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Chaos and chaotic fluid mixing

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Very simple mathematical equations can give rise to surprisingly complicated, *chaotic* dynamics, with behavior that is sensitive to small deviations in the initial conditions. We illustrate this with a single recurrence equation that can be easily simulated, and with mixing in simple fluid flows.

1 Introduction to chaos

People used to believe that simple mathematical equations had simple solutions and complicated equations had complicated solutions. However, Henri Poincaré (1854–1912) and Edward Lorenz [4] showed that some remarkably simple mathematical systems can display surprisingly complicated behavior. An example is the well-known *logistic map equation*:

$$x_{n+1} = Ax_n(1 - x_n), \tag{1}$$

where x_n is a quantity (between 0 and 1) at some time n, x_{n+1} is the same quantity one time-step later and A is a given real number between 0 and 4. One of the most common uses of the logistic map is to model variations in the size of a population. For example, x_n can denote the population of adult mayflies in a pond at some point in time, measured as a fraction of the maximum possible number the adult mayfly population can reach in this pond. Similarly, x_{n+1} denotes the population of the next generation (again, as a fraction of the maximum) and the constant A encodes the influence of various factors on the size of the population (ability of pond to provide food, fraction of adults that typically reproduce, fraction of hatched mayflies not reaching adulthood, overall mortality rate, etc.)

As seen in Figure 1, the behavior of the logistic system changes dramatically with changes in the parameter A. For a small value of the parameter A (Figure 1(a)), the behavior settles down quickly to a *fixed point*, meaning x_n have the same value for large n. Increasing the value of A results in transitions to well-ordered *period-2* (Figure 1(b)) and *period-4* (Figure 1(c)) oscillations (where, after a while, x_n alternates between only 2 and 4 values, respectively). Increasing A even further, the behavior ultimately reaches a chaotic state (Figure 1(d)) where the sequence of values of x_n never repeats. A transition from well-ordered, predictable behavior to unpredictable, erratic behavior when a value of a parameter is changed, is common in many mathematical systems having a wide range of applications in science, engineering, and economics.

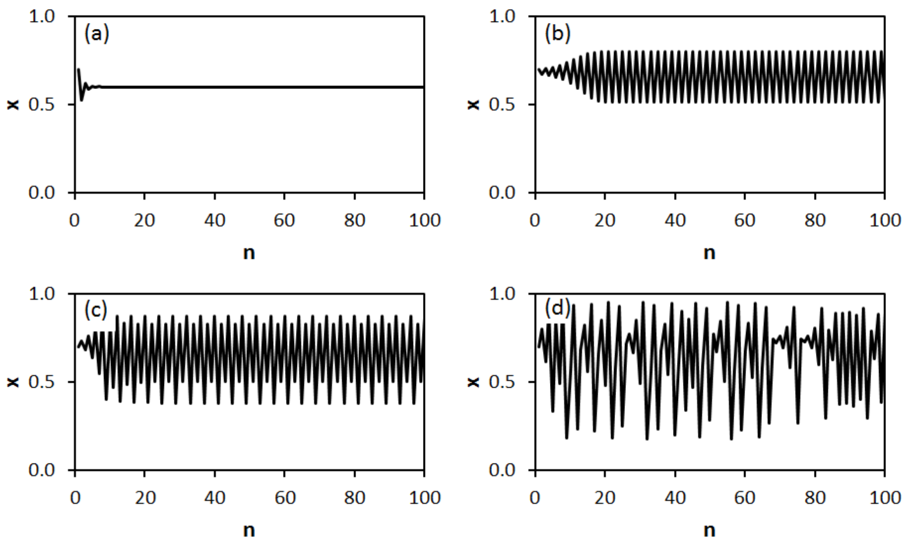

Figure 1: Logistic map, simulated with Microsoft Excel for $x_0 = 0.75$. (a) $A = 2.5$: the system settles down quickly to a stationary state, remaining fixed at 0.6; (b) $A = 3.2$: period-2 – the system oscillates periodically between 0.513 and 0.799; (c) A=3.5: period-4 – the system goes through a periodic cycle of 4 different values before repeating; (d) $A = 3.8$: chaotic behavior – there is no repeating pattern.

The ability of a simple, deterministic equation or set of equations to display remarkably complicated behavior is one of the defining features of *deterministic chaos.* The other defining feature is *sensitive dependence on initial conditions*, also referred to in popular culture as the *butterfly effect* [2, 6], where small differences in the initial state of a system result in a significant change in the outcome; for example, inaccuracy in measurement leads to a dramatic deviation of the calculated long-term behavior from the observed one. We will illustrate sensitive dependence on initial conditions in the next section on chaotic mixing.

2 Chaotic mixing

Another example of deterministic chaos in a real-world system can be found when an impurity is mixed into a fluid. Say, for example, that you want to stir some pepper into a large pot of mushroom soup.[1] You might be tempted to stir the soup in a circular motion, but that is a very inefficient way to mix in the pepper, which will mostly go around in a circle in the soup. On the other hand, if you stir in a circle away from the center, then stir in a similar circle on the other side, and then alternate back and forth between the two stirring circles, the pepper will mix *much* more efficiently in the soup. This protocol for mixing – called the *blinking vortex flow* [1, 5] – is shown in Figure 2 (see [1]).

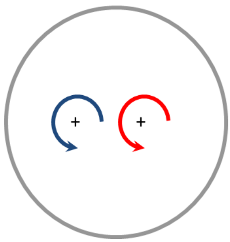

Figure 2: Blinking vortex flow. The flow circulates alternately back and forth around the left and right + marks.

