} + \langle F_N(u(t)),1\rangle - 1) \ge \dfrac{1}{c}(\|z(t)\|^2_{\mathcal{H}^1} - 1),\\ E(t) \le c(\|z(t)\|^2_{\mathcal{H}^1} + \langle F_N(u(t)),1\rangle + 1),\end{cases} \tag{26}$$

thus, by (12), possibly reducing ν, there holds

$$\frac{\mathrm{d}}{\mathrm{dt}}E(t) + \nu E(t) + \varepsilon^2\|\partial_t z(t)\|^2_{\mathcal{L}^2} \le c. \tag{27}$$

Then, applying Gronwall's lemma,

$$\begin{aligned}&\|z(t)\|^2_{\mathcal{H}^1} + \langle F_N(u(t)),1\rangle + \varepsilon^2\int_1^t e^{-\nu(t-s)}\|\partial_t z(s)\|^2_{\mathcal{L}^2}ds\\&\le ce^{-\nu(t-1)}\left(\|z(1)\|^2_{\mathcal{H}^1} + \langle F_N(u(1)),1\rangle\right) + c,\end{aligned} \tag{28}$$

where now c also depends on ν, and Lemma 3.3 is proved. □

Lemma 3.4. *We have, for $0 < t \le 1$ and N large enough,*

$$\int_0^t \left(\|z(s)\|^2_{\mathcal{H}^1} + \langle F_N(u(s)),1\rangle\right)ds \le c(1 + \|z_0\|^2_{\mathcal{L}^2}) \tag{29}$$

and

$$t\left(\|z(t)\|^2_{\mathcal{H}^1} + \langle F_N(u(t)),1\rangle\right) + \int_0^t s\|\partial_t z(s)\|^2_{\mathcal{L}^2}ds \le c(1 + \|z_0\|^2_{\mathcal{L}^2}). \tag{30}$$

Proof. The first inequality is obtained by integrating (23) over $(0,t)$, for $t \in (0,1]$, in view of (18) and (22). In particular, using (26), there holds

$$\int_0^t E(s)ds \le c(1 + \|z_0\|^2_{\mathcal{L}^2}).$$

Then, multiplying (27) by $t \in (0,1]$, we obtain

$$\frac{\mathrm{d}}{\mathrm{dt}}(tE(t)) - E(t) + \nu tE(t) + t\varepsilon^2\|\partial_t z(s)\|^2_{\mathcal{L}^2} \le c$$

and an integration between 0 and t furnishes (30). □

Lemma 3.5. *We have, for all $t \ge 1$ and N large enough,*

$$\|z(t)\|^2_{\mathcal{H}^1} + \int_1^t e^{-\nu(t-s)}\|\partial_t z(s)\|^2_{\mathcal{L}^2}ds \le c\|z_0\|^2_{\mathcal{L}^2}e^{-\nu t} + c.$$

Moreover, for any $t > 0$ and N large enough, there holds

$$\|z(t)\|^2_{\mathcal{H}^1} \leq c\frac{t+1}{t}(1 + \|z_0\|^2_{\mathcal{L}^2}). \tag{31}$$

Proof. The first formula is a direct consequence of Lemmas 3.3 and 3.4. Indeed, from (30), we have

$$\|z(1)\|^2_{\mathcal{H}^1} + \langle F_N(u(1)), 1\rangle \leq c(1 + \|z_0\|^2_{\mathcal{L}^2}),$$

which, plugged into (28), allows to conclude. Inequality (31) then follows from this first estimate and (30). □

Lemma 3.6. *There holds, for all $t \geq 0$ and N large enough,*

$$\begin{aligned}&\|w(t)\|^2_{H^2(\Omega)} + \|\partial_t z(t)\|^2_{\mathcal{L}^2} + \int_t^{t+1} \|\partial_t z(s)\|^2_{\mathcal{H}^1} ds \\ &\leq c\left(\|u_0\|^2 + \|\psi_0\|^2_\Gamma + \|w_0\|^2_{H^2(\Omega)} + \|\partial_t u(0)\|^2 + \|\partial_t \psi(0)\|^2_\Gamma\right) e^{-\nu t} + c.\end{aligned} \tag{32}$$

Moreover, for $t > 0$, we have the smoothing property

$$\|w(t)\|^2_{H^2(\Omega)} + \|\partial_t z(t)\|^2_{\mathcal{L}^2} + \int_t^{t+1} \|\partial_t z(s)\|^2_{\mathcal{H}^1} ds \leq c\frac{t^2+1}{t^2}\left(1 + \|z_0\|^2_{\mathcal{L}^2}\right). \tag{33}$$

Proof. Having set $\theta = \partial_t u$, $\zeta = \partial_t \psi$, a differentiation of the second and third equations of (14) with respect to time gives

$$\begin{cases} \partial_t \theta - \Delta\theta + \tilde{f}'_N(u)\theta = \partial_t w, & t > 0, \quad x \in \Omega, \\ \partial_t \zeta - \Delta_\Gamma \zeta + \zeta + g'_0(\psi)\zeta + \partial_n \theta = 0, & t > 0, \quad x \in \Gamma. \end{cases} \tag{34}$$

Taking the product of the first equation of (34) by θ in $L^2(\Omega)$, we obtain

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{dt}}(\|\theta\|^2 + \|\zeta\|^2_\Gamma) + 2\|\nabla\theta\|^2 + 2\|\zeta\|^2_{H^1(\Gamma)} + 2\langle \tilde{f}'_N(u)\theta, \theta\rangle + 2\langle g'_0(\psi)\zeta, \zeta\rangle_\Gamma \\ &= 2\langle \partial_t w, \theta\rangle.\end{aligned} \tag{35}$$

Then, multiplying the first equation of (14) by $-\varepsilon\Delta\partial_t w$ in $L^2(\Omega)$, we have

$$\frac{\mathrm{d}}{\mathrm{dt}}(\varepsilon\|\Delta w\|^2) + 2\varepsilon^2\|\nabla\partial_t w\|^2 = 2\varepsilon\langle\theta, \Delta\partial_t w\rangle = -2\varepsilon\langle\nabla\theta, \nabla\partial_t w\rangle.$$

Adding the above equations to (24), thanks to (4) and (5), we are led to

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{dt}}(\|\theta\|^2 + \|\zeta\|^2_\Gamma + \varepsilon\|\Delta w\|^2 + \|\nabla w\|^2) \\ &\quad + 2\|\nabla\theta\|^2 + 2\|\zeta\|^2_{H^1(\Gamma)} + 2\varepsilon^2\|\nabla\partial_t w\|^2 + 2\varepsilon\|\partial_t w\|^2 + \|\nabla w\|^2 \\ &= -2\varepsilon\langle\nabla\theta, \nabla\partial_t w\rangle - 2\langle\tilde{f}'_N(u)\theta, \theta\rangle - 2\langle g'_0(\psi)\zeta, \zeta\rangle_\Gamma + \|\nabla w\|^2 \\ &\leq \|\nabla\theta\|^2 + \varepsilon^2\|\nabla\partial_t w\|^2 + 2|\lambda|\|\theta\|^2 + 2C_0\|\zeta\|^2_\Gamma + \|\nabla w\|^2.\end{aligned}$$

Introducing the energy functional

$$\mathcal{S}(t) = \varepsilon\|\Delta w(t)\|^2 + \|\nabla w(t)\|^2 + \|\theta(t)\|^2 + \|\zeta(t)\|^2_\Gamma$$

and noting that $\Delta w = \varepsilon\partial_t w + \theta$, it is straightforward to check that, for some $c > 1$,

$$\begin{cases} \mathcal{S}(t) & \geq \dfrac{1}{c}(\|\Delta w(t)\|^2 + \|\nabla w(t)\|^2 + \|\partial_t z(t)\|^2_{\mathcal{L}^2}), \\ \mathcal{S}(t) & \leq c(\|\nabla w(t)\|^2 + \|\partial_t z(t)\|^2_{\mathcal{L}^2}) \\ & \leq c(\|\nabla w(t)\|^2 + \|\Delta w(t)\|^2 + \|\partial_t z(t)\|^2_{\mathcal{L}^2}). \end{cases} \tag{36}$$

Thus, from (12), we have the differential inequality

$$\frac{\mathrm{d}}{\mathrm{dt}}\mathcal{S} + \nu\mathcal{S} + c\|\partial_t z\|^2_{\mathcal{H}^1} \le c(\|\nabla w\|^2 + \|\partial_t u\|^2 + \|\partial_t \psi\|^2_\Gamma) \le c\mathcal{S}, \tag{37}$$

for some $\nu > 0$ small enough. We first consider the case $t \ge 1$. Possibly reducing ν, in order to exploit Lemma 3.2 and the first estimate in Lemma 3.5, the Gronwall lemma applied to (37) furnishes

$$\begin{aligned}\mathcal{S}(t) &\le \mathcal{S}(1)e^{-\nu(t-1)} + c\int_1^t e^{-\nu(t-s)}[\|\nabla w(s)\|^2 + \|\partial_t u(s)\|^2 + \|\partial_t\psi(s)\|^2_\Gamma]ds \qquad (38)\\ &\le \mathcal{S}(1)e^{-\nu(t-1)} + c\|z_0\|^2_{\mathcal{L}^2}e^{-\nu t} + c, \qquad t \ge 1.\end{aligned}$$

We notice that, integrating (27) over $(t, t+1)$, for $t \ge 1$, owing to Lemma 3.3, (26) and (30), we have

$$\begin{aligned}\int_t^{t+1} \|\partial_t z(s)\|^2_{\mathcal{L}^2} ds &\le c[1 + E(t)] \qquad (39)\\ &\le c[1 + (\|z(1)\|^2_{\mathcal{H}^1} + 2\langle F_N(u(1)), 1\rangle)e^{-\nu(t-1)}]\\ &\le c(1 + \|z_0\|^2_{\mathcal{L}^2}e^{-\nu t}).\end{aligned}$$

Then, an integration of (37) over $(t, t+1)$, again for $t \ge 1$, gives, in view of the previous inequality, (19) and (38),

$$\begin{aligned}\int_t^{t+1} \|\partial_t z(s)\|^2_{\mathcal{H}^1} ds &\le c\mathcal{S}(t) + c\int_t^{t+1} [\|\nabla w(s)\|^2 + \|\partial_t u(s)\|^2 + \|\partial_t\psi(s)\|^2_\Gamma]ds \qquad (40)\\ &\le c\mathcal{S}(1)e^{-\nu(t-1)} + c\|z_0\|^2_{\mathcal{L}^2}e^{-\nu t} + c.\end{aligned}$$

Since (37), in particular, yields

$$\frac{\mathrm{d}}{\mathrm{dt}}\mathcal{S} \le c\mathcal{S}, \tag{41}$$

we obtain, multiplying this inequality by t^2 and integrating by parts over $(0, t)$, for $t \in (0, 1]$,

$$t^2\mathcal{S}(t) \le c\int_0^t \sigma^2\mathcal{S}(\sigma)d\sigma + 2\int_0^t \sigma\mathcal{S}(\sigma)d\sigma \le c\int_0^t \sigma\mathcal{S}(\sigma)d\sigma \le c(1 + \|z_0\|^2_{\mathcal{L}^2}), \tag{42}$$

since, owing to (29), (30) and (36), we have

$$\begin{aligned}\int_0^t \sigma\mathcal{S}(\sigma)d\sigma &\le c\int_0^t \sigma\|w(\sigma)\|^2_{H^1(\Omega)}d\sigma + c\int_0^t \sigma\|\partial_t z(\sigma)\|^2_{\mathcal{L}^2}d\sigma\\ &\le c\int_0^t \|z(\sigma)\|^2_{\mathcal{H}^1}d\sigma + \int_0^t \sigma\|\partial_t z(\sigma)\|^2_{\mathcal{L}^2}d\sigma\\ &\le c(1 + \|z_0\|^2_{\mathcal{L}^2}), \qquad \forall t \in (0, 1).\end{aligned}$$

Then, plugging $\mathcal{S}(1) \le c(1 + \|z_0\|^2_{\mathcal{L}^2})$ into (38) and (40), we are led to

$$\mathcal{S}(t) + \int_t^{t+1} \|\partial_t z(s)\|^2_{\mathcal{H}^1} ds \le c\|z_0\|^2_{\mathcal{L}^2}e^{-\nu t} + c, \quad t \ge 1, \tag{43}$$

which, together with Lemma 3.2 and (36), leads to (32) for $t \ge 1$ (in fact, we have a better control). Besides, this last estimate and (42) yield

$$\mathcal{S}(s) \le c\Big(1 + \frac{1}{s^2}\Big)(1 + \|z_0\|^2_{\mathcal{L}^2}), \quad s > 0. \tag{44}$$

Thus, integrating (37) over $(t, t+1)$, for $t > 0$, we have, owing to (44),

$$\int_t^{t+1} \|\partial_t z(s)\|^2_{\mathcal{H}^1} ds \le c\mathcal{S}(t) + c\int_t^{t+1} \mathcal{S}(s)ds \le c\Big(1+\frac{1}{t^2}\Big)(1+\|z_0\|^2_{\mathcal{L}^2}), \quad t>0,$$

whence (33) follows.

