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THE TRIANGULAR NUMBERS IN ACTIONS

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ABSTRACT

The triangular numbers are formed by partial sum of the series 1+2+3+4+5+6+7…. +n [2]. In other words, triangular numbers are those counting numbers that can be written as T_n *= 1+2+3+…+ n. So,*

$T_1 = 1$

$T_2 = 1+2=3$

$T_3 = 1+2+3=6$

$T_4 = 1+2+3+4=10$

$T_5 = 1+2+3+4+5=15$

$T_6 = 1+2+3+4+5+6= 21$

$T_7 = 1+2+3+4+5+6+7= 28$

$T_8 = 1+2+3+4+5+6+7+8= 36$

$T_9 =1+2+3+4+5+6+7+8+9=45$

$T_{10} =1+2+3+4+5+6+7+8+9+10=55$

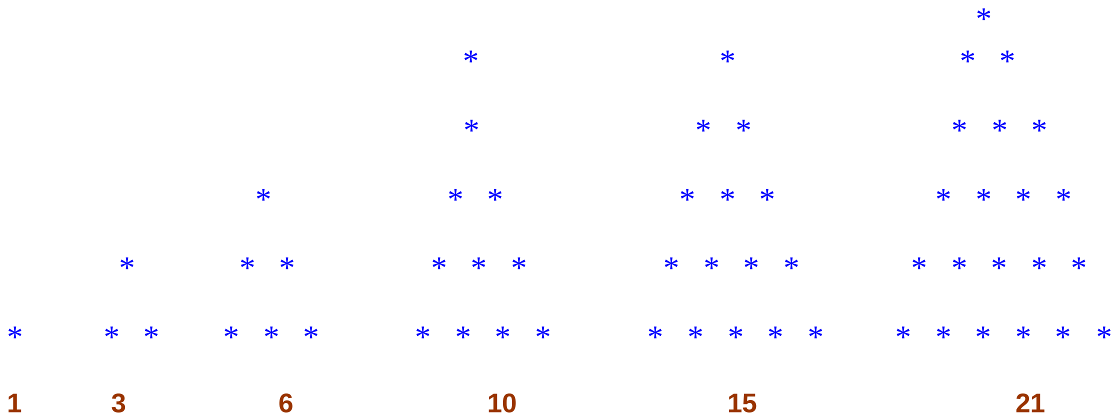
In this paper we investigate some important properties of triangular numbers. Some important results dealing with the mathematical concept of triangular numbers will be proved. We try our best to give short and readable proofs. Most of the results are supplemented with examples.

KEYWORDS

Triangular numbers, Perfect square, Pascal Triangles, and perfect numbers.

1. Introduction :

The sequence 1, 3, 6, 10, 15, …, $n(n + 1)/2$, … shows up in many places of mathematics[1] . The Greek called them triangular numbers [1]. The triangular number T_n is a figurate number that can be represented in the form of a triangular grid of points where the first row contains a single element and each subsequent row contains one more element than the previous one as shown below [2].



Mathematicians have been fascinated for many years by the properties and patterns of triangular numbers [2]. We can easily hunt for triangular numbers using the formula:

$$T_n = \frac{n(n+1)}{2}, \quad n > 0.$$

T_1	T_2	T_3	T_4	T_5	T_6	T_7	T_8	T_9	T_{10}	T_{11}	T_{12}	T_{13}	T_{14}	T_{15}	T_{16}	T_{17}	T_{18}	T_{19}	T_{20}
1	3	6	10	15	21	28	36	45	55	66	78	91	105	120	136	153	171	190	210

The first 20 triangular numbers are as follows.

2. The Main Results:

Theorem 1: Every triangular number is a binomial coefficient .

Thoerem2. Every T triangular number is an arithmetic progression.

Thoerem 3.

Theorem 4

Theorem 5. Every perfect number is a triangular number.

Proof without words Refer to the following Pascal's Triangle [2] and see the red colored numbers.

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1
1 9 36 84 126 126 84 36 9 1

Remark1: Also note that $T_n = \dfrac{n(n+1)}{2} = \dbinom{n+1}{2}$ which is a binomial coefficient for each $\mathbf{n} \geq 1$.

Theorem 1: T is a triangular number $\Leftrightarrow$ 8T+1 is a perfect square.

Proof: **(i)** ($\Rightarrow$) Assume T is a triangular number.

