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DGS Cui-Rods: Reinventing Mathematical Concepts

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ABSTRACT

Curriculum materials are instructional materials produced to be used for teaching and learning. Successful teaching requires new materials and innovative approaches. In the present study, I shall present DGS Cui-rods an instructional material for effective teaching and learning of mathematical concepts. I created them in the Geometer's Sketchpad environment. Dynamic geometry environments allow students to discover and reinvent a wide range of mathematical concepts and their relationships. During problem-solving situations, students are able to construct meanings. They are led to create their personal representations of mathematical concepts and transform them. The design of activities in the learning environment as a part of the instruction thus has a crucial role to play. In the next sections, I shall describe how learning through DGS Cui-Ros affects students' cognitive structure's transformations and consequently their cognitive growth.

KEYWORDS

Dynamic Reinvention; Curriculum materials; DGS Cui-rods; Problem-solving.

1. Introduction: Learning as a 'dynamic' reinvention

The term 'curriculum materials' is used by many researchers (e.g., Remillard, 1999, 2005). These are instructional materials produced to be used for the teaching and learning. Freudenthal emphasized that "*curriculum theory is not a fixed, pre-stated set of theories, aims and means, contents, and methods*." (Gravemeijer & Terwel, 2000, p.779). Remillard (2005) also refers on "how teachers interact, draw on, refer to and are influenced by teaching materials designed to guide instruction" (p. 212)". This is in accordance with Freudenthal's proposed educational development of mathematics, his own alternative to curriculum development which centres on the development of curriculum materials, and seeks to foster actual change in classroom teaching (Gravemeijer & Terwel, 2000, p.779). Remillard (1999) also reports on curriculum materials

> "*Regardless of how teachers draw on and use curriculum materials, their work in relation to planning and teaching mathematics can be viewed as curriculum development—the processes by which teachers develop curricular plans and ideals and translate them into classroom events. Through the curriculum development process, teachers plan and shape students' experiences in the classroom. The term "curriculum development" is often used to describe the writing of curriculum materials. In referring to teachers as curriculum developers, I suggest that the curriculum development process does not stop when textbooks are printed, but continues in the classroom*" (p.319).

As viewed by Freudenthal, "*mathematics was first and foremost a human activity, […] a process*". (Gravemeijer & Terwel, 2000, p.780). Freudenthal (1973) supports that mathematics education should be a process of *guided reinvention*. Guided reinvention for Freudenthal means a faithful reproduction of a scientific activity by the student, and is thus an elaboration on the Socratic Method ("maieftiki" in Greek). The teachers' task is to design a course "of action that fits anticipated student *reactions. More precisely, the idea is that teaching matter is re-invented by students in such interaction*" (Gravemeijer & Terwel, 2000, p.786). Remillard (1999) also emphasizes "the substantial role that teachers play in shaping the curriculum experienced by students".

In the present paper it will be presented curriculum materials "*particularly those designed to promote curricular and pedagogical change*" (Remillard, 1999). Which is to say, constructing meaningful tasks/activities and problems for the learners by imagining how they might interact with the instructional materials, what obstacles they had to overcome, the possible (or multiple) solutions they could find, how their thinking could be raised due to the evolution of mathematical discussions they participate in. This is in accordance with what Freudenthal argues that "*doing mathematics is more important than mathematics as a ready-made product*" (Gravemeijer & Terwel, 2000, p.780). In accordance to Olive (1999), Olive & Steffe (2002), Olive & Makar (2010) the mathematical knowledge which children build up during their engagement in a mathematical activity, is distinguished among others to (a) 'children's mathematics – the mathematics that children […] construct for themselves and is available to them as they engage in mathematical activity'; (b) 'mathematics for children – the mathematical activities that curriculum developers/writers and teachers design to engage students in meaningful mathematical activity' (Olive & Makar, 2010, p.136)

Many researchers argue that working in a dynamic geometry environment allows students to reinvent their personal knowledge by interacting with the other members of the group or with the teacher (or the participating researcher). For example, Furringhetti & Paola (2003) support that "in this case, the reinvention is guided, […] by the use of the [dynamic geometry] environment". For this, I think that *dynamic reinvention* (Patsiomitou, 2012, p. 57)of knowledge is the kind of knowledge the students could reinvent by interacting with the artefacts made in a DGS environment, "*knowledge for which they themselves are responsible*" (Gravemeijer & Terwel, 2000, p.786).

