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THE FASCINATING GAMMA FUNCTION IN ACTIONS

Mulatu Lemma, Issaac Wrigth, Agegnehu Atena & Jonathan Lambright

College of Science and Technology, Department of Mathematics, Savannah State University, USA.

***Corresponding author**:* *Mulatu Lemma

Tel**.*:+1 ***Email: lemmam@savannahstate.edu

ABSTRACT

The gamma function has several properties that define it. In this paper, I will present proofs for those properties and give example using real numbers. The properties I will prove are as followed:

$\Gamma(n) = (n-1)\,\Gamma(n-1)$

$\Gamma(n+1) = n * \Gamma(n)$

$\Gamma(1) = 1$

$\Gamma(n) = (n-1)!$

$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

$\Gamma(n+1) = n!$

Main Results

The Gamma Function is defined as

$$\Gamma(n) = \int_0^\infty x^{n-1}\, e^{-x}\, dx$$

The first property of the Gamma Function is $\Gamma(n) = (n-1)\,\Gamma(n-1)$

Proof: $\Gamma(n)$ = (n -1) $\Gamma(n-1)$

To integrate this function we use integration by parts with respect to x

$\Gamma(n) = \int_0^\infty x^{n-1}\, e^{-x} dx$

Integration by Parts: uv - $\int vdu$

u = x^{n-1} du = (n-1)x^{n-2}

dv = e^{-x} dx v = -e^{-x}

Once it has been integrated by parts it can be written as

$(-x^{n-1}e^{-x})l_0^\infty$ - $\int_0^\infty (n-1)x^{n-2} - e^{-x}dx$

This function can be reduced to

(n-1)$\int_0^\infty x^{n-2}\, e^{-x}dx$

From this function we now know that

$\Gamma(n)$ = (n -1) $\Gamma(n-1)$

The second property of the Gamma Function is $\Gamma(n+1)$ = n $*$ $\Gamma(n)$

Proof: $\Gamma(n+1)$ = n $*$ $\Gamma(n)$

$$\Gamma(n+1) = \int_0^\infty x^{(n+1)-1}\, e^{-x}dx$$

This can be reduced to

$$\Gamma(n+1) = \int_0^\infty x^{n}\, e^{-x}dx$$

To integrate this function we need to integrate by parts respect to x again.

u = x^n du = nx^{n-1}

dv = $e^{-x}dx$ v = -e^{-x}

Combing uv - $\int vdu$

$(-x^{n}e^{-x})l_0^\infty$ - $\int_0^\infty (nx^{n-1} - e^{-x}dx$

This can be reduced to

$$n\int_0^\infty x^{n-1}\, e^{-x} dx$$

Thus, we have proven that

$\mathbf{\Gamma(n+1) = n * \Gamma(n)}$

The third property of the Gamma Function is $\Gamma(1) = 1$

Proof: $\Gamma(1) = 1$

One of the properties of the gamma function is $\Gamma(1) = 1$. The next step is to prove this

$\Gamma(1) = \int_0^\infty x^{1-1}\, e^{-x} dx$

This can be simplified to the equation below

$\Gamma(1) = \int_0^\infty e^{-x}\, dx$

When we integrate with respect to x

$\int_0^\infty e^{-x}\, dx = 1$

So $\mathbf{\Gamma(1) = 1}$

Using one of the properties we proved earlier, $\Gamma(n+1)$=n $*\ \Gamma(n)$, we can develop a formula for the gamma function that gives a new property related to factorial values. We first review the factorial values for n = 1, 2, 3, 4, 5, 6.

1! = 1

2! = 2 * 1 = 2

3! = 3*2*1 = 6

4! = 4*3*2*1 = 24

5! = 5*4*3*2*1 = 120

The fourth property of the Gamma Function is $\Gamma(n)$= (n − 1)!