Let T= $\dfrac{n(n+1)}{2}$, n a positive integer.

$$\Rightarrow 8T = 8\frac{n(n+1)}{2}$$

$$\Rightarrow 8T + 1 = 8\frac{n(n+1)}{2} + 1$$

$$\Rightarrow 8T + 1 = 8\frac{n(n+1)}{2} + \frac{2}{2}$$

$$\Rightarrow 8T + 1 = \frac{8n^2 + 8n + 2}{2}$$

$$\Rightarrow 8T + 1 = 2\frac{(4n^2 + 4n + 1)}{2}$$

$$= (2n+1)(2n+1)$$

$$= (2n+1)^2$$

Hence, 8T+1 is a perfect square.

(ii) Assume 8T+1 is a perfect square. Then 8T+1 is odd $\Rightarrow$ for some positive integer n, we have 8T+1= $(2n+1)^2 = 4n^2 + 4n + 1$ implies that T== $\dfrac{n(n+1)}{2}$.

Hence, T is a triangular number.

By (i) and (ii) the theorem is proved.

Example 1. 6 is a triangular number implies that 8(6) +1 =49 is a perfect square

Example 2. 8(15)+1= 121, a perfect square implies that 15 is a triangular number.

Corollary 1. T is a triangular number $\Leftrightarrow$ n = $\dfrac{\sqrt{8T+1}-1}{2}$ is an integer.

Proof: The corollary easily follows by **Theorem 1.**

Theorem 2: If T_m and T_n are triangular numbers, then

$$T_{m+n} = T_m + T_n + mn$$

for m and n positive integers.

Proof:

Note: $T_m = \dfrac{m(m+1)}{2}$ & $T_n = \dfrac{n(n+1)}{2}$. Then

$$T_m + T_n + mn = \frac{m(m+1)}{2} + \frac{n(n+1)}{2} + mn$$

$$= \frac{m^2 + m + n^2 + n}{2} + mn$$

$$= \frac{m^2 + m + n^2 + n + 2mn}{2} = \frac{m^2 + 2mn + n^2 + m + n}{2}$$

$$= \frac{(m+n)(m+n) + (m+n)}{2} = \frac{(m+n)[m+n+1]}{2} = T_{m+n}$$

Example 3. Consider T_3 and T_4. Note that $T_3 = 6$ and $T_4 = 10$. Observe that $T_{3+4} = T_7 = 28$ and $T_3 + T_4 + 3(4) = 6 + 10 + 12 = 28$.

Hence, $T_{3+4} = T_3 + T_4 + 3(4)$

Theorem 3: If T_m and T_n are triangular numbers, then

$$T_{mn} = T_m T_n + T_{m-1} T_{n-1}$$

Proof: Note: $T_m = \frac{m(m+1)}{2}$ and $T_n = \frac{n(n+1)}{2}$. Then

$$T_m T_n + T_{m-1} T_{n-1} = \frac{m(m+1)}{2}\frac{n(n+1)}{2} + \frac{(m-1)m}{2}\frac{(n-1)n}{2}$$

$$= \left(\frac{m^2+m}{2}\right)\left(\frac{n^2+n}{2}\right) + \left(\frac{m^2-m}{2}\right)\left(\frac{n^2-n}{2}\right)$$

$$= \left[\frac{m^2n^2+mn^2+nm^2+mn}{4}\right] + \left[\frac{m^2n^2-mn^2-nm^2+mn}{4}\right]$$

$$= \frac{2m^2n^2+2mn}{4} = \frac{2mn(mn+1)}{4} = \frac{mn(mn+1)}{2}$$

$$= T_{mn}$$

Example 4. Let m =6 and n =7. Then $T_6 = 21 \text{ and } T_7 = 28.$

By using $T_n = \frac{n(n+1)}{2}$, we get $T_{(6)(7)} = T_{42} = \frac{42(43)}{2} = 903$

We also have $T_5 T_6 = 15(21) = 315$ and $T_6 T_7 + T_5 T_6 =$ 588+315=903.

Hence, $T_{(6)(7)} = T_{42} = T_6 T_7 + T_5 + T_6$

Lemma 1. The sum of two consecutive triangular numbers is a perfect square

Proof: Let T_{n-1} and T_n be any two consecutive triangular numbers, such that

$$T_{n-1} = \frac{(n-1)(n)}{2} \quad \text{and} \quad T_n = \frac{n(n+1)}{2}$$

Then,

$$T_{n-1} + T_n = \frac{(n-1)n}{2} + \frac{n(n+1)}{2}$$

$$= \frac{n^2 - n + n^2 + n}{2} = \frac{2n^2}{2} = n^2$$

Which is a perfect square.