Dynamic geometry systems help children develop personally meaningful ways of reasoning "*that enable them to carefully analyze spatial problems and situations*" (Battista, 2001, p. 74). Generally speaking, a computer learning environment such as a DGS scaffolds students' co-building of the meanings introduced in the teaching and learning activity. The design of activities in the learning environment as a part of the instruction thus has a crucial role to play in the comprehension of mathematical meanings. Jackiw, & Sinclair (2009) also argue that: *"[…] A Dynamic Geometry [object] is not an illustration, in other words— not an example of some more abstract, general, or encompassing idea—it is that idea and fully manifests its extent. At the same time, the dragged [object] implies a dragging intelligence. And this hidden actor, in whose hands the [object] comes alive, is the other focus of research attention" (p.414).*

2. Students' cognitive development through a dynamic reinvention process

A constructivist framework of learning considers the student as an *active participant* and learning as an *active process*. Many important philosophers and theorists (e.g., Kant, 1965; Dewey, 1938/1988; Piaget 1937/1971, 1970; von Glasersfeld, 1991, 1995;Vygotsky, 1934/1962, 1978; Skemp, 1987)gradually changed the traditional "*route by memorization*", the behaviourists' view of learning mathematics, to a sociocultural-constructivist view of learning mathematics. From an epistemological point of view, constructivism emphasizes the construction of meanings in collaboration between the instructor (or /teacher-action researcher) and the student (e.g., Hayes & Oppenheim, 1997).Students cannot directly be given knowledge or concepts; they must construct it from their own perceptions, experience, and inquiry. Fuys et al (1988) stress the importance of *instructional activities* in which students explore and discuss concepts. Language facilitates communication between the students in a group, allowing them to describe, clarify and discuss the structures they observe and re-conceptualize identified misconceptions. Students discuss how to solve problems and learning occurs in a context of collaborative, social interactions that leads to understanding (Roehler & Cantlon, 1997). Fuys et al. (1988) consider language to be a crucial factor in moving students through the hierarchy of van Hiele levels. The students in the gaps between levels are presented with *disequilibration* situations that force them to re-organize their schemes and cognitive structures. The notion of *cognitive equilibration* is borrowed from Piaget (1937), who used it to refer to an individual re-organizing his/her *schemata* when his/her experience does not fit within a conceptual structure or does not act in line with his/her expectations. Piaget believed that, to equilibrate, the individual has to modify his/her conceptual structures or schemes in order to better organize his/her experiences. The reorganization of the individual's schemata involves the sub-processes of *accommodation and assimilation,* which correspond to modifying and adding to extant schemata. Von Glasersfeld (1995, p.68) describes *equilibration* as the process "*when a scheme, instead of producing the expected result, leads to a perturbation, and perturbation, in turn, to an accommodation that maintains or re-establishes equilibrium*". Vygotsky's theory also in educational research led to studies of how children learned through collaborative interaction with adults, and it became common to use the term "scaffolding" to describe the interaction between adult and child (e.g.,Rogoff & Wertsch, 1984).

Many researchers (e.g., Artigue, 2000, 2002; Trouche, 2004; Drijvers et al., 2013) have reported on the dual interactive process involved in *instrumental genesis* (Verillon & Rabardel, 1995), a theoretical framework appropriate to describing the interactions which occur from the integration of technological tools into mathematics education. In my opinion, student learning does not work as a machine into which data, information and the principles of a learning theory are entered and the expected results come out. On the other hand, is the merging of constructivist and sociocultural perspectives a theory we can apply to instructional processes and the everyday teaching of mathematics? Can we construct *instrumental learning trajectories (Patsiomitou, 2021* a, b) to apply the principles of constructivism to student's learning? Can we view the *instrumental trajectories* or a set of learning trajectories as an evolution of the meaning of a curriculum? Certainly, the learning trajectory process allows the students to determine what the next sequential instructional activity will be, whether it is to overcome an obstacle or to form the next cognitive step in their understanding of a concrete concept.