Proof: $\mathbf{\Gamma(n) = (n-1)!}$

Let n = 1 for $\Gamma(n+1)$ = n $*\ \Gamma(n)$

$\Gamma(1+1) = 1 * \Gamma(1)$

$\Gamma(2) = 1 * 1$

$\Gamma(2) = 1$

Let n = 2 for $\Gamma(n+1) = n * \Gamma(n)$

$\Gamma(2+1) = 2 * \Gamma(2)$

$\Gamma(3) = 2 * \Gamma(2)$

$\Gamma(3) = 2 * 1$

$\Gamma(3) = 2$

Let n = 3 for $\Gamma(n+1) = n * \Gamma(n)$

$\Gamma(3+1) = 3 * \Gamma(3)$

$\Gamma(4) = 3 * \Gamma(3)$

$\Gamma(4) = 3 * 2$

$\Gamma(4) = 6$

Let n = 4 for $\Gamma(n+1) = n * \Gamma(n)$

$\Gamma(4+1) = 4 * \Gamma(4)$

$\Gamma(5) = 4 * \Gamma(4)$

$\Gamma(5) = 4 * 6$

$\Gamma(5) = 24$

Let n = 5 for $\Gamma(n+1) = n * \Gamma(n)$

$\Gamma(5+1) = 5 * \Gamma(5)$

$\Gamma(6) = 5 * \Gamma(5)$

$\Gamma(6) = 5 * 24$

$\Gamma(6) = 120$

Thus, we have now shown that

$\mathbf{\Gamma(n) = (n-1)!}$

The fifth property of the Gamma Function is $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

Proof: $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty x^{\frac{1}{2}-1}\, e^{-x} dx$$

$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty x^{-\frac{1}{2}}\, e^{-x} dx$$

Let x = u^2

Then dx = 2u * du

$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty (u^2)^{-\frac{1}{2}}\, e^{-u^2}\, 2u * du$$

$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty u^{-1}\, e^{-u^2}\, 2u * du$$

$$\Gamma\left(\frac{1}{2}\right) = 2\int_0^\infty e^{-u^2}\, du$$

To solve the integral we will need to change it to polar coordinates because no integral technique exist for this function.