We are left to show (32) for $t \in (0,1]$. Applying the Gronwall lemma to (41) in $[0,t]$, for any $t \in (0,1]$, we obtain, thanks to (36),

$$\begin{aligned} \mathcal{S}(t) &\le ce^{ct}[\|w_0\|^2_{H^2(\Omega)} + \|\partial_t u(0)\|^2 + \|\partial_t\psi(0)\|^2] \\ &\le ce^{-\nu t}[\|w_0\|^2_{H^2(\Omega)} + \|\partial_t u(0)\|^2 + \|\partial_t\psi(0)\|^2], \quad t\in(0,1]. \end{aligned} \tag{45}$$

Furthermore, integrating (37) over $(t, t+1)$, for $t \in (0,1]$, from (43) and (45), we have

$$\begin{aligned} \int_t^{t+1} \|\partial_t z(s)\|^2_{\mathcal{H}^1} ds &\le c\mathcal{S}(t) + c\int_t^{t+1} \mathcal{S}(s)ds = c\mathcal{S}(t) + c\int_t^1 \mathcal{S}(s)ds + \int_1^{t+1} \mathcal{S}(s)ds \\ &\le ce^{-\nu t}[\|z_0\|^2_{\mathcal{L}^2} + \|w_0\|^2_{H^2(\Omega)} + \|\partial_t u(0)\|^2 + \|\partial_t\psi(0)\|^2] + c. \end{aligned}$$

This last estimate, Lemma 3.2 and (45) complete the proof of (32). □

We conclude this section with an estimate on the difference of two solutions to problem (14) which furnishes the Lipschitz continuous dependence of the solutions on the initial data and, in particular, the uniqueness of solutions to problem (14).

Lemma 3.7. *Let f and g satisfy the above assumptions. Given two solutions $z^i = (u^i, \psi^i, w^i)$ departing from $z_i = (u_i, \psi_i, w_i)$, $i = 1, 2$, we have the following estimate on the difference $z^1(t) - z^2(t) = (\bar u(t), \bar\psi(t), \bar w(t))$ in terms of the initial datum $z_1 - z_2 = (\bar u_0, \bar\psi_0, \bar w_0)$:*

$$\begin{aligned} &\|\bar u(t)\|^2 + \|\bar\psi(t)\|^2_\Gamma + \varepsilon^2\|\bar w(t) - \langle \bar w(t)\rangle\|^2_{[\tilde H^1(\Omega)]'} \\ &+ \int_t^{t+1} [\|\bar u(s)\|^2_{H^1(\Omega)} + \|\bar\psi(s)\|^2_{H^1(\Gamma)} + \varepsilon\|\bar w(s) - \langle \bar w(s)\rangle\|^2]ds \\ &\le c\, e^{ct}\left(\|\bar u_0\|^2 + \|\bar\psi_0\|^2_\Gamma + \|\bar w_0 - \langle \bar w_0\rangle\|^2_{[\tilde H^1(\Omega)]'} + I^2\right), \end{aligned}$$

where $I := \langle \bar u_0 + \varepsilon \bar w_0\rangle$. Moreover,

$$\begin{aligned} &\|z^1(t) - z^2(t)\|^2_{\mathcal{L}^2} \\ &+ \int_t^{t+1} [\|\bar u(s)\|^2_{H^1(\Omega)} + \|\bar\psi(s)\|^2_{H^1(\Gamma)} + \|\bar w(s) - \langle \bar w(s)\rangle\|^2_{H^1(\Omega)}]ds \\ &\le ce^{ct}\|z_1 - z_2\|^2_{\mathcal{L}^2}. \end{aligned} \tag{46}$$

Finally,

$$\int_t^{t+1} [\|\bar u(s) + \varepsilon\bar w(s)\|^2_{H^1(\Omega)} + \|\partial_t(\bar u(s) + \varepsilon \bar w(s))\|^2_{[\tilde H^1(\Omega)]'}]ds \le ce^{ct}\|z_1 - z_2\|^2_{\mathcal{L}^2}. \tag{47}$$

Proof. It is immediate to check that $\bar z(\cdot) = (\bar u(\cdot), \bar\psi(\cdot), \bar w(\cdot))$ satisfies the system

$$\begin{cases} \partial_t(\bar u + \varepsilon\bar w - I) - \Delta(\bar w - \langle \bar w\rangle) = 0, & t>0, \quad x\in\Omega, \\ \partial_t \bar u - \Delta \bar u + \tilde f_N(u^1) - \tilde f_N(u^2) = \bar w, & t>0, \quad x\in\Omega, \\ \partial_t\bar\psi - \Delta_\Gamma\bar\psi + \bar\psi + g_0(\psi^1) - g_0(\psi^2) + \partial_n \bar u = 0, & t>0, \quad x\in\Gamma, \\ \partial_n \bar w|_\Gamma = 0, \quad \bar u|_\Gamma = \bar\psi, \\ \bar w|_{t=0} = \bar w_0, \quad \bar u|_{t=0} = \bar u_0, \quad \bar\psi|_{t=0} = \bar\psi_0. \end{cases} \tag{48}$$

The product of the first equation by $(-\Delta)^{-1}(\bar u+\varepsilon\bar w-I)$, added to the second one multiplied by $\bar u$ leads, on account of the third equation and of the boundary conditions, to

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{dt}}[\|\bar u\|^2+\|\bar\psi\|_\Gamma^2+\|\bar u+\varepsilon\bar w-I\|^2_{[\tilde H^1(\Omega)]'}+I^2]\\&\quad+2\|\nabla\bar u\|^2+2\|\bar\psi\|^2_{H^1(\Gamma)}+2\varepsilon\|\bar w-\langle\bar w\rangle\|^2\\&\quad+2\langle\tilde f_N(u^1)-\tilde f_N(u^2),\bar u\rangle+2\langle g_0(\psi^1)-g_0(\psi^2),\bar\psi\rangle_\Gamma=2|\Omega|\langle\bar w\rangle\langle\bar u\rangle.\end{aligned}\tag{49}$$

Having set

$$E(t)=\|\bar u(t)\|^2+\|\bar\psi(t)\|_\Gamma^2+\|\bar u(t)+\varepsilon\bar w(t)-I\|^2_{[\tilde H^1(\Omega)]'}+I^2,$$

by (4), (5) and (8), we obtain

$$\frac{\mathrm{d}}{\mathrm{dt}}E+2\|\nabla\bar u\|^2+2\|\bar\psi\|^2_{H^1(\Gamma)}+2\varepsilon\|\bar w-\langle\bar w\rangle\|^2\le2|\lambda|\|\bar u\|^2+2C_0\|\bar\psi\|_\Gamma^2+|\Omega|\frac{I^2}{\varepsilon}.\tag{50}$$

Thus, by (12), we have the differential inequality

$$\frac{\mathrm{d}}{\mathrm{dt}}E+c\|\bar u\|^2_{H^1(\Omega)}+c\|\bar\psi\|^2_{H^1(\Gamma)}+2\varepsilon\|\bar w-\langle\bar w\rangle\|^2\le cE$$

and the Gronwall lemma furnishes

$$E(t)\le e^{ct}E(0).\tag{51}$$

By (9) and (11), we have

$$\begin{aligned}\varepsilon^2\|\bar w(t)-\langle\bar w(t)\rangle\|^2_{[\tilde H^1(\Omega)]}&\le2\|\bar u(t)+\varepsilon\bar w(t)-I\|^2_{[\tilde H^1(\Omega)]'}+2\|\bar u(t)-\langle\bar u(t)\rangle\|^2_{[\tilde H^1(\Omega)]'}\\&\le2\|\bar u(t)+\varepsilon\bar w(t)-I\|^2_{[\tilde H^1(\Omega)]'}+2c_\Omega\|\bar u(t)\|^2_{L^2(\Omega)},\end{aligned}\tag{52}$$

yielding, by (11) and (51),

$$\begin{aligned}&\|\bar u(t)\|^2+\|\bar\psi(t)\|_\Gamma^2+\varepsilon^2\|\bar w(t)-\langle\bar w(t)\rangle\|^2_{[\tilde H^1(\Omega)]'}+I^2\\&\le2(1+c_\Omega)E(t)\\&\le2(1+c_\Omega)e^{ct}[\|\bar u_0\|^2+\|\bar\psi_0\|_\Gamma^2+\|\bar u_0+\varepsilon\bar w_0-I\|^2_{[\tilde H^1(\Omega)]'}+I^2]\\&\le ce^{ct}[\|\bar u_0\|^2+\|\bar\psi_0\|_\Gamma^2+\varepsilon^2\|\bar w_0-\langle\bar w_0\rangle\|^2_{[\tilde H^1(\Omega)]'}+I^2].\end{aligned}$$

To conclude the proof of the first estimate, we integrate the differential inequality for E over $(t,t+1)$, which yields

$$\begin{aligned}&\int_t^{t+1}[\|\bar u(s)\|^2_{H^1(\Omega)}+\|\bar\psi(s)\|^2_{H^1(\Gamma)}+\varepsilon\|\bar w(s)-\langle\bar w(s)\rangle\|^2]ds\\&\le cE(t)+c\int_t^{t+1}E(s)ds\\&\le ce^{ct}[\|\bar u_0\|^2+\|\bar\psi_0\|_\Gamma^2+\varepsilon^2\|\bar w_0-\langle\bar w_0\rangle\|^2_{[\tilde H^1(\Omega)]'}+I^2].\end{aligned}$$

In order to prove the second formula, we multiply the first equation of (48) by $\varepsilon(\bar u+\varepsilon\bar w)$ and add the resulting equation to (49), getting