3. The DGS Cui-Rods: learning through problem-solving

The young learners learn how to represent numbers using coloured manipulatives (e.g., Cuisenaire rods, abacuses, Dienes cubes, Montessori colour beads, fraction circles, Geoboards, pattern blocks).Manipulatives or concrete representations are objects (e.g., Cuisenaire rods) which are designed to mediate between a particular mathematical concept and the way pupils learn the concept. Pupils can manipulate them by touching or moving, and thus are concrete means (Dienes, 1960; Baroody, 1989; Van de Walle et al., 2005). Ross (2004) defines manipulatives as: "[…] materials that represent explicitly and concretely mathematical ideas that are abstract. They have visual and tactile appeal and can be manipulated by pupils through hands-on experiences" (p. 5). Generally speaking, a virtual manipulative is defined as "*an interactive, web-based visual representation of a dynamic object that presents opportunities for constructing mathematical knowledge*" (Moyer et al., 2002, p. 373).Numerous researchers present the advantages/ key benefits of using computer manipulatives, and rethink the meaning of "concrete" manipulatives (e.g., Clements & Mcmillen, 1996; Clements & Sarama, 2007).

As I was investigating for computer manipulatives in Google, I found many websites which I could use with young learners. Some researchers used dynamic geometry environments to investigate the understanding of mathematical concepts of primary school pupils (e.g., Sinclair & Crespo, 2006; Ng & Sinclair, 2015). Cuisenaire-Rods are mathematics learning aids for pupils. They provide an interactive, hands-on way to explore mathematics and learn mathematical concepts (e.g., the four basic arithmetical operations, working with fractions and finding divisors). In this study, I am presenting a DGS version of Cuisenaire rods (DGS Cui-Rods) that I created in the Geometer's Sketchpad (Jackiw, 1991/2001) DGS environment. Moreover, we shall examine the DGS Cui-rods in terms of the way in which pupils dynamically reinvent mathematical concepts through problem solving situations.

As we build models of children's mathematical tasks and activities, "*it is important to identify the used cognitive operations*" (Wheatley, & Reynolds, 1996, p. 67). Students' understanding of meanings often led me to note the sequence of steps or stages through which they gathered information from the [computing] environment as stimuli. The information from the computer environment goes through a modification, linked to students' minds stored information (or is modified in the light of the information stored in their mind) so they can answer the teacher's questions or participate in a class discussion

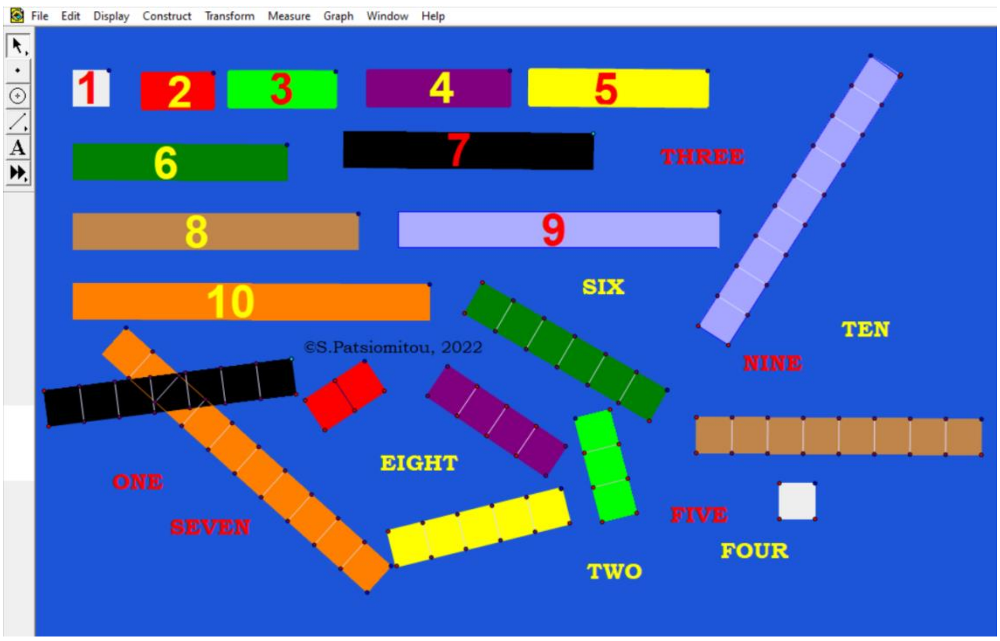

Figure 1. The DGS Cui-Rods set in one-to-one correspondence to number's verbal expression

