THE FASCINATING MATHEMATICAL BEAUTY OF THE SUM OF SQUARES

Mulatu Lemma
Department of Mathematics
Savannah State University
Savannah, GA 31404 USA

Abstract: We will investigate which integers can be written as the sum of squares. Different examples are given to supplement each given theorems.

Introduction: We say that a positive integer $\boldsymbol{n}$ is representable as asum of two squares if $\boldsymbol{n}=\boldsymbol{a}^2+\boldsymbol{b}^2$ for some integers $\boldsymbol{a}$ and. We include $\mathbf{0}$ as a possible value of $\boldsymbol{a}$ and $\boldsymbol{b}$. We also say that a positive integer $\boldsymbol{n}$ is representable as a sum of $\boldsymbol{m}$ squares if $\boldsymbol{n}=\boldsymbol{a}_1^2+\boldsymbol{a}_2^2+\boldsymbol{a}_3^2+\cdots+\boldsymbol{a}_m^2$ for some integers m and a_i.

1. The Sum of Two Squares

Theorem 1. An integer $\boldsymbol{n}$ is the sum of two squares $\Leftrightarrow$ $\mathbf{2}\boldsymbol{n}$ is the sum of the squares.

Proof (1) $\Rightarrow$ Assume $\boldsymbol{n}$ is the sum of two squares. Let $\boldsymbol{n}=\boldsymbol{a}^2+\boldsymbol{b}^2$ for integers $\boldsymbol{a}$ and $\boldsymbol{b}$.

$$\text{Then } \mathbf{2}\boldsymbol{n}=\mathbf{2}(\boldsymbol{a}^2+\boldsymbol{b}^2)$$

$$\Rightarrow \mathbf{2}\boldsymbol{n}=(\boldsymbol{a}+\boldsymbol{b})^2+(\boldsymbol{a}-\boldsymbol{b})^2$$

$$\Rightarrow \mathbf{2}\boldsymbol{n} \text{ is the sum of two squares}$$

(2) $\Leftarrow$ Assume $\mathbf{2}\boldsymbol{n}=\boldsymbol{c}^2+\boldsymbol{d}^2$. Since $\boldsymbol{c}$ and $\boldsymbol{d}$ are both even or both odd $\boldsymbol{c}+\boldsymbol{d}$ and $\boldsymbol{c}-\boldsymbol{d}$ are even integers.

$$\boldsymbol{n}=\left(\frac{\boldsymbol{c}+\boldsymbol{d}}{\mathbf{2}}\right)^2+\left(\frac{\boldsymbol{c}-\boldsymbol{d}}{\mathbf{2}}\right)^2$$

$$\Rightarrow \boldsymbol{n} \text{ is the sum of two squares.}$$

The theorem follows by (1) and (2).

Example 1. Let $n = 29$then

$$n^2 = 5^2 + 2^2 \text{ and}$$

$$2n = 58 = 7^2 + 3^2$$

Theorem 2. If n a triangular number, prove that even if each of the three consecutive integers $8n^2$,$8n^2 + 1$, and $8n^2 + 2$ can be expressed as a sum of two squares.

Proof

1) $8n^2 = (2n)^2 + (2n)^2$, hence sum of two squares

2) n is a triangular number

$$\Rightarrow n = \frac{m(m+1)}{2}$$

$$\Rightarrow 8n = 4m(m+1)$$

$$\Rightarrow 8n = 4(m)(m+1) + 1$$

$$= 4m^2 + 4m + 1$$

$$= (2m+1)^2$$

Hence $8n + 1$ is a perfect square.

Let $8n + 1 = k^2$

Now observe that

$$2(8n^2 + 1) = (4n+1)^2 + (8n+1)$$

$$= (4n+1)^2 + k^2$$

$\Rightarrow$by Theorem 1, $8n^2 + 1$ is a sum of two squares

3) Note that

$$8n^2 + 2 = (m(m+1) + 1)^2 + (m(m+1) - 1)^2$$

a sum of two squares also.

Theorem 3. If each of the natural numbers x and y is a sum of two squares then so is xy.

Proof Let $x = a^2 + b^2$ and $= c^2 + d^2$. Then

$$xy = (a^2 + b^2)(c^2 + d^2)$$
$$= a^2c^2 + a^2d^2 + b^2c^2 + b^2d^2$$
$$= a^2c^2 + 2abcd + b^2d^2 + a^2d^2 - 2abcd + b^2 + c^2$$
$$= (ac + bd)^2 + (ad - bc)^2.$$ Thus the theorem is proved.