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{dt}}Y+2\|\nabla\bar u\|^2+2\|\bar\psi\|^2_{H^1(\Gamma)}+2\varepsilon\|\bar w-\langle\bar w\rangle\|^2+2\varepsilon^2\|\nabla\bar w\|^2\\&=-2\langle\tilde f_N(u^1)-\tilde f_N(u^2),\bar u\rangle-2\langle g_0(\psi^1)-g_0(\psi^2),\bar\psi\rangle_\Gamma+2|\Omega|\langle\bar w\rangle\langle\bar u\rangle-2\varepsilon\langle\nabla\bar w,\nabla\bar u\rangle,\end{aligned}$$

where Y coincides with (20), with (u,ψ,w) replaced by $(\bar u,\bar\psi,\bar w)$, namely,

$$Y(t)=\|\bar u(t)\|^2+\|\bar\psi(t)\|_\Gamma^2+\|\bar u(t)+\varepsilon\bar w(t)-I\|^2_{[\tilde H^1(\Omega)]'}+\varepsilon\|\bar u(t)+\varepsilon\bar w(t)\|^2.$$

The right-hand side of the differential inequality is easily controlled by

$$-2\varepsilon\langle\nabla\bar{w},\nabla\bar{u}\rangle \le \|\nabla\bar{u}\|^2+\varepsilon^2\|\nabla\bar{w}\|^2,$$

which, together with (4), (5), (8) and (9), leads to

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{dt}}Y+\|\nabla\bar{u}\|^2+2\|\bar{\psi}\|^2_{H^1(\Gamma)}+2\varepsilon\|\bar{w}-\langle\bar{w}\rangle\|^2+\varepsilon^2\|\nabla\bar{w}\|^2\\&\le 2|\lambda|\|\bar{u}\|^2+2C_0\|\bar{\psi}\|^2_\Gamma+\frac{1}{\varepsilon}\|\bar{u}+\varepsilon\bar{w}\|^2.\end{aligned}$$

Since Y satisfies (22), we have

$$\frac{\mathrm{d}}{\mathrm{dt}}Y+\|\nabla\bar{u}\|^2+2\|\bar{\psi}\|^2_{H^1(\Gamma)}+2\varepsilon\|\bar{w}-\langle\bar{w}\rangle\|^2+\varepsilon^2\|\nabla\bar{w}\|^2\le cY.$$

A further application of the Gronwall lemma, followed by an integration over $(t,t+1)$, entails the second inequality.

We now go back to the first equation of (48) written as

$$\partial_t(\bar{u}+\varepsilon\bar{w})-\Delta(\bar{w}-\langle\bar{w}\rangle)=0$$

and we multiply it by $(-\Delta)^{-1}(\partial_t(\bar{u}+\varepsilon\bar{w}))$ (here, $\langle\partial_t(u+\varepsilon w)\rangle=0$), getting

$$\|\partial_t(\bar{u}+\varepsilon\bar{w})\|^2_{[\tilde{H}^1(\Omega)]'}+\langle\bar{w}-\langle\bar{w}\rangle,\partial_t(\bar{u}+\varepsilon\bar{w})\rangle=0,$$

which, viewing $\langle\cdot,\cdot\rangle$ as the duality pairing between $H^1(\Omega)$ and $(H^1(\Omega))^*$, gives, owing to the Young inequality,

$$\|\partial_t(\bar{u}+\varepsilon\bar{w})\|^2_{[\tilde{H}^1(\Omega)]'}\le\|\bar{w}-\langle\bar{w}\rangle\|^2_{H^1(\Omega)}.$$

Inequality (47) finally follows from (46). □

4. **Variational formulation and well-posedness.** This section is devoted to the definition of a suitable notion of a solution to the limit problem, that is, the problem obtained by letting $N\to+\infty$ and which formally coincides with (2). The difficulty is that we should allow the solutions to reach the singular values (i.e., the pure phases) on the boundary and, at the same time, get a well-posed problem, whose solutions coincide with the classical ones under proper assumptions (e.g., sign conditions). All the following computations are formal.

The first equation of (2) can be rewritten as

$$\partial_t(u+\varepsilon w)-\Delta(w-\langle w\rangle)=0$$

and the product by $(-\Delta)^{-1}(v_1-\langle v_1\rangle)$, for any $v_1\in L^2(\Omega)$, gives

$$\langle\partial_t(u+\varepsilon w),v_1-\langle v_1\rangle\rangle_{[\tilde{H}^1(\Omega)]'}+\langle w,v_1-\langle v_1\rangle\rangle=0.$$

Next, we introduce the bilinear form $B(u,v)=\langle\nabla u,\nabla v\rangle+\langle\nabla_\Gamma u,\nabla_\Gamma v\rangle_\Gamma$ which satisfies

$$B(u,u-v)\ge B(v,u-v),\qquad\forall\,u,v\in H^1(\Omega)\otimes H^1(\Gamma).\tag{53}$$

Taking advantage of this notation and multiplying the second equation of (2) by $u-v_2$, for any $v_2\in H^1(\Omega)\otimes H^1(\Gamma)$, we get

$$\begin{aligned}&\langle\partial_t u,u-v_2\rangle+\langle\partial_t u,u-v_2\rangle_\Gamma+B(u,u-v_2)+\langle f(u),u-v_2\rangle\\&=\langle w,u-v_2\rangle+\lambda\langle u,u-v_2\rangle-\langle g(u),u-v_2\rangle_\Gamma.\end{aligned}$$

Then, from (53) and the monotonicity of f, we infer

$$\langle \partial_t u, u - v_2\rangle + \langle \partial_t u, u - v_2\rangle_\Gamma + B(v_2, u - v_2) + \langle f(v_2), u - v_2\rangle$$
$$\leq \langle w, u - v_2\rangle + \lambda\langle u, u - v_2\rangle - \langle g(u), u - v_2\rangle_\Gamma, \qquad \forall v_2 \in H^1(\Omega) \otimes H^1(\Gamma).$$

If we consider the solutions to (14) departing from initial data in

$$\Phi = \{(u, \psi, w) \in L^\infty(\Omega) \times L^\infty(\Gamma) \times L^2(\Omega) : \ \|u\|_{L^\infty(\Omega)} \leq 1, \ \|\psi\|_{L^\infty(\Gamma)} \leq 1\}$$

and then pass to the limit $N \to +\infty$, we find functions living in Φ for any time, as we will rigorously see in this section (cf. Theorem 4.3). These functions are not necessarely solutions to (2) in the usual sense and, arguing as in [19, Section 3], we suitably modify the notion of a solution as follows.

Definition 4.1. Given $z_0 = (u_0, \psi_0, w_0) \in \Phi$, we say that $z(t) = (u(t), \psi(t), w(t))$ is a variational solution to problem (2) originating from z_0 if

- $z(0) = z_0$;
- $u(t)_{|\Gamma} = \psi(t)$, for almost all $t > 0$;
- $-1 < u(x,t) < 1$, for almost all $(x,t) \in \Omega \times \mathbb{R}^+ = \Omega \times [0,\infty)$;
- $z \in C([0,+\infty), \mathcal{L}^2) \cap L^2([0,T], \mathcal{H}^1)$, for any $T > 0$;
- $f(u) \in L^1(\Omega \times [0,T])$, for any $T > 0$;
- $\partial_t z \in L^2([\tau,T], \mathcal{H}^1)$, for any $\tau \in (0,T]$, $T > 0$;
- $\langle u(t) + \varepsilon w(t)\rangle = \langle u_0 + \varepsilon w_0\rangle$, for any $t > 0$;

and

$$\langle \partial_t(u(t) + \varepsilon w(t)), v_1 - \langle v_1\rangle\rangle_{[\tilde{H}^1(\Omega)]'} + \langle w(t), v_1 - \langle v_1\rangle\rangle = 0, \tag{54}$$
$$\langle \partial_t u(t), u(t) - v_2\rangle + \langle \partial_t u(t), u(t) - v_2\rangle_\Gamma + B(v_2, u(t) - v_2) + \langle f(v_2), u(t) - v_2\rangle \tag{55}$$
$$\leq \langle w(t), u(t) - v_2\rangle + \lambda\langle u(t), u(t) - v_2\rangle - \langle g(u(t)), u(t) - v_2\rangle_\Gamma,$$

for almost all $t > 0$ and any pair of test functions $(v_1, v_2) \in L^2(\Omega) \times [H^1(\Omega) \otimes H^1(\Gamma)]$ such that $f(v_2) \in L^1(\Omega)$.

We emphasize that we do not assume in the definition that ψ_0 is the trace of u_0.

In order to show the uniqueness of a variational solution, we consider (54)-(55) in terms of test functions $v_1 = v_1(x,t)$ and $v_2 = v_2(x,t)$, with v_1, v_2 satisfying the regularity assumptions in Definition 4.1, and integrate (54)-(55) with respect to t, a legitimate step, since all terms are L^1 in time. This gives, for $t > s > 0$,

$$\int_s^t [\langle \partial_t(u + \varepsilon w), v_1 - \langle v_1\rangle\rangle_{[\tilde{H}^1(\Omega)]'} + \langle w, v_1 - \langle v_1\rangle\rangle] d\sigma = 0, \tag{56}$$
$$\int_s^t [\langle \partial_t u, u - v_2\rangle + \langle \partial_t u, u - v_2\rangle_\Gamma + B(v_2, u - v_2) + \langle f(v_2), u - v_2\rangle] d\sigma \tag{57}$$
$$\leq \int_s^t [\langle w, u - v_2\rangle + \lambda\langle u, u - v_2\rangle - \langle g(u), u - v_2\rangle_\Gamma] d\sigma.$$

Arguing as in [19], the function $v_\alpha = (1 - \alpha)u + \alpha v_2$, where $\alpha \in (0,1]$ and v_2 is an arbitrary admissible test function, is an admissible test function for (57) as well (indeed, $f(v_\alpha(t)) \in L^1(\Omega)$ follows from $(3)_4$ which implies that $|f(\cdot)|$ is convex). Then, taking the corresponding (57), where we recall that u is absolutely continuous

on $[s,t]$ with values in $[L^2(\Omega)\otimes L^2(\Gamma)]$, we simplify by α and pass to the limit $\alpha\to 0$, getting

$$\int_s^t [\langle \partial_t u, u-v_2\rangle + \langle \partial_t u, u-v_2\rangle_\Gamma + B(u,u-v_2) + \langle f(u), u-v_2\rangle]d\sigma \qquad (58)$$
$$\leq \int_s^t [\langle w, u-v_2\rangle + \lambda\langle u, u-v_2\rangle - \langle g(u), u-v_2\rangle_\Gamma]d\sigma,$$

thanks to the Lebesgue dominated convergence theorem.

We can now state the following

Lemma 4.2. *For every two variational solutions $z^i(t)=(u^i(t),\psi^i(t),w^i(t))$ departing from $z_i=(u_i,\psi_i,w_i)$, $i=1,2$, we have the following estimate on the difference $z^1(t)-z^2(t)=(\bar u(t),\bar\psi(t),\bar w(t))$ in terms of the initial datum $z_1-z_2=(\bar u_0,\bar\psi_0,\bar w_0)$:*

$$\|\bar u(t)\|^2 + \|\bar\psi(t)\|_\Gamma^2 + \varepsilon^2\|\bar w(t)-\langle \bar w(t)\rangle\|^2_{[\tilde H^1(\Omega)]'} + 2\varepsilon\int_0^t \|\bar w(\sigma)-\langle \bar w(\sigma)\rangle\|^2 d\sigma$$
$$\leq\ ce^{ct}(\|\bar u_0\|^2 + \|\bar\psi_0\|_\Gamma^2 + \varepsilon^2\|\bar w_0-\langle \bar w_0\rangle\|^2_{[\tilde H^1(\Omega)]'} + I^2),$$

where $I:=\langle \bar u_0+\varepsilon\bar w_0\rangle$.