In terms of the current study, special attention was given to the construction of abstract units from rectangular shapes which represent the DGS Cui-Rods. Particularly, I created the Cui-Rods using parameters of concrete length (Figure 1). Below, I am describing the DGS Cui-rods construction in the Geometer's Sketchpad dynamic geometry software and explain a few rules they follow:

- The rectangular shapes are constructed on the screen using the parametrical mode (i.e., using different parameters for each number). The Cui-Rod "1" is a square with sides equal to 1cm, created by the parameter "ONE= 1cm". The Cui-Rod "2" is a rectangle with sides equal to 1cm and 2cm.
- Every Cui-Rod can be moved from the "blue" point and can turn its orientation.

The even numbers have been highlighted in yellow and the odd numbers in red, allowing the student to see the colours at once and visualize a categorization of the numbers in two discrete sequences.

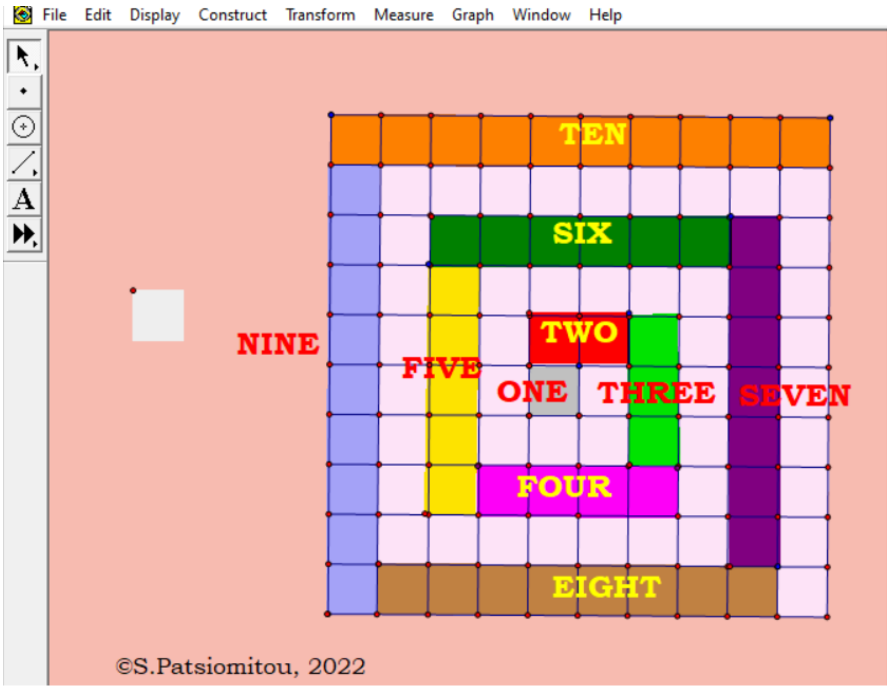

Figure 2. Problem solving with DGS Cui-Rods

Through modeling with DGS Cui-Rods young learners can recognize the structure and rules of the numeration system. Furthermore, they can develop number sense (Verschaffel et al., 2000). Problem solving is very important to teaching and learning mathematics. During problem solving situations students are able to reinvent meanings and processes. As a teacher of mathematics, I have often asked myself the following questions (Patsiomitou, 2019b, p. 151):

- During the problem-solving process, do students demonstrate significant, meaningful and appropriately organized connections between pieces of information in the problem?
- What conceptual considerations need to be taken into account when designing problems in a dynamic geometry environment? How do these conceptual considerations impact on our students' learning and understanding of mathematics?