Remark1 xy can an also be written as

$$xy = (ac - bd)^2 + (ad + bc)^2$$

Example 2.

$$65 = 5 \cdot 13$$ Note that

$$5 = (2^2 + 1) \; and \; 13 = 3^2 + 2^2$$

$$a = 2 \quad b = 1 \quad c = 3 \quad d = 2$$

So, we have

$$65 = (6 + 2)^2 + (4 - 3)^2 = 8^2 + 1^2$$
$$= (6 - 2)^2 + (4 + 3)^2 = 4^2 + 7^2$$

We state the following two Theorem without proof and use them.

Theorem 4. If the prime $P \equiv 1(mod\ 4)$ then there exist unique integers x and y such that $x > y > 0$ and $p = x^2 + y^2$.

Example 3. Let p= 97. Then $P \equiv 1(mod\ 4)$ and 97 can be expressed as sum of two squares. Note **97= =**$9^2 + 4^2$**.**

Theorem 5. Let n be a positive integer. Then n can be expressed as the sum of two squares if and only if all prime factors of n of the form 4t+3 have even exponents in the factorization of n.

Example 4. Take n= 162. Then n= $2(3^4)$ and 3 is a prime factor of the form 4t+ 3 with even exponent 4 and hence can be expressed as the sum of two squares. Note that **162=**$9^2 + 9^2$**.**

2. The sum of three squares.

Lemma 1: Every number can be expressed as the sum of 3 triangular numbers.

Theorem 6 Every number of the form $8k+3$ can be expressed as the sum of three squares.

Proof By Lemma 1, K can be written as the sum of three triangular numbers. That is,

$$K = \frac{a(a+1)}{2} + \frac{b(b+1)}{2} + \frac{c(c+1)}{2}$$

$$\Rightarrow 8K + 3 = 4a(a+1) + 4b(b+1) + 4c(c+1)$$

$$\Rightarrow 8K + 3 = 4a^2 + 4a + 4b^2 + 4b + 4c^2 + 4c + 1$$

$$= (2a+1)^2 + (2b+1)^2 + (2c+1)^2$$

Hence the theorem is proved

Remark2: A number can be expressed as the sum of three squares in only one way.

We state the following important theorem without proof and use it.

A natural number can be represented as the sum of three squares of integers.

$$n = a^2 + b^2 + c^2 \Leftrightarrow n \text{ is of the form}$$

$$n = 4^m(8k+7) \text{ for integers } m \text{ and } k$$

Example 5 List five integers that can be expressed as the sum of three square integers using $n = 8k+3$

$$k = 0 \Rightarrow n = 3 = 1^2 + 1^2 + 1^2$$

$$k = 1 \Rightarrow n = 11 = 3^2 + 1^2 + 1^2$$

$$k = 2 \Rightarrow n = 19 = 2^2 + 3^2 + 1^2$$

$$k = 3 \Rightarrow n = 27 = 3^2 + 3^2 + 3^2$$

$$k = 4 \Rightarrow n = 35 = 5^2 + 3^2 + 1^2$$

Theorem 7 Let n be a positive integer. Then n can be expressed as the sum of three squares if and only if n is not of the form 4^k (8t + 7).

Example 6. Let n= 15. Then 15 is of the form 4^k (8t + 7) and cannot be expressed as the sum of three squares.

3. The sum of four squares.

Lagrange's Theorem: We state the theorem without proof and use it.

Theorem 8 Every natural number is the sum of four squares.

Example 4:

$$(1)\quad 5 = 2^2 + 1^2 + 0^2$$

$$(2)\quad 21 = 4^2 + 2^2 + 1^2 + 0^2$$

$$(3)\quad 28 = 5^2 + 1^2 + 1^2 + 1^2$$

(4) Sum of squares of consecutive integers

Theorem 8 The sum of the squares of the first n natural numbers is given by

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

Proof: Easily follows using induction.

Corollary 1: The sum of the squares of the first n **even** natural numbers is given by

$$\sum_{k=1}^{n}(2n)^{2} = \frac{2n(n+1)(2n+1)}{3}$$

Corollary 2. The sum of the squares of the first **even odd natural** numbers is given by

$$\sum_{k=1}^{n}(2n-1)^{2} = \frac{n(2n+1)(2n-1)}{3}$$

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