Proof. We sum (58), with $u=u^2$, $w=w^2$, $v_2=u^1$, and the variational inequality (57), with $u=u^1$, $w=w^1$, $v_2=u^2$, obtaining

$$\int_s^t [\langle \partial_t\bar u,\bar u\rangle + \langle \partial_t\bar\psi,\bar\psi\rangle_\Gamma]d\sigma \leq \int_s^t [\langle \bar w,\bar u\rangle + \lambda\|\bar u\|^2 - \langle g(\psi^1)-g(\psi^2),\bar\psi\rangle_\Gamma]d\sigma.$$

This provides, after obvious simplifications,

$$\|\bar u(t)\|^2 + \|\bar\psi(t)\|_\Gamma^2 \leq \|\bar u(s)\|^2 + \|\bar\psi(s)\|_\Gamma^2 \qquad (59)$$
$$+ 2\int_s^t [\langle \bar w(\sigma),\bar u(\sigma)\rangle + c(\|\bar u(\sigma)\|^2 + \|\bar\psi(\sigma)\|_\Gamma^2)]d\sigma.$$

Then, recalling (7), we use (56), with $w=w^1$, $u=u^1$, $v_1=\bar u+\varepsilon\bar w$ (respectively, with $w=w^2$, $u=u^2$, $v_1=\bar u+\varepsilon\bar w$), and find, after subtracting the two resulting equalities and since $\langle\langle \bar w\rangle, \bar u+\varepsilon\bar w - \langle \bar u+\varepsilon\bar w\rangle\rangle = 0$,

$$\int_s^t \frac{\mathrm{d}}{\mathrm{d}\sigma}\|(\bar u+\varepsilon\bar w-I)(\sigma)\|^2_{[\tilde H^1(\Omega)]'}d\sigma + 2\int_s^t \langle \bar w(\sigma)-\langle \bar w(\sigma)\rangle, (\bar u-\langle \bar u\rangle+\varepsilon(\bar w-\langle \bar w\rangle))(\sigma)\rangle d\sigma = 0,$$

that is,

$$\|\bar u(t)+\varepsilon\bar w(t)-I\|^2_{[\tilde H^1(\Omega)]'} + 2\int_s^t \langle \bar w(\sigma),\bar u(\sigma)\rangle d\sigma + 2\varepsilon\int_s^t \|\bar w(\sigma)-\langle \bar w(\sigma)\rangle\|^2 d\sigma$$
$$= \|\bar u(s)+\varepsilon\bar w(s)-I\|^2_{[\tilde H^1(\Omega)]'} + 2|\Omega|\int_s^t \langle \bar w(\sigma)\rangle\langle \bar u(\sigma)\rangle d\sigma.$$

Adding this equation to (59), we see that the functional

$$\mathcal{Y}(t) = \|\bar u(t)\|^2 + \|\bar\psi(t)\|_\Gamma^2 + \|\bar u(t)+\varepsilon\bar w(t)-I\|^2_{[\tilde H^1(\Omega)]'} + |\Omega|\frac{I^2}{\varepsilon}$$

satisfies, owing to (8),

$$\mathcal{Y}(t) + 2\varepsilon\int_s^t \|\bar w(\sigma)-\langle \bar w(\sigma)\rangle\|^2 d\sigma \leq \mathcal{Y}(s) + c\int_s^t \Big(\|\bar u(\sigma)\|^2 + \|\bar\psi(\sigma)\|_\Gamma^2 + |\Omega|\frac{I^2}{\varepsilon}\Big)d\sigma$$
$$\leq \mathcal{Y}(s) + c\int_s^t \mathcal{Y}(\sigma)\,d\sigma$$

and, thanks to the Gronwall lemma, we find

$$\mathcal{Y}(t) \le ce^{c(t-s)}\mathcal{Y}(s),$$

for some positive constant c which is independent of s and t. Passing to the limit $s \to 0$ and owing to the continuity of z^1, z^2 (cf. Definition 4.1), we get the desired estimate on $\|\bar{u}(t)\|^2 + \|\bar{\psi}(t)\|_\Gamma^2$. Arguing exactly as in the proof of Lemma 3.7, we control the w-term and we finally have the estimate of Lemma 4.2, which, in particular, gives the uniqueness of a variational solution. □

We now prove the existence of a variational solution in the following

Theorem 4.3. *For every initial datum $z_0 = (u_0, \psi_0, w_0) \in \Phi$, problem (2) possesses a unique variational solution $z(t) = (u(t), \psi(t), w(t))$ in the sense of Definition 4.1. Such a solution regularizes as $t > 0$ and all the uniform estimates obtained above (except for the one on $\|F_N(u_N(\cdot))\|_{L^1(\Gamma)}$) hold. More precisely, for every $\delta > 0$ and $t > 0$, we have*

$$\begin{aligned} &\|u(t)\|^2_{H^1(\Omega)} + \|u(t)\|^2_{C^\alpha(\Omega)} + \|\nabla D_\tau u(t)\|^2 + \|u(t)\|^2_{H^2(\Omega_\delta)} + \|u(t)\|^2_{H^2(\Gamma)} \\ &\quad + \|w(t)\|^2_{H^2(\Omega)} + \|f(u(t))\|_{L^1(\Omega)} + \|\partial_t z(t)\|^2_{\mathcal{L}^2} + \int_t^{t+1} \|\partial_t z(s)\|^2_{\mathcal{H}^1} ds \\ &\le c\,\frac{t^2+1}{t^2}\,(1 + \|z_0\|^2_{\mathcal{L}^2}). \end{aligned} \tag{60}$$

Furthermore, for any pair of initial data z_1, z_2, the difference of the corresponding solutions $z^1(t)$ and $z^2(t)$ satisfies

$$\begin{aligned} &\|z^1(t) - z^2(t)\|^2_{\mathcal{L}^2} \\ &\quad + \int_t^{t+1} [\|\bar{u}(s)\|^2_{H^1(\Omega)} + \|\bar{\psi}(s)\|^2_{H^1(\Gamma)} + \|\bar{w}(s) - \langle \bar{w}(s)\rangle\|^2_{H^1(\Omega)}] ds \\ &\le ce^{ct}\|z_1 - z_2\|^2_{\mathcal{L}^2}, \end{aligned} \tag{61}$$

where $(\bar{u}(t), \bar{\psi}(t), \bar{w}(t)) = z^1(t) - z^2(t)$. Finally,

$$\int_t^{t+1} [\|\bar{u}(s) + \varepsilon\bar{w}(s)\|^2_{H^1(\Omega)} + \|\partial_t(\bar{u}(s) + \varepsilon\bar{w}(s))\|^2_{[\tilde{H}^1(\Omega)]'}] ds \le ce^{ct}\|z_1 - z_2\|^2_{\mathcal{L}^2}. \tag{62}$$

Proof. We follow the rationale of the proof of [19, Theorem 3.3]. Repeating the derivation of the variational inequalities (56) and (57), we can easily check that (u_N, ψ_N, w_N), which is a smooth solution to problem (14), satisfies

$$\int_s^t [\langle \partial_t(u_N + \varepsilon w_N), v_1 - \langle v_1\rangle\rangle_{[\tilde{H}^1(\Omega)]'} + \langle w_N - \langle w_N\rangle, v_1 - \langle v_1\rangle\rangle] d\sigma = 0, \tag{63}$$

$$\begin{aligned} &\int_s^t [\langle \partial_t u_N, u_N - v_2\rangle_{L^2(\Omega)\otimes L^2(\Gamma)} + B(v_2, u_N - v_2) + \langle f_N(v_2), u_N - v_2\rangle] d\sigma \\ &\qquad \le \int_s^t [\langle w, u_N - v_2\rangle + \lambda\langle u_N, u_N - v_2\rangle - \langle g(u_N), u_N - v_2\rangle_\Gamma] d\sigma, \end{aligned} \tag{64}$$

for every admissible test functions v_1 and v_2 and $t > s > 0$. Our task is to pass to the limit $N \to +\infty$.

We start with the case when

$$u_0 \in H^2(\Omega) \otimes H^2(\Gamma), \quad w_0 \in H^2_N(\Omega), \quad |u_0(x)| \le 1 - \eta, \quad \psi_0 = u_0|_\Gamma, \tag{65}$$

for some $\eta \in (0,1)$, so that $\partial_t u(0) \in L^2(\Omega) \otimes L^2(\Gamma)$. Then, by (15) in Theorem 3.1, it is straightforward to check that, at least for a subsequence,

$$
\begin{aligned}
&u_N \overset{*}{\rightharpoonup} u \quad \text{in} \quad L^\infty([0,T], H^1(\Omega) \otimes H^2(\Gamma) \cap H^2(\Omega_\delta)),\\
&u_N \to u \quad \text{in} \quad C^\gamma(\Omega \times [0,T]),\\
&(\partial_t u_N, \partial_t u_N|_\Gamma, \partial_t w_N) \overset{*}{\rightharpoonup} (\partial_t u, \partial_t u|_\Gamma, \partial_t w) \quad \text{in} \quad L^\infty([0,T], \mathcal{L}^2),\\
&(\partial_t u_N, \partial_t u_N|_\Gamma, \partial_t w_N) \rightharpoonup (\partial_t u, \partial_t u|_\Gamma, \partial_t w) \quad \text{in} \quad L^2([0,T], \mathcal{H}^1),\\
&D_\tau^2 u_N \overset{*}{\rightharpoonup} D_\tau^2 u \quad \text{in} \quad L^\infty([0,T], L^2(\Omega)),\\
&w_N \overset{*}{\rightharpoonup} w \quad \text{in} \quad L^\infty([0,T], H^2_N(\Omega)),\\
&w_N \to w \quad \text{in} \quad C(\Omega \times [0,T]),
\end{aligned}
$$

for some $\gamma > 0$. These convergences allow to pass to the limit in (63)-(64) and prove that the limit $(u, u|_\Gamma, w)$ satisfies (56)-(57), for any admissible test functions (v_1, v_2). The crucial point $-1 < u(x,t) < 1$, for almost all $(x,t) \in \Omega \times \mathbb{R}^+$, can then be checked as in [19]. Indeed, taking into account the definition of f_N and the fact that $f_N(u_N)$ is uniformly bounded in $L^1(\Omega \times [T, T+1])$, then, for each $M \geq N$, there holds

$$
\text{meas}\{(x,t) \in \Omega \times [T, T+1] : |u_M(x,t)| \geq 1 - 1/N\} \leq \varphi(1/N), \tag{66}
$$

where

$$
\varphi(x) = \frac{c}{\min\{|f(1-x)|, |f(-1+x)|\}}
$$

and c is a positive constant which is independent of $T > 0$, of N and $M \geq N$. Since $\varphi(x)$ goes to zero as $x \to 0$, then, passing to the limit $M, N \to +\infty$ in (66), we have

$$
\text{meas}\{(x,t) \in \Omega \times [T, T+1] : |u(x,t)| = 1\} = 0,
$$

so that $-1 < u(x,t) < 1$, for almost all $(x,t) \in \Omega \times \mathbb{R}^+$. This, combined with the strong convergence of u_N to u in $C^\gamma(\Omega \times [0,T])$, yields the almost everywhere convergence of $f_N(u_N)$ to $f(u)$. Therefore, the Fatou lemma gives

$$
\|f(u)\|_{L^1(\Omega \times [0,T])} \leq \liminf_{N \to +\infty} \|f_N(u_N)\|_{L^1(\Omega \times [0,T])} < +\infty. \tag{67}
$$

Thus, $f(u) \in L^1(\Omega \times [0,T])$ and (u, ψ, w) is a variational solution to problem (2). In particular, the bound on $f(u)$ in $L^1(\Omega)$ follows from (67). Since the separation from the singularities is not ensured on the boundary, we are not allowed to pass to the limit in $\|F_N(u_N(t))\|_{L^1(\Gamma)}$ (see the subsequent Proposition 1). Finally, we are allowed to pass to the limit in (16), proving (60), and in (46)-(47), obtaining the desired Lipschitz continuous dependence (61)-(62).