Aamodt (1991) states that "*A mathematical problem may be structured [or divided] in sub-problems, in which case the problem-solving process may be correspondingly split into sub-processes*" (Aamodt, 1991, p.31). Learning through problem solving can be addressed by both open-ended complex geometric problems and non-open strict geometric problems, presented in a static or dynamic environment. The problem-solving process, including diagram construction, can be experienced using a "*brainstorming technique*" session, which is regarded as the most effective tools we know about creative problem-solving (e.g., Osborn, 1953). In a "brainstorming technique" session, students express/formulate what they know with the teacher helping them by introducing the concepts through essential questions, writing their ideas on the board and organizing them into a "concept map" (e.g., Novak, 1990).Clements (2000)reports among others, the following characteristics that have good mathematics problems for students (adapted from Russell, Magdalene, & Rubin, 1989; Wheatley, 1991, cited in Clements, 2000): (a) "Are meaningful to the students; (b) stimulate curiosity about a mathematical or nonmathematical domain, not just an answer; (c) engage knowledge that students already have, about mathematics or about the world, but challenges them to think harder or differently about what they know; (d) invite students to make decisions; (e) open discussion to multiple ideas and participants and (f) are amenable to continuing investigation, and generation of new problems and questions." (p. 12)

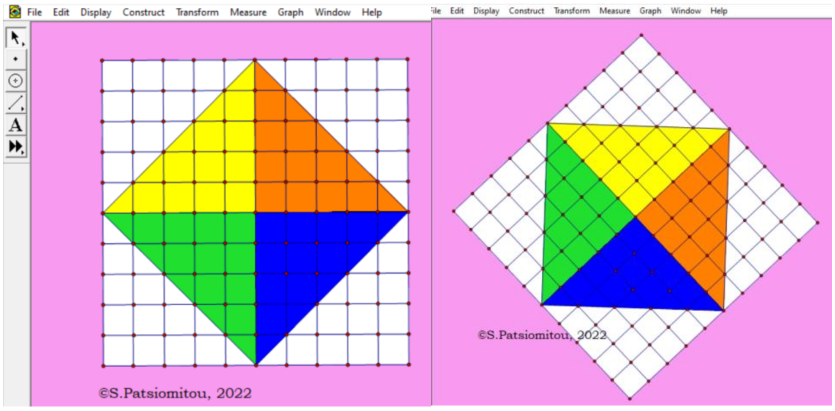

Figure 3 a,b. Transforming the shapes using a mixed 'trial and error' and 'guided reinvention" method (see also, Patsiomitou, 2019a, p. 4).

In Figure 3a,b, the problem in question is "*How can you double the area of the square?".* Many times, we ask students to experiment using transformations (e.g., using a squared paper). This will help them understand that if they double the side of the square, the area of the square this creates is quadrupled. (Figure 3).

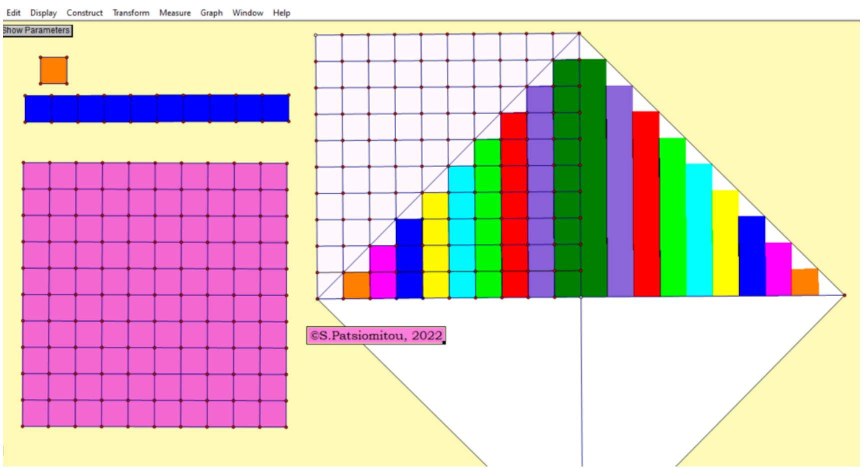

Figure 4. Combining Dienes blocks and DGS-Cui-Rods in a dynamic reinvention method for young learners

Many researchers (e.g., Cobb & Bauersfeld, 1995; Hiebert et al., 1996; quoted in Fuson et al., 2000, p.277) agree that "*in contrast to traditional textbook instruction focused primarily on rote learning and practice of skills, instruction is envisioned through which students construct meaning for the mathematical concepts* *and procedures they are investigating and engage in meaningful problem-solving activities*" (Fuson et al., 2000, p.277).

Measuring the area of a square could help pupils to reinvent multiplication as an easier way to find the area. Combining Dienes blocks and DGS Cui-Rods in elementary mathematics, children could learn geometrical concepts and develop their spatial sense as they interact with these activities (Figure 4). They are able also to reinvent relationships among mathematical objects (e.g., shapes and angles) as they explore these activities and tasks or solve problems. Mathematical problem solving in DGS diagrams activate creativity and reasoning.