Example 5. Let T_6 and T_7 be any consecutive triangular numbers. Then $T_6 + T_7 = 21+28= 49$, which is a perfect square.

Lemma 2. $1^2 + 2^2 + 3^2 + 4^2 + ... + k^2 = \frac{k(k+1)(2k+1)}{6}$

Proof. We can easily prove the lemma using induction.

Example 6. Let k = 5 . Then $1^2 + 2^2 + 3^3 + 4^2 + 5^2 = 1+4+9+16+25 = 55$.

Also, we have $\frac{5(6)(11)}{6} = \mathbf{55}$ and hence $1^2 + 2^2 + 3^3 + 4^2 + 5^2 = \frac{5(6)(11)}{6}$.

Theorem 4. If T_k be triangular numbers fork $k > 0$, then we have

$$\sum_{k=1}^{n} T_k = \frac{n(n+1)(n+2)}{6}$$

Proof: To prove the theorem, we apply *divide and conquer method* by considering two cases:

(1) If n is even, say n =2k, then

$$T_1 + T_2 + ... + T_n = (T_1 + T_2) + (T_3 + T_4) + ... + (T_{2k-1} + T_{2k})$$

$$= 2^2 + 4^2 + ... + (2k)^2 \quad \text{(by Lemma 1)}$$

$$= 4(1^2 + 2^2 + ... + k^2)$$

$$= \frac{4k(2k+1)(k+1)}{6} \quad \text{(by Lemma 2).}$$

$$= \frac{n(n+1)(n+2)}{6} \quad \text{as } n = 2k$$

(2) If n is odd, say n= 2k+1, then

$$T_1 + T_2 + \dots + T_n = (T_1 + T_2) + (T_3 + T_4) + \dots + (T_{2k-1} + T_{2k}) + T_{2k+1}$$

$$= 2^2 + 4^2 + \dots + (2k)^2 + \frac{(2k+1)(2k+2)}{2}$$ (**by Lemma 1 and definition of** T_k)

$$= 4(1^2 + 2^2 + \dots + k^2) + \frac{(2k+1)(2k+2)}{2}$$

$$= \frac{4k(2k+1)(k+1)}{6} + \frac{3(2k+1)(2k+2)}{6}$$ (by Lemma 2.)

$$= \frac{2k(2k+1)(2k+2)}{6} + \frac{3(2k+1)(2k+2)}{6}$$

$$= \frac{(2k+1)(2k+2)(2k+3)}{6}$$

$$= \frac{n(n+1)(n+2)}{6}$$ as n = 2k+1

By (1) and (2) the Theorem is proved.

Example 7. Let n = 5. Then $\sum_{k=1}^{5} T_k = T_1 + T_2 + T_3 + T_4 + T_5$ = 1+3+6+10+15= 35. We also have,

$$\frac{5(6)(7)}{6} = 35 \text{ and hence } \sum_{k=1}^{5} T_k = \frac{5(6)(7)}{6}.$$

Theorem 5 For any natural number n**,** the number

1 + 9 + $\mathbf{9^2}$ **+** $\mathbf{9^3}$ **+ ... +** $\mathbf{9^n}$ is a triangular number.

Proof: Let T =**1 + 9 +** $\mathbf{9^2}$ **+** $\mathbf{9^3}$ **+ ... +** $\mathbf{9^n}$**.** Then **T=** $\frac{9^{n+1}-1}{8}$.

By **Theorem 1,** it is suffice to prove that **8T+1** is a perfect square. We will apply the divide_and conquer method as in **Theorem 4.**

(1) If n is even, say n =2k, then

8T+1 = 8 $(\frac{9^{n+1}-1}{8})+1 = 9^{n+1} = 9^{2k+1} = 9\left(9^{2k}\right)=\left(3^{2k+1}\right)^2$, which a perfect square.

(2) If n is odd, say n= 2k+1, then

8T+1 = 8 $(\frac{9^{n+1}-1}{8})+1 = 9^{n+1} = 9^{2k+2} = \left(9^{k+1}\right)^2$, which a perfect square.

By (1)and (2), the theorem is proved.

Theorem 6. If T_n be triangular numbers for $n \geq 1$, then we have

$$\sum_{n=1}^{\infty}\frac{1}{T_n} = 2$$

Proof: $\sum_{n=1}^{\infty}\frac{1}{T_n}$

$$= \sum_{n=1}^{\infty}\frac{2}{n(n+1)}$$

$$= 2\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right)$$

$$= 2(1) = 2$$

Proposition 2. The difference of the squares of two consecutive triangular numbers is a cube.