We now remove assumption (65). In that case, we approximate the initial datum $(u_0, \psi_0, w_0) \in \Phi$ by a sequence (u_0^k, ψ_0^k, w_0^k) satisfying (65), together with

$$
\|u_0^k - u_0\| + \|\psi_0^k - \psi_0\|_\Gamma + \|w_0^k - w_0\| \to 0, \qquad \text{as} \quad k \to +\infty. \tag{68}
$$

Let $(u_k(t), \psi_k(t), w_k(t))$, where $u_k|_\Gamma = \psi_k$, be a sequence of variational solutions to (2) such that $(u_k(0), \psi_k(0), w_k(0)) = (u_0^k, \psi_0^k, w_0^k)$ (the existence and the regularity of this solution have been proved above). Then, by (61) and (68), we can see that (u_k, ψ_k, w_k) is a Cauchy sequence in $C([0,T], \mathcal{L}^2)$, so that the limit function $(u, \psi, w) = \lim_{k \to +\infty} (u_k, \psi_k, w_k)$ exists and belongs to $C([0,T], \mathcal{L}^2)$. Then, the proof finishes as above. □

5. **Sufficient conditions for a variational solution to be a classical one.** We have the following

Lemma 5.1. *Let $(u(t),\psi(t),w(t))$ be a variational solution to problem (2). Then, there holds $\psi(t)=u(t)_{|\Gamma}$, for any $t>0$. Moreover, w and u solve (2) in the usual sense, that is, for any $\varphi\in C_0^\infty(\Omega\times(0,T))$, there hold*

$$\int_{\mathbb{R}^+}\langle\partial_t u(t)+\varepsilon\partial_t w(t),\varphi(t)\rangle dt=\int_{\mathbb{R}^+}\langle\Delta w(t),\varphi(t)\rangle dt, \tag{69}$$

$$\int_{\mathbb{R}^+}\langle\partial_t u(t),\varphi(t)\rangle dt=\int_{\mathbb{R}^+}[\langle\Delta u(t),\varphi(t)\rangle-\langle\tilde f(u(t)),\varphi(t)\rangle-\langle w(t),\varphi(t)\rangle]dt. \tag{70}$$

Finally, $w\in L^\infty((\tau,T],H^2_N(\Omega))$ and $u\in L^\infty([\tau,T],W^{2,1}(\Omega))$, so that $[\partial_n u]_{int}:=\partial_n u_{|\Gamma}\in L^\infty([\tau,T],L^1(\Gamma))$, for any $0<\tau<T$.

Proof. Arguing as in [19, Proposition 3.5] and exploiting the facts that the sequences w_N and u_N are uniformly bounded in $L^\infty((\tau,T],H^2(\Omega))$ and in $L^\infty((\tau,T],H^2(\Omega_\delta))$, for any $\delta>0$, respectively, we are allowed to pass to the limit in the equations corresponding to (69) and (70) for w_N and u_N. In particular, u is solution to

$$\partial_t u-\Delta u+f(u)-\lambda u=w,\quad \text{in } L^2_{loc}(\Omega\times(\tau,T)).$$

Moreover, since $f(u)\in L^\infty((\tau,T],L^1(\Omega))$, we deduce that $\Delta u\in L^\infty((\tau,T],L^1(\Omega))$. This, combined with the control of $\nabla D_\tau u$, leads to $u\in L^\infty((\tau,T],W^{2,1}(\Omega))$ and we infer the regularity claimed in Lemma 5.1 for the trace $\partial_n u_{|\Gamma}$. □

Concerning the third equation of (2), we deduce from Theorem 3.1 and Lemma 3.6 that any regular solution $(u_N(t),\psi_N(t),w_N(t))$ to (14) satisfies

$$\|\psi_N\|_{L^\infty([\tau,T],H^2(\Gamma))}+\|\partial_t\psi_N\|_{L^\infty([\tau,T],L^2(\Gamma))}\le c,$$

where the constant c is independent of N. Thus, we obtain from the third equation in (14) a uniform (in N) bound on $\partial_n u_N$ in $L^\infty([\tau,T],L^2(\Gamma))$. Passing then to the limit $N\to+\infty$, we have the weak* convergence in $L^\infty([\tau,T];L^2(\Gamma))$

$$[\partial_n u]_{ext}:=\lim_{N\to+\infty}\partial_n u_{N\,|\Gamma}\in L^\infty([\tau,T],L^2(\Gamma)) \tag{71}$$

and

$$\partial_t\psi-\Delta_\Gamma\psi+g(\psi)+[\partial_n u]_{ext}=0,\quad \text{on }\Gamma,\quad T>\tau>0.$$

In order to verify that the variational solution $(u(t),\psi(t),w(t))$ satisfies (2) in the usual sense, there remains to check whether

$$[\partial_n u]_{int}=[\partial_n u]_{ext},\qquad \text{for almost all } (x,t)\in\Gamma\times\mathbb{R}^+.$$

The two following results are obtained in [19, Proposition 4.1 and Corollary 4.3] for the Cahn-Hilliard equation. Since the regularity of w is now established, they still hold in our case and can be summarized as

Theorem 5.2. *Let (u,ψ,w) be a variational solution to (2). Assume, in addition, that*

$$|u(x_0,t_0)|<1,$$

for some $(x_0,t_0)\in\Gamma\times(0,+\infty)$. Then, there exists a neighborhood $V\times(t_0-\epsilon,t_0+\epsilon)$ of $(x_0,t_0)\in\Gamma\times\mathbb{R}^+$ such that $[\partial_n u]_{ext}(x,t)=[\partial_n u]_{int}(x,t)$, for all $(x,t)\in V\times(t_0-\epsilon,t_0+\epsilon)$. In particular, if u satisfies

$$|u(x,t)|<1,\quad \textit{for almost all}\quad (x,t)\in\Gamma\times\mathbb{R}^+, \tag{72}$$

then, the equality $[\partial_n u]_{ext} = [\partial_n u]_{int}$ *holds almost everywhere in* $\Gamma\times\mathbb{R}^+$ *and* (u,ψ,w) *solves* (2) *in the usual sense.*

Proof. Owing to the Hölder continuity of u with respect to both x and t, there exists $\epsilon > 0$ such that

$$|u(x,t)| \le 1-\epsilon,$$

in a neighborhood $V_\epsilon \times (t_0-\epsilon, t_0+\epsilon)$ of (x_0,t_0) in $\Omega\times\mathbb{R}^+$. Applying Proposition 2, the approximate solutions u_N to (14) (converging to u) satisfy

$$\|u_N\|_{L^\infty([t_0-\epsilon,\, t_0+\epsilon],\, H^2(V_\epsilon))} \le c,$$

where the positive constant c is independent of N, but depends on ϵ and (x_0,t_0). Then, without loss of generality, we can assume that $u_N \overset{*}{\rightharpoonup} u$ in $L^\infty([t_0-\epsilon, t_0+\epsilon], H^2(V_\epsilon))$. In turn, this yields $\partial_n u_N|_\Gamma \rightharpoonup \partial_n u|_\Gamma = [\partial_n u]_{int}$ in $L^2(V\times[t_0-\epsilon, t_0+\epsilon])$, for some proper neighborhood V of x_0. This, together with (71), leads to the first statement. The second one is just a straightforward consequence of the first one. □

As in [19], we can show that, provided that f is strongly singular or that proper sign conditions hold for g, any variational solution u to our problem satisfies (72). The proof is identical to the one in [19] and is thus omitted.

Proposition 1. *We assume that either* $\lim_{s\to\pm1} F(s) = +\infty$ *or* $g(-1) < 0 < g(1)$. *Then,* (72) *holds. Thus, for all* $T > 0$, $[\partial_n u]_{ext} = [\partial_n u]_{int}$, *almost everywhere in* $\Gamma\times[T,+\infty]$, *and, for* $t \ge T$, (u,ψ,w), *solves* (2) *in the usual sense.*

Remark 1. Unfortunately, the thermodynamically relevant function

$$\tilde f(s) = -2\kappa_0 s + \kappa_1 \ln\frac{1+s}{1-s}, \quad s\in(-1,1), \quad 0<\kappa_1<\kappa_0,$$

does not satisfy the first assumption above, as $\tilde F$ is bounded in that case.

Theorem 5.3. *Assume that there exist* $M > 0$ *and* $p > 2$ *such that*

$$\frac{\kappa_1}{(1-s^2)^{p-1}} \le \frac{f(s)}{s} \le \frac{\kappa_2}{(1-s^2)^M}, \tag{73}$$

for some positive constants κ_i, $i = 1,2$. *Then, any variational solution* u *is strictly separated from the singularities* ±1 *in* Ω, *namely, for any* $T > 0$, *there exists* $\delta_T \in (0,1)$ *such that*

$$|u(x,t)| \le 1-\delta_T, \qquad \forall t\ge T, \quad \forall x\in\Omega. \tag{74}$$

The proof is written in Appendix for the reader's convenience, since it essentially goes as the one of [19, Theorem 4.7].

Remark 2. We emphasize that (74) implies the full $H^2(\Omega)$-regularity on u (see Proposition 2).

Remark 3. Under assumptions which are similar to (73) above, the well-posedness of standard solutions to (2) has already been studied in [2].

6. **Existence of finite-dimensional attractors.** In this section, we introduce the space $\Psi = L^2(\Omega)\times L^2(\Gamma)\times L^2(\Omega)$, endowed with the following norm:

$$\|(u,\psi,w)\|_\Psi^2 = \|u\|^2 + \|\psi\|_\Gamma^2 + \|u+\varepsilon w - I\|^2_{[\tilde H^1(\Omega)]'} + \varepsilon\|u+\varepsilon w\|^2 \quad (I := \langle u+\varepsilon w\rangle) \tag{75}$$

which is equivalent to the standard $\mathcal{L}^2$-norm (see (22)).

In view of the conservation law (7), we consider, for each $M > 0$, the following subset of the phase space Φ:

$$\Phi_M = \{(u, \psi, w) \in \Phi : \quad |\langle u + \varepsilon w \rangle| \le M \}.$$

Lemma 6.1. *Problem (2) generates a solution semigroup $S(t) : \Phi_M \to \Phi_M$, where $S(t)z_0 = z(t)$ is the unique variational solution to (2) departing from z_0. Furthermore, this semigroup is Lipschitz continuous in the Ψ-topology,*

$$\begin{aligned} &\|S(t)z_1 - S(t)z_2\|_\Psi^2 \qquad (76)\\ &\quad + \int_t^{t+1} [\|\bar{u}(s)\|_{H^1(\Omega)}^2 + \|\bar{\psi}(s)\|_{H^1(\Gamma)}^2 + \|\bar{w}(s) - \langle \bar{w}(s) \rangle\|_{H^1(\Omega)}^2] ds \\ &\le c e^{ct} \|z_1 - z_2\|_\Psi^2, \quad t \ge 0, \end{aligned}$$

where $(\bar{u}(t), \bar{\psi}(t), \bar{w}(t)) = z^1(t) - z^2(t)$, for any $z_i = (u_i, \psi_i, w_i) \in \Phi_M$, $i = 1, 2$. Here, the positive constant c depends on M, but not on t.