4. Conclusions

"*Much has been written about the work of teaching, but relatively few efforts have been devoted to examining and conceptualizing curriculum materials*" (Remillard, 2005, p.229).How can we provide students with a "*journey of knowledge*" that is full of experiences through the educational design of the material, when this presupposes that the sequence in which the tools used are properly designed?

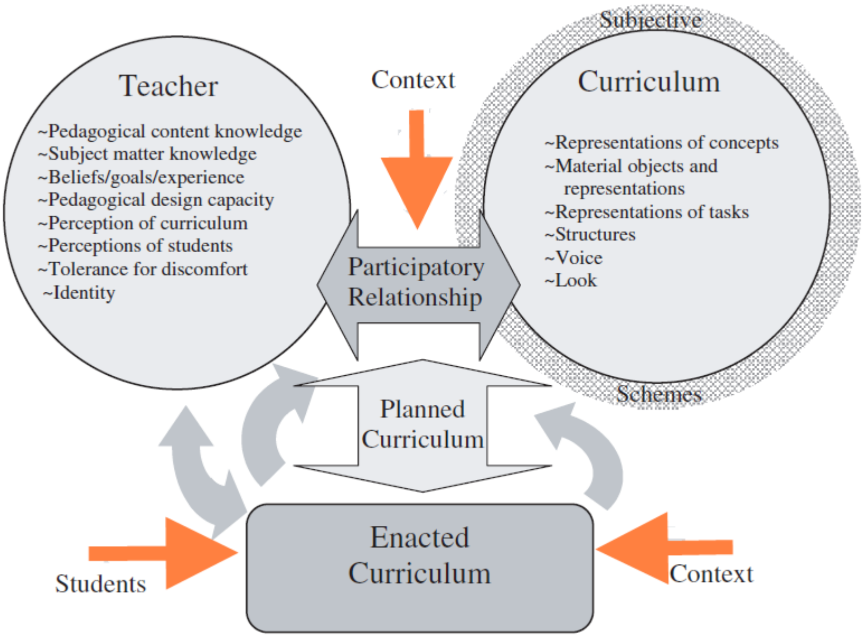

Figure 5. Framework of components of teacher–curriculum relationship (Remillard, 2005, p. 235) (adapted)

Figure 5 illustrates the different factors that impact the teacher-curriculum framework (Remillard, 2005). According to Remmilard (2005) "*The four principal constructs of the framework are (a) the teacher, (b) the curriculum,(c) the participatory relationship between them, and (d) the resulting planned and enacted curricula. Its design is grounded in two assumptions central to the previous account of teaching: that teaching involves curriculum design and that it is multifaceted.*" (p. 236).

Many students do not have the ability to dynamically visualize and mentally manipulate math-objects, which is an important skill for solving problems in mathematics. Without it, they cannot reflect on or anticipate a possible solution to the problem. Freudenthal (1971) supports that

> "[…] *In a thought experiment the teacher has been reinventing the subject matter as though he himself was the student, and this is what he teaches. […] This is a modern reinforcement of the socratic idea*" (p.416).

For example, if we are not able to have access to mathematical objects, which processes could become mental objects whose aim is to reinforce students' cognitive development in mathematical thinking? Thus, we have to act or operate on external objects or on external representations of these objects or on their external symbols. This is in accordance with what Piaget (1970) stated about mathematical knowledge which can be abstracted either directly from objects or the external experiences we have in relation to the objects, or from operations that are mentally performed on objects.

Are the students able to grasp logical operations on abstract mathematical objects? What does it mean to obtain access to an abstract mathematical object or a mathematical entity? This assumption imposes a series of questions about the nature of the mathematical curriculum materials.

"*No matter how well curriculum materials are tested and how many times they are revised, each school brings its own resources and barriers; each classroom brings its own needs, styles, experience, and interests.[. . .] And each day in the classroom brings its own set of issues, catastrophes, and opportunities.[…] At some point, we have to decide that the curriculum materials themselves are good enough—ready for teachers to use and revise in their own classrooms*".(Susan Jo Russell, 1997, p. 251, cited in Remillard, 2005).

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