Proof: Consider $T_{n-1} = \frac{(n-1)n}{2}$ and $T_n = \frac{(n+1)n}{2}$

Then, $(T_n)^2-(T_{n-1})^2 = \left(\frac{n(n+1)}{2}\right)^2-\left(\frac{(n-1)n}{2}\right)^2$

$$= \frac{n^4+2n^2+n^2}{4} - \frac{n^4-2n^3+n^2}{4} = \frac{4n^3}{4} = n^3$$

Example 8. Let T_6 and T_7 be any two consecutive triangular numbers. Then

$(T_7)^2-(T_6)^2 = 28^2-21^2$ =(28+-21)(28-21)=(49)(7) = 7^3, which is a perfect cube.

Proposition 3: T is a triangular is number $\Longrightarrow$ 9T+1 is a triangular number .

Proof: Assume T is a triangular number.

Let $T = \frac{n(n+1)}{2}$

$$\Rightarrow 9T = \frac{9n(n+1)}{2}$$

$$\Rightarrow 9T + 1 = \frac{9n(n+1)}{2} + 1$$

$$= \frac{9n(n+1)}{2} + \frac{2}{2}$$

$$= \frac{9n(n+1)+2}{2}$$

$$= \frac{9n^2 + 9n + 2}{2}$$

$$= \frac{(3n+2)(3n+1)}{2}$$

$$= \frac{m(m+1)}{2}$$, where **m = 3n+1**.

Hence, 9T+1 is a triangular number.

Example 9. Let T_8 be the triangular number. Then 9 T_8 +1= 45, which is a triangular number.

Proposition 4: n = $2^{k-1} + 2^k + 2^{k+1} + \ldots + 2^{2k-2}$ is a triangular number.

Proof: Note that n = $2^{k-1} + 2^k + \ldots.. + 2^{2k-2} = 2^{k-1}(1 + 2 + 2^2 + \ldots.. + 2^{k-1})$

$$= 2^{k-1}(2^k - 1)$$

$$= \frac{2^k(2^k - 1)}{2}$$

$$= \frac{m(m+1)}{2}$$, where m = $2^k - 1$.

Hence, n is a triangular number

Example 10. Let k =3. Then **n** = $2^{3-1} + 2^3 + 2^4$ = 28, which is a triangular number.

Proposition 5. *n = 1+2+3+4+…+(*$2^k - 1$*)* is a triangular number.

Proof : Note that n =

$$\frac{2^k(2^k - 1)}{2}$$

$$= \frac{m(m+1)}{2}, \quad \text{where } m = 2^k - 1.$$

Hence, n is a triangular number

Example 11. Let k = 3. Then 1+2+3+4+5+6+7=28, which is a triangular numbers.

Proposition 6. Every Perfect number [3] is a triangular number.

Proof: Let n be a perfect number. Then n= $2^{k-1}(2^k - 1)$ where $2^k - 1$ is prime [3]. Note that n = $2^{k-1}(2^k - 1) = \frac{2^k(2^k - 1)}{2} = \frac{m(m+1)}{2}$, where m = $2^k - 1$. Hence n is a triangular number.

Proposition 7. Let T_n be a triangular number. Then:

(1) $\mathbf{T_n^2 = T_n + T_{n-1} * T_{n+1}}$

(2) $\mathbf{T_{n^2-1} = 2*T_n * T_{n-1}}$

Proof:

(1) We have $\mathbf{T_n + T_{n-1} * T_{n+1}} = \frac{n(n+1)}{2} + \frac{(n-1)n}{2} * \frac{(n+1)(n+2)}{2}$

$$= \frac{n(n+1)}{2} + \frac{n^4 + 2n^3 - n^2 - 2n}{4} \; \frac{2n^2 + 2n + n^4 + 2n^3 - n^2 - 2n}{4}$$

$$= \frac{n^4 + 2n^3 + n^2}{4}$$

$$= \left(\frac{n(n+1)}{2}\right)^2$$

$$= T_n^2$$

(2) Note that $T_{n^2-1} = \frac{(n^2-1)(n^2)}{2}$

$$= \frac{((n-1)n)(n(n+1))}{2}$$

$$= \frac{2((n-1)n)(n(n+1))}{4}$$

$$= 2 * \frac{(n-1)n}{2} * \frac{n(n+1)}{2}$$

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