This lemma is a direct consequence of Theorem 4.3.

Moreover, the semigroup $S(t)$ is dissipative since, by Lemma 3.5 (which also holds for the variational solutions), we infer the existence of $R_0 = R_0(M) > 0$ such that $B_{\mathcal{H}^1}(R_0)$, i.e., the $\mathcal{H}^1$-ball centered at zero with radius R_0, is absorbing in Φ_M and compact in the Ψ-topology. In particular, there exists a time $t_0 \ge 1$ such that $S(t)B_{\mathcal{H}^1}(R_0) \subset B_{\mathcal{H}^1}(R_0)$, for any $t \ge t_0$. As a consequence, the set

$$\mathbb{B}_0 := \overline{\cup_{t \ge t_0} S(t) B_{\mathcal{H}^1}(R_0)}^\Psi$$

is absorbing and positively invariant. Thus, we have the

Lemma 6.2. *The semigroup $(S(t), \Phi_M)$, associated with the variational solutions to problem (2), possesses the global attractor $\mathcal{A}(M)$ which is bounded in $C^\alpha(\Omega) \times C^\alpha(\Gamma) \times C^\alpha(\Omega)$, for some positive $\alpha < 1/4$.*

The finite fractal dimensionality of the global attractor then follows from the next (see, e.g., [18])

Theorem 6.3. *For each $M \in \mathbb{R}^+$, the semigroup $S(t)$ possesses a $(\Phi_M, L^\infty(\Omega) \times L^\infty(\Gamma) \times L^\infty(\Omega))$-exponential attractor $\mathcal{E}(M) \subset \Phi_M$ which is bounded in $C^\alpha(\Omega) \times C^\alpha(\Gamma) \times C^\alpha(\Omega)$, for some positive $\alpha < 1/4$.*

We first construct a $(\Phi_M, \Psi)-$ exponential attractor, recovering the natural topology of Φ_M by interpolation, thanks to the Hölder continuity (cf. [19, Proof of Theorem 5.2]).

First, there exists a positive constant $R = R(M, R_0)$ such that

$$\|u(t)\|_{C^\alpha(\Omega \times [t,t+1])}^2 + \|u(t)\|_{H^2(\Gamma)}^2 + \|w(t)\|_{H^2(\Omega)}^2 + \|f(u(t))\|_{L^1(\Omega)} + \|\partial_t z\|_{L^2([t,t+1], \mathcal{H}^1)}^2 \le R,$$

for any initial datum in $\mathbb{B}_0$. Moreover, by definition of $\mathbb{B}_0$, there holds $u|_\Gamma = \psi$.

For any fixed $z_0 = (u_0, \psi_0, w_0) \in \mathbb{B}_0$, let $\theta \in C^\infty(\mathbb{R}^3, [0, 1])$ be a smooth cut-off function such that

$$\theta(x) = \begin{cases} 0, & x \in \overline{\Omega}_\delta(z_0), \\ 1, & x \in \Omega_{2\delta}(z_0), \end{cases}$$

where $\delta > 0$ is a sufficiently small parameter and

$$\begin{aligned} \Omega_\delta(z_0) &= \{x \in \Omega : \quad |u_0(x)| < 1 - \delta\}, \\ \overline{\Omega}_\delta(z_0) &= \{x \in \Omega : \quad |u_0(x)| > 1 - \delta\}. \end{aligned}$$

Furthermore, θ satisfies, together with its derivatives,

$$\|\theta\|_{C^k(\mathbb{R}^3)} \leq c_k, \tag{77}$$

for any $k \in \mathbb{N}$, where the constants c_k only depend on δ and on the structural data of the problem. The existence of such a cut-off function is ensured by the uniform Hölder continuity of u_0 in Ω, giving the strict separation between $\partial\Omega_{\delta_1}$ and $\partial\Omega_{\delta_2}$, for any $\delta_1 \neq \delta_2$. Here, we dropped z_0 for the sake of brevity.

We denote by $B_\Psi(z_0, \rho)$ the ball in the space $\mathbb{B}_0$, endowed with the metric of Ψ, centered at z_0 with radius $\rho > 0$. We then define $\mathbb{K}_{z_0}$ as the operator

$$\begin{aligned} \mathbb{K}_{z_0} : B_\Psi(z_0, \rho) &\to \mathcal{L}^2 \\ z &\mapsto (\theta u(\cdot), 0, u(\cdot) + \varepsilon w(\cdot)), \end{aligned}$$

where $(u(\cdot), \psi(\cdot), w(\cdot))$ is the variational solution departing from z.

Lemma 6.4. *For any fixed $z_0 \in \mathbb{B}_0$, there exist $\delta > 0$, $T = T(\delta) > 0$, $\beta(\delta) > 0$ and $\rho_0 = \rho_0(\delta) \in (0, 1]$ such that*

$$\|S(T)z_1 - S(T)z_2\|_\Psi^2 \leq e^{-\beta T}\|z_1 - z_2\|_\Psi^2 + c\int_0^T \|\mathbb{K}_{z_0}(z_1) - \mathbb{K}_{z_0}(z_2)\|_{\mathcal{L}^2}^2 dt,$$

where $S(t)z_i = (u^i(t), \psi^i(t), w^i(t))$, $i = 1, 2$, for any $z_1, z_2 \in B_\Psi(z_0, \rho)$, for any $\rho \in (0, \rho_0]$, and the positive constants β and c are independent of the concrete choice of z_1, z_2 and z_0.

Proof. Having fixed $\delta > 0$, our first aim is to find $T(\delta) > 0$ and $\rho_0 = \rho_0(\delta) \in (0, 1]$ such that

$$\begin{cases} |u(x,t)| \leq 1 - \delta/4, & x \in \Omega_\delta(z_0), \quad t \in [0, T], \\ |u(x,t)| \geq 1 - 4\delta, & x \in \overline{\Omega}_{2\delta}(z_0), \quad t \in [0, T], \end{cases} \tag{78}$$

where $S(t)z = (u(t), \psi(t), w(t))$, for any $z \in B_\Psi(z_0, \rho), \forall \rho \leq \rho_0$. This is possible since, setting $S(t)z_0 = (u^0(t), \psi^0(t), w^0(t))$, the uniform Hölder continuity of $u^0(t)$ in space and time allows to select $T = T(\delta) = \mathcal{O}(\delta^{1/\alpha})$, for any $\delta > 0$, such that

$$\begin{cases} |u^0(x,t)| \leq 1 - \delta/2, & x \in \Omega_\delta(z_0), \quad t \in [0, T], \\ |u^0(x,t)| \geq 1 - 3\delta, & x \in \overline{\Omega}_{2\delta}(z_0), \quad t \in [0, T]. \end{cases} \tag{79}$$

Furthermore, from this very same continuity, we infer

$$\|u^1(t) - u^2(t)\|_{C(\Omega)} \leq C\|u^1(t) - u^2(t)\|^\kappa \|u^1(t) - u^2(t)\|_{C^\alpha(\Omega)}^{1-\kappa} \leq C_T \rho^\kappa,$$

having set $S(t)z_i = (u^i(t), \psi^i(t), w^i(t))$, $i = 1, 2$, for any $z_1, z_2 \in B_\Psi(z_0, \rho)$. Thus, we can take $\rho_0 > 0$ small enough such that, for any $\rho \in (0, \rho_0]$, for any $z \in B_\Psi(z_0, \rho)$, the first component of $S(t)z = (u(t), \psi(t), w(t))$ satisfies (78). Actually, the final purpose of this argument is to select δ small enough to ensure that

$$f'(u(x,t)) \geq \Lambda(\delta), \qquad x \in \overline{\Omega}_{2\delta}(z_0), \quad t \in [0, T], \tag{80}$$

for any $z \in B_\Psi(z_0, \rho)$, having set $\Lambda(\delta) := \min\{f'(1-4\delta), f'(-1+4\delta)\}$. Assumption $(3)_3$ allows to make Λ as large as we want, provided that we take δ close enough to zero.

Having these preliminary results at our disposal, in order to formally prove the estimate in Lemma 6.4, we consider the difference equations. Adopting the notation

$$l(t) := \int_0^1 f'(su^1(t) + (1-s)u^2(t))ds \quad \text{and} \quad m(t) := \int_0^1 g_0'(s\psi^1(t) + (1-s)\psi^2(t))ds,$$

the difference $(\bar u(t),\bar\psi(t),\bar w(t)) = (u^1(t)-u^2(t),\psi^1(t)-\psi^2(t),w^1(t)-w^2(t))$ solves the following problem:

$$\begin{cases} \partial_t(\bar u+\varepsilon\bar w)-\Delta(\bar w-\langle\bar w\rangle)=0, \quad \text{in} \quad \Omega,\\ \partial_t\bar u-\Delta\bar u+l(t)\bar u-\lambda\bar u=\bar w, \quad \text{in} \quad \Omega,\\ \partial_t\bar\psi-\Delta_\Gamma\bar\psi+\bar\psi+m(t)\bar\psi=-\partial_n\bar u, \quad \text{on} \quad \Gamma, \end{cases} \tag{81}$$

supplemented with the conservation law $\langle\bar u(t)+\varepsilon\bar w(t)\rangle = I_1-I_2 := I$ and with obvious initial and boundary conditions.

Since the first equation of (81) is linear, arguing exactly as in the proof of Lemma 3.2, we obtain

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{dt}}(\|\bar u+\varepsilon\bar w-I\|^2_{[\tilde H^1(\Omega)]'}+\varepsilon\|\bar u+\varepsilon\bar w\|^2)+2\varepsilon\|\bar w-\langle\bar w\rangle\|^2+2\varepsilon^2\|\nabla\bar w\|^2\\ &=-2\langle\bar w,\bar u\rangle+2|\Omega|\langle\bar w\rangle\langle\bar u\rangle-2\varepsilon\langle\nabla\bar w,\nabla\bar u\rangle. \end{aligned}$$

Next, the product of the second and the third equations by $\bar u$ and $\bar\psi$, respectively, leads to

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{dt}}(\|\bar u\|^2+\|\bar\psi\|^2_\Gamma)+2\|\nabla\bar u\|^2+2\|\bar\psi\|^2_{H^1(\Gamma)}+2\langle l(t)\bar u,\bar u\rangle\\ &=2\lambda\|\bar u\|^2+2\langle\bar w,\bar u\rangle-2\langle m(t)\bar\psi,\bar\psi\rangle_\Gamma. \end{aligned} \tag{82}$$

Thus, summing the last two equations and introducing the energy functional

$$\begin{aligned} W(t)&=\|S(t)z_1-S(t)z_2\|^2_\Psi\\ &=\|\bar u(t)\|^2+\|\bar\psi(t)\|^2_\Gamma+\|\bar u(t)+\varepsilon\bar w(t)-I\|^2_{[\tilde H^1(\Omega)]'}+\varepsilon\|\bar u(t)+\varepsilon\bar w(t)\|^2, \end{aligned}$$

we have

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{dt}}W+2\|\nabla\bar u\|^2+2\|\bar\psi\|^2_{H^1(\Gamma)}+2\varepsilon\|\bar w-\langle\bar w\rangle\|^2+2\varepsilon^2\|\nabla\bar w\|^2+2\langle l(t)\bar u,\bar u\rangle\\ &=2|\Omega|\langle\bar w\rangle\langle\bar u\rangle-2\varepsilon\langle\nabla\bar w,\nabla\bar u\rangle+2\lambda\|\bar u\|^2-2\langle m(t)\bar\psi,\bar\psi\rangle_\Gamma. \end{aligned}$$