Let I = $\int_0^\infty e^{-x^2}\, du$

We need to square both sides of the equation

$I^2 = [\int_0^\infty e^{-x^2}\, du]^2$

Simplify this to

$=\iint_0^\infty e^{-x^2} * e^{-y^2} dydx$

We combine the bases together

$=\iint_0^\infty e^{-(x^2+y^2)} dydx$

In polar coordinates we know

$r^2 = (x^2 + y^2)$

da = dydx = $rdrd\theta$

If wee substitute these into the equation we have

I² $= \int_0^{\frac{\pi}{2}} \int_0^{\infty} e^{-r^2} r dr d\theta$

We can now use U substitution to integrate this equation

Let u = r^2

2rdr = du

dr = $\frac{du}{2r}$

Substitute these into the equation

$$\int_0^{\frac{\pi}{2}} \int_0^{\infty} e^{-u} r \frac{du}{2r} d\theta$$

Simplifying the equation we are left with

$$\int_0^{\frac{\pi}{2}} \frac{1}{2} \int_0^{\infty} e^{-u} du d\theta$$

If we integrate the equation with respect to u we have

$$\int_0^{\frac{\pi}{2}} \frac{1}{2} [(-e^{-u})] l_0^{\infty} d\theta$$

Replace U with our original definition

$$\int_0^{\frac{\pi}{2}} \frac{1}{2} (-e^{-r^2}) l_0^{\infty} \, d\theta$$

Substitute in the limits for r

$$\int_0^{\frac{\pi}{2}} \frac{1}{2} [-e^{-\infty^2} - (-e^{-0^2})] \, d\theta$$

$= \int_0^{\frac{\pi}{2}} \frac{1}{2} [-e^{-\infty^2} - (-e^{-0^2})] \, d\theta$

$= \int_0^{\frac{\pi}{2}} \frac{1}{2} [0 - (-1)] \, d\theta$

$= \int_0^{\frac{\pi}{2}} \frac{1}{2} d\theta$

We integrate with respect to θ

$$[\frac{1}{2} \theta] l_0^{\frac{\pi}{2}}$$

$= [\frac{1}{2}\left(\frac{\pi}{2}\right) - \frac{1}{2}(0)]$

$= \frac{\pi}{4}$

Therefore

$I^2 = \frac{\pi}{4}$

We take the square root of both sides and have

$I = \int_0^\infty e^{-u^2}\, du = \frac{\sqrt{\pi}}{2}$

If we take this value and return to the equation

$$\Gamma\left(\frac{1}{2}\right) = 2\int_0^\infty e^{-u^2}\, du$$

We will have

$\Gamma\left(\frac{1}{2}\right) = 2 * \frac{\sqrt{\pi}}{2}$

This simplifies to

$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

Therefore,

$\boldsymbol{\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}}$

Using the properties $\Gamma(n+1) = \text{n} * \Gamma(n)$ and $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$, we can find the gamma of non-integers.

Let n = $\frac{1}{2}$ for $\Gamma(n+1) = \text{n} * \Gamma(n)$

$\Gamma\left(\frac{1}{2}+1\right) = \frac{1}{2} * \Gamma\left(\frac{1}{2}\right)$

$\Gamma\left(\frac{3}{2}\right) = \frac{1}{2} * \sqrt{\pi}$

$\Gamma\left(\frac{3}{2}\right) = \frac{\sqrt{\pi}}{2}$

Let n = $\frac{3}{2}$ for $\Gamma(n+1) = \text{n} * \Gamma(n)$

$\Gamma\left(\frac{3}{2}+1\right) = \frac{3}{2} * \Gamma\left(\frac{3}{2}\right)$

$\Gamma\left(\frac{5}{2}\right) = \frac{3}{2} * \frac{\sqrt{\pi}}{2}$

$\Gamma\left(\frac{5}{2}\right) = \frac{3\sqrt{\pi}}{4}$

Let n = $\frac{5}{2}$ for $\Gamma(n+1) = \text{n} * \Gamma(n)$

$\Gamma\left(\frac{5}{2}+1\right) = \frac{5}{2} * \Gamma\left(\frac{5}{2}\right)$

$\Gamma\left(\frac{7}{2}\right) = \frac{5}{2} * \frac{3\sqrt{\pi}}{4}$

$\Gamma\left(\frac{7}{2}\right) = \frac{15\sqrt{\pi}}{8}$

We can also use the property $\Gamma(n+1)$ = n! to find factorial values for non-integers.

Let n = $\frac{1}{2}$ for $\Gamma(n+1)$ = n!

$\Gamma\left(\frac{1}{2}+1\right) = \frac{1}{2}!$

$\Gamma\left(\frac{3}{2}\right) = \frac{1}{2}!$

$\frac{\sqrt{\pi}}{2} = \frac{1}{2}!$

Let n = $\frac{3}{2}$ for $\Gamma(n+1)$ = n!

$\Gamma\left(\frac{3}{2}+1\right) = \frac{3}{2}!$

$\Gamma\left(\frac{5}{2}\right) = \frac{3}{2}!$

$\frac{3\sqrt{\pi}}{4} = \frac{3}{2}!$

Let n = $\frac{5}{2}$ for $\Gamma(n+1)$ = n!

$\Gamma\left(\frac{5}{2}+1\right) = \frac{5}{2}!$

$\Gamma\left(\frac{7}{2}\right) = \frac{5}{2}!$

$\frac{15\sqrt{\pi}}{8} = \frac{5}{2}!$

We can also use this property to define factorial values for negative non-integers.

Let n = -$\frac{1}{2}$ for $\Gamma(n+1)$ = n!

$\Gamma\left(-\frac{1}{2}+1\right) = -\frac{1}{2}!$

$\Gamma\left(\frac{1}{2}\right) = -\frac{1}{2}!$

$\sqrt{\pi} = -\frac{1}{2}!$

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