Thanks to (80) and $(3)_3$, by definition of θ,

$$\begin{aligned} \langle l(t)\bar u,\bar u\rangle &\geq \int_{\overline\Omega_{2\delta}} l(x,t)|\bar u(x,t)|^2dx \geq \Lambda\|\bar u\|^2_{L^2(\overline\Omega_{2\delta})}=\Lambda\|\bar u\|^2_{L^2(\Omega)}-\Lambda\|\bar u\|^2_{L^2(\Omega_{2\delta})}\\ &\geq\Lambda\|\bar u\|^2-\Lambda\|\theta\bar u\|^2. \end{aligned}$$

In order to suitably control the first term on the right-hand side of the above differential equality, we exploit (8), whence

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{dt}}W+2\|\nabla\bar u\|^2+2\|\bar\psi\|^2_{H^1(\Gamma)}+2(\Lambda-\lambda)\|\bar u\|^2+2\varepsilon\|\bar w-\langle\bar w\rangle\|^2+2\varepsilon^2\|\nabla\bar w\|^2\\ &\leq\|\nabla\bar u\|^2+\varepsilon^2\|\nabla\bar w\|^2-2\langle m(t)\bar\psi,\bar\psi\rangle_\Gamma+2\Lambda\|\theta\bar u\|^2+\frac{1}{\varepsilon}\|\bar u+\varepsilon\bar w\|^2. \end{aligned}$$

The third term in the right-hand side can be controlled, thanks to the assumptions on g_0 and by the trace interpolation inequality, namely,

$$2\langle m(t)\bar\psi,\bar\psi\rangle_\Gamma\leq 2C_0\|\bar\psi\|^2_\Gamma\leq 2C_0c\|\bar u\|_{H^1(\Omega)}\|\bar u\|\leq\frac{C_0c}{\sqrt{\Lambda-\lambda}}\|\bar u\|^2_{H^1(\Omega)}+C_0c\sqrt{\Lambda-\lambda}\|\bar u\|^2,$$

where C_0 is the constant defined in (5), since we can take Λ large enough to ensure that $\Lambda-\lambda>0$ and, actually, as large as we want. Replacing these computations in

the differential inequality, we obtain

$$\frac{\mathrm{d}}{\mathrm{dt}}W + \|\nabla\bar{u}\|^2 + 2\|\bar{\psi}\|^2_{H^1(\Gamma)} + 2(\Lambda-\lambda)\|\bar{u}\|^2 + 2\varepsilon\|\bar{w}-\langle\bar{w}\rangle\|^2 + \varepsilon^2\|\nabla\bar{w}\|^2$$
$$\leq \frac{C_0 c}{\sqrt{\Lambda-\lambda}}\|\bar{u}\|^2_{H^1(\Omega)} + C_0 c\sqrt{\Lambda-\lambda}\|\bar{u}\|^2 + 2\Lambda\|\theta\bar{u}\|^2 + \frac{1}{\varepsilon}\|\bar{u}+\varepsilon\bar{w}\|^2.$$

Moreover, there holds, for some $\omega \in (0,2)$,

$$\|\nabla\bar{u}\|^2 + 2\|\bar{\psi}\|^2_{H^1(\Gamma)} \geq \omega\|\bar{u}\|^2_{H^1(\Omega)} + \omega\|\bar{\psi}\|^2_{H^1(\Gamma)},$$

leading to

$$\frac{\mathrm{d}}{\mathrm{dt}}W + \omega\|\bar{u}\|^2_{H^1(\Omega)} + \omega\|\bar{\psi}\|^2_{H^1(\Gamma)} + 2(\Lambda-\lambda)\|\bar{u}\|^2 + 2\varepsilon\|\bar{w}-\langle\bar{w}\rangle\|^2 + \varepsilon^2\|\nabla\bar{w}\|^2$$
$$\leq \frac{C_0 c}{\sqrt{\Lambda-\lambda}}\|\bar{u}\|^2_{H^1(\Omega)} + C_0 c\sqrt{\Lambda-\lambda}\|\bar{u}\|^2 + 2\Lambda\|\theta\bar{u}\|^2 + \frac{1}{\varepsilon}\|\bar{u}+\varepsilon\bar{w}\|^2.$$

By (11), we have the control

$$\|\bar{u}+\varepsilon\bar{w}-I\|^2_{[\tilde{H}^1(\Omega)]'} \leq c_\Omega\|\bar{u}+\varepsilon\bar{w}-I\|^2 \leq 2c_\Omega(\|\bar{u}\|^2 + \varepsilon^2\|\bar{w}-\langle\bar{w}\rangle\|^2).$$

Thus, taking $\beta > 0$ small enough and possibly reducing δ, we can require that

$$C_0 c \leq \omega\frac{\sqrt{\Lambda-\lambda}}{2}.$$

Then, simplifying the last differential inequality, we are led to

$$\frac{\mathrm{d}}{\mathrm{dt}}W + \beta W \leq c(\|\bar{u}+\varepsilon\bar{w}\|^2 + \|\theta\bar{u}\|^2).$$

Applying the Gronwall lemma, we finally obtain

$$W(t) \leq e^{-\beta t}W(0) + c\int_0^t e^{-\beta(t-s)}[\|\bar{u}(s)+\varepsilon\bar{w}(s)\|^2 + \|\theta\bar{u}(s)\|^2]ds$$
$$\leq e^{-\beta t}W(0) + c\int_0^t \|\mathbb{K}_{z_0}(z_1) - \mathbb{K}_{z_0}(z_2)\|^2_{\mathcal{L}^2}ds,$$

which yields the thesis, by definition of W. □

Lemma 6.5. *There exists a positive constant c, independent of the concrete choice of z_0, and $z_i \in B_\Psi(z_0,\rho)$, $i=1,2$, such that*

$$\|\partial_t(\theta\bar{u})\|_{L^2(0,T;[H^1(\Omega)]^*)} + \|\theta\bar{u}\|_{L^2(0,T;H^1(\Omega))} \leq ce^{cT}\|z_1 - z_2\|_\Psi,$$

where $(\bar{u}(t),\bar{\psi}(t),\bar{w}(t)) = S(t)z_1 - S(t)z_2$.

Proof. The proof goes as the one of [19, Lemma 5.1], that is, the control on the latter norm is a straightforward consequence of (76) and (77). Concerning the former one, we can prove the boundedness of the functional by testing the second equation in (81) by $\theta\varphi$, for any $\varphi \in C_0^\infty(\Omega)$. Since $\operatorname{supp}\theta \subset \Omega_\delta(z_0)$, (78) ensures that $|\langle l(t)\bar{u}, \theta\varphi\rangle| \leq C\|\bar{u}\|\|\varphi\|$ and there holds

$$\langle\partial_t(\theta\bar{u}),\varphi\rangle = \langle\partial_t\bar{u},\theta\varphi\rangle = -\langle\nabla\bar{u},\nabla(\theta\varphi)\rangle - \langle l(t)\bar{u},\theta\varphi\rangle + \lambda\langle\bar{u},\theta\varphi\rangle + \langle\bar{w},\theta\varphi\rangle$$
$$\leq c(\|\bar{u}\|_{H^1(\Omega)} + \|\bar{w}\|)\|\varphi\|_{H^1(\Omega)},$$

which implies

$$\|\partial_t(\theta\bar{u}(t))\|_{[H^1(\Omega)]^*} \leq c(\|\bar{u}(t)\|_{H^1(\Omega)} + \|\bar{w}(t)\|) \leq c(\|\bar{u}(t)\|_{H^1(\Omega)} + \|\bar{u}(t)+\varepsilon\bar{w}(t)\|),$$

whence the desired inequality, thanks to (76) (and Theorem 4.1). □

Proof of Theorem 6.3. Having fixed an arbitrary $z_0 \in \mathbb{B}_0$, for δ, $T > 0$ and $\rho_0 > 0$ as in Lemma 6.4, we introduce the spaces

$$\mathbb{H}_1 = L^2([0,T], \mathcal{H}^1) \cap H^1([0,T], (\tilde{\mathcal{H}}^1)') \Subset \mathbb{H} = L^2([0,T], \mathcal{L}^2).$$

Thus, we prove that $(S(t), \mathbb{B}_0)$ admits an exponential attractor $\mathcal{E}(M)$ by exploiting the ℓ-trajectories method [15] (see also [19]). Roughly speaking, we aim to show that, for any $\rho \in (0, \rho_0)$, the difference of solutions departing from $B_\Psi(z_0, \rho)$ can be decomposed into the sum of a contraction and a smoothing map.

According to Lemma 6.4, for any $\rho \in (0, \rho_0)$, for any $z_1, z_2 \in B_\Psi(z_0, \rho)$, there holds

$$\|S(T)z_1 - S(T)z_2\|_\Psi \leq \gamma \|z_1 - z_2\|_\Psi + c\|\mathbb{K}_{z_0}(z_1) - \mathbb{K}_{z_0}(z_2)\|_{\mathbb{H}}, \tag{83}$$

for some $\gamma \in (0,1)$. Here, by (62) and Lemma 6.5, the map $\mathbb{K}_{z_0}$ satisfies

$$\|\mathbb{K}_{z_0}(z_1) - \mathbb{K}_{z_0}(z_2)\|_{\mathbb{H}^1} \leq c\|z_1 - z_2\|_\Psi. \tag{84}$$

Thus, exploiting the ℓ-trajectories method as in [19], we deduce the existence of a $(\Phi_M, \Psi)-$exponential attractor $\mathcal{E}(M)$, with basin of attraction $\mathbb{B}_0$. In particular, $\mathcal{E}(M)$ is bounded in $C^\alpha(\Omega) \times C^\alpha(\Gamma) \times C^\alpha(\Omega)$. Thus, as in [19], by interpolation (between Ψ and $C^\alpha(\Omega) \times C^\alpha(\Gamma) \times C^\alpha(\Omega)$), we see that $\mathcal{E}(M)$ has finite dimension and exponentially attracts the bounded sets of Φ_M in its natural topology $L^\infty(\Omega) \times L^\infty(\Gamma) \times L^\infty(\Omega)$. Actually, due to (76) and the properties of $\mathbb{B}_0$, the transitivity of exponential attraction devised in [6] applies, so that the basin of attraction extends to the whole phase space. □

7. **Numerical results.** As far as the numerical simulations are concerned, we use a P_1 finite element approach for the space discretization, together with a semi-implicit Euler time discretization (i.e., implicit for the linear terms and explicit for the nonlinear ones).

The numerical simulations are performed with the software Freefem++ [10]. In the numerical results presented below, Ω is a $(0,10) \times (0,4)$-rectangle and (2) is endowed with periodic boundary conditions for u and w in the first direction and Neumann for w and dynamic for u boundary conditions in the second one. We further take $\varepsilon = 0.3$, $\tilde{f}(u) = -3u + \ln(\frac{1+u}{1-u})$ and g affine. Finally, the initial value consists of uniformly distributed random fluctuations of amplitude ± 0.5.

The first two pictures below represent the isovalues of the solution u for different choices of the function g. In the first figure, we take $g(u) = u - 0.8$, so that the sign conditions are satisfied. As proved in section 5, the solution u stays away from the singularities ± 1, for every time, and is thus a classical solution. At time $t = 20$, u and w have almost converged to a steady state (for w, the steady state is a constant). In Figures 2 and 3, the sign conditions are not satisfied. However, the solution still is classical in Figure 2, where $g(u) = u - 1.5$. On the contrary, in Figure 3a, where $g(u) = u - 3$, u reaches the singular value 1 on the boundaries corresponding to dynamic boundary conditions at time $t = 0.82$ and the simulation stops. Figure 3.b represents the isovalues of w at time $t = 0.82$. This shows that boundary singularities can appear and suggests nonexistence of classical solutions. Similar simulations were performed in [5] to illustrate nonexistence of classical solutions for the Cahn-Hilliard equation with singular potentials and dynamic boundary conditions.

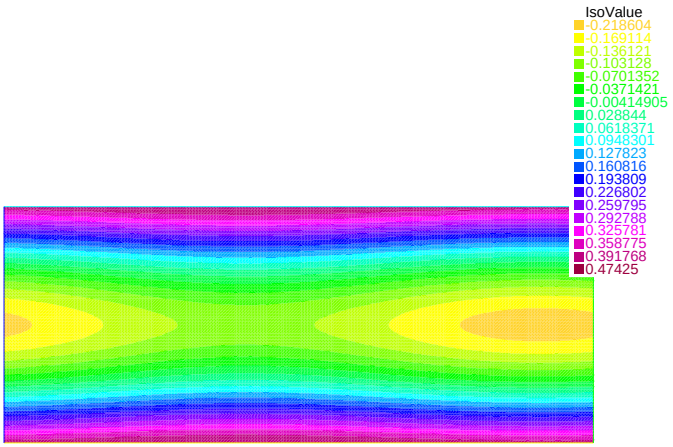

FIGURE 1. Isovalues of the solution u, at time $t = 20$, when $g(s) = s - 0.8$.

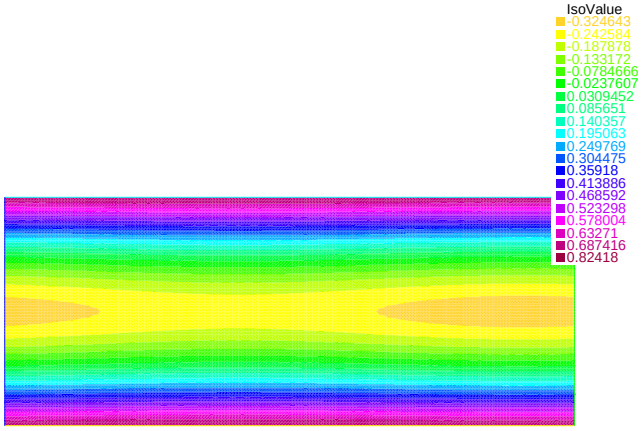

FIGURE 2. Isovalues of the solution u, at time $t = 20$, when $g(s) = s - 1.5$.

8. **Appendix.**

Here, we write [19, Theorem 6.1], for the reader's convenience. In this statement, $F(s) = \int_0^s f(r)dr$.

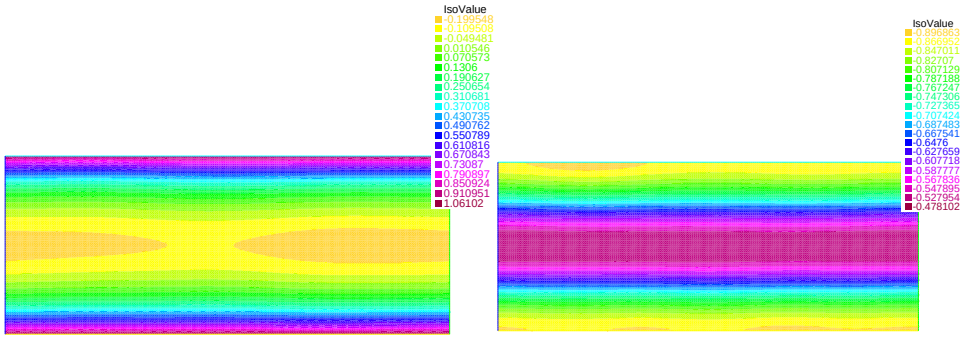

FIGURE 3. Isovalues of the solutions u (left) and w (right), at time $t = 0.82$, when $g(s) = s - 3$.

Theorem 8.1. *We assume that $\tilde{h}_1 \in L^2(\Omega)$ and $\tilde{h}_2 \in L^2(\Gamma)$. Then, the solution (u, ψ) to the problem*

$$\begin{cases} -\Delta u + f(u) + u = \tilde{h}_1, & \text{in} \quad \Omega, \\ -\Delta_\Gamma \psi + \psi + \partial_n u = \tilde{h}_2, & \text{on} \quad \Gamma, \end{cases} \tag{85}$$

is such that, for some $\alpha \in (0, 1/4)$, $u \in C^\alpha(\Omega) \otimes H^2(\Gamma)$, $F(u) \in L^1(\Gamma)$, $\nabla D_\tau u \in [L^2(\Omega)]^6$, $u \in H^2(\Omega_\delta)$, where $\Omega_\delta = \{x \in \Omega : \quad \text{dist}(x, \Gamma) > \delta\}$, for every $\delta > 0$. Moreover, u satisfies

$$\begin{aligned} &\|u\|^2_{H^1(\Omega)} + \|u\|^2_{C^\alpha(\Omega)} + \|\nabla D_\tau u\|^2 + \|u\|^2_{H^2(\Omega_\delta)} \\ &\quad + \|u\|^2_{H^2(\Gamma)} + \|f(u)\|_{L^1(\Omega)} + \|F(u)\|_{L^1(\Gamma)} \\ &\leq c(1 + \|\tilde{h}_1\|^2_{L^2(\Omega)} + \|\tilde{h}_2\|^2_{L^2(\Gamma)}), \end{aligned} \tag{86}$$

for some positive constant c which only depends on δ and the structural parameters of the problem. In particular, c is independent of $\tilde{h}_1$ and $\tilde{h}_2$.

The proof is based on localization techniques and the continuous embeddings $H^2(\Omega_\epsilon) \subset C^\alpha(\Omega_\epsilon)$, for $\alpha < 1/2$, and $L^2(\mathbb{R}, H^2(\mathbb{R}^2)) \cap H^1(\mathbb{R}, H^1(\mathbb{R}^2)) \subset C^\alpha(\mathbb{R}^3)$, for $\alpha < 1/4$.

For the sake of clarity, we recall [19, Proposition 4.1].

Proposition 2. *Let (u, ψ, w) be a variational solution to (2). For any $\epsilon, T > 0$, we set*

$$\Omega_\epsilon(T) = \{x \in \Omega : \quad |u(x, T)| < 1 - \epsilon\}.$$

Then, $u(T) \in H^2(\Omega_\epsilon(T))$ and there holds

$$\|u(T)\|_{H^2(\Omega_\epsilon(T))} \leq Q_{\epsilon,T},$$

where the positive constant $Q_{\epsilon,T}$ only depends on ϵ and T, but is independent of the concrete choice of the solution.

The main ingredients of the proof are the Hölder continuity of u and a localization technique, together with regularity results for linear elliptic problems.

We now give the proof of Theorem 5.3 which relies on the following lemma, whose rationale is slightly different from the one of [19, Lemma 4.8].

Lemma 8.2. *We assume that (73) holds. Then, given any variational solution (u,ψ,w), there holds $f(u)\in L^q(\Omega\times[t,t+1])$, for any $q>1$ and $t>0$. Moreover, for any fixed $q>1$ and $T>0$, there exists $c_{T,q}>0$ (which is independent of u and t) such that*

$$\|f(u)\|_{L^q(\Omega\times[t,\,t+1])}\le c_{T,q},\qquad \forall t\ge T.$$

Proof. We rewrite the second and third equations of the problem as

$$\begin{cases}\partial_t u-\Delta u+f(u)+u=h_1:=(1+\lambda)u+w, & \text{in}\quad \Omega,\\ \partial_t u-\Delta_\Gamma u+u+\partial_n u=h_2:=-g_0(u), & \text{on}\quad \Gamma,\end{cases}$$

where

$$\|h_1(t)\|_{L^3(\Omega)}+\|h_2(t)\|_{L^\infty(\Gamma)}=\|(1+\lambda)u(t)+w(t)\|_{L^3(\Omega)}+\|g_0(u(t))\|_{L^\infty(\Gamma)}\le c_T,$$

thanks to Theorem 4.3.

Then, we multiply the first equation by $u\varphi(u)^{n+1}$, where

$$\varphi(u):=\frac{1}{1-u^2}\tag{87}$$

and $n>1$ is an arbitrary integer, and we integrate over Ω. After simplifications, see [19, Lemma 4.8], we obtain

$$\frac{d}{dt}[\|\varphi(u)\|^n_{L^n(\Omega)\otimes L^n(\Gamma)}]+\kappa\|\varphi(u)\|^{n+2}_{L^{n+2}(\Omega)}+\kappa'\|\varphi(u)\|^{n+1}_{L^{n+1}(\Gamma)}\le C_{T,n},$$

where $C_{T,n}$ is independent of time, due to the uniform bounds on h_1 and h_2, but, of course, depends on n and T. The product by $(t-T)^{n+1}$, followed by an integration over $(T,T+2)$, gives

$$\int_T^{T+2}(t-T)^{n+1}\|\varphi(u(t))\|^{n+2}_{L^{n+2}(\Omega)}dt\le C_{T,n},$$

which, thanks to the second inequality of (73) and the arbitrariness of n, allows to conclude. □

Proof of Theorem 5.3. By (60) and a proper Sobolev embedding theorem, we obtain

$$\|u(t)\|_{W^{2-1/3,3}(\Gamma)}\le c\|u(t)\|_{H^2(\Gamma)}\le c_T,\quad t\ge T.\tag{88}$$

Furthermore, we find, owing to the above lemma and the second equation of (2),

$$\|\Delta u\|_{L^q([t,t+1],L^3(\Omega))}\le c_T,$$

for any $q\ge1$. Thus, by the maximal regularity of the Laplacian in L^3 and a further application of a Sobolev embedding theorem, it follows that

$$\|\nabla u\|_{L^q(\Omega\times[t,t+1])}\le c_{T,q},\quad t\ge T,$$

again for an arbitrary $q\ge1$. This bound, together with (60) and Lemma 8.2, yields

$$\begin{cases}\|\varphi(u)\|_{L^r([t,t+1],W^{1,r}(\Omega))}\le c_{T,r}, & t\ge T,\\ \|\partial_t\varphi(u)\|_{L^{2-\eta}([t,t+1],L^{6-\eta}(\Omega))}\le c_{T,\eta}, & t\ge T,\end{cases}\tag{89}$$

where $\eta>0$, $r>0$, φ is defined as in (87) and both constants $c_{T,r}$ and $c_{T,\eta}$ are independent of t and u. Then, provided that r is large enough and η is close enough to zero, we have

$$W^{1-\eta}(\Omega\times[t,t+1])\cap L^r([t,t+1],\,W^{1,r}(\Omega))\subset C(\Omega\times[t,t+1])$$

and, from (89), it follows that

$$\sup_{(x,s)\in\Omega\times[t,t+1]}\left|\frac{1}{1-u^2(x,s)}\right| = \|\varphi(u)\|_{C(\Omega\times[t,t+1])} \le c_T, \quad t \ge T,$$

which leads to (74). □

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