Tracers (particles of substances that trace trajectories; for example, the pepper in the mushroom soup) moving in the blinking vortex flow can undergo either ordered trajectories (if far enough away from the vortex centers) or chaotic trajectories (if closer). This is illustrated in Figure 3(a), which shows the trajectory of a single tracer in the flow, determined from the equations describing the mixing portrayed in Figure 2.

[1] For simplicity, we consider only 2-dimensional mixing, where the pepper stays at the surface of the soup and does not sink towards the bottom of the pot.

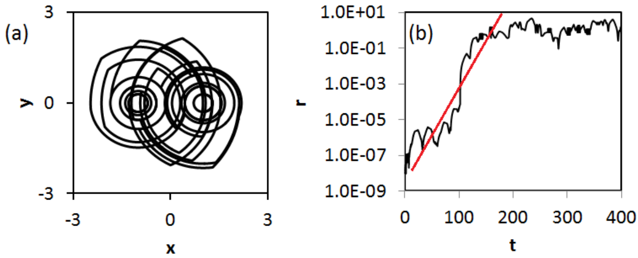

Figure 3: (a) Chaotic particle trajectory in a blinking vortex flow. (b) Separation r between two nearby tracers in the blinking vortex flow, plotted logarithmically[2] as a function of time to emphasize the rapid growth. The tracers separate roughly exponentially in time until the separation reaches a scale close to the size of the system, as indicated by the straight line in the diagram.

Sensitive dependence on initial conditions for these chaotic trajectories can be observed by simulating the motion of two tracers which are initially very close together (the two tracers simulated for Figure 3(b) are initially 0.00000001 apart). Figure 3(b) shows a plot of the separation of these two tracers. The separation between the tracers fluctuates erratically, but grows roughly exponentially in time until saturating at a distance comparable to the size of the system.

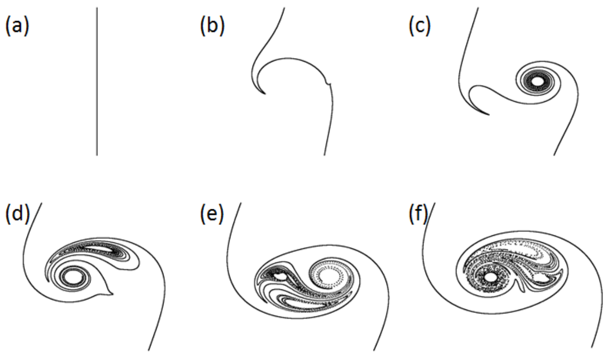

Figure 4: Simulation of mixing of a line of tracers in the blinking vortex flow. Images are 0, 0.5, 1.0, 1.5, 2.0 and 2.5 blinking periods after start.

[2] A *logarithmic* scale is one where each mark represents a multiple of the quantity represented by the previous mark. In Figure 3(b), each mark on the r axis represents a distance 100 times the distance denoted by the one before it (and 10000 times the one before that), whereas the difference between two marks on the time (t) axis is a constant of 100 units. It is used to compress the large distances into a convenient graph.

Since chaotic trajectories separate rapidly in time, a large collection of tracers (pepper particles) will mix efficiently. Figure 4 shows a sequence of images of the evolution of an initial line of 10,000 tracer particles in the blinking vortex flow. After a few periods of the blinking, a well-mixed region has formed in the center region of the flow.

3 Chaotic mixing in three dimensions

To get chaotic mixing in a system varying continuously with time, you need three independent dimensions to describe the trajectories in the equations of motion (as was determined by researchers in what is called 'chaos theory' even though many of the studies are experimental). That can be achieved in a two-dimensional (2D) flow if it is time-dependent, with the third dimension being time. In the case of the blinking vortex flow in the previous section, time dependence is introduced via the alternation (blinking) of the circulating flow between the two poles, resulting in chaotic mixing in the central region. One can think of switching between the poles as mixing also along the time dimension. But if the flow is three-dimensional (3D), chaotic mixing is possible even if the flow is time-*independent*. That is, we don't need to change the way we mix over time (as we did in the previous example) in order for the mixing to be chaotic. For example, we combine simultaneous horizontal and vertical mixing to get a flow which consists of two nested chains of mixing vortices, as shown in Figure 5 (see [3]).

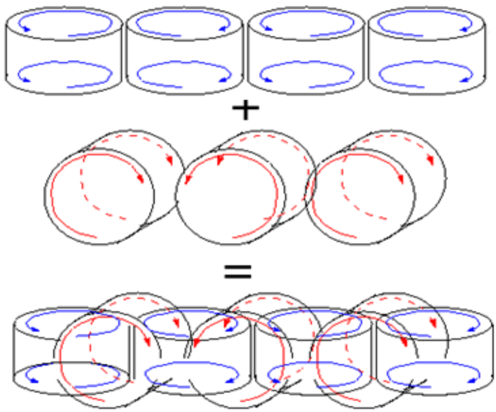

Figure 5: Nested vortex flow, composed of the superposition of a horizontal chain of vortices and a vertical chain of vortices.

Tracers moving in this flow follow trajectories that are either chaotic (upper left of Figure 6) or ordered (upper right), depending on their location. The coexistence of both ordered and chaotic mixing regions can be seen easily by making a so-called 2D *Poincaré section.* We construct it, in this case, by plotting a dot on an horizontal plane located mid-height, every time a tracer crosses this plane. In fact, the black dots showing the chaotic region in Figure 6 are all from the same trajectory. The "holes" in the Poincaré section show the regions where trajectories are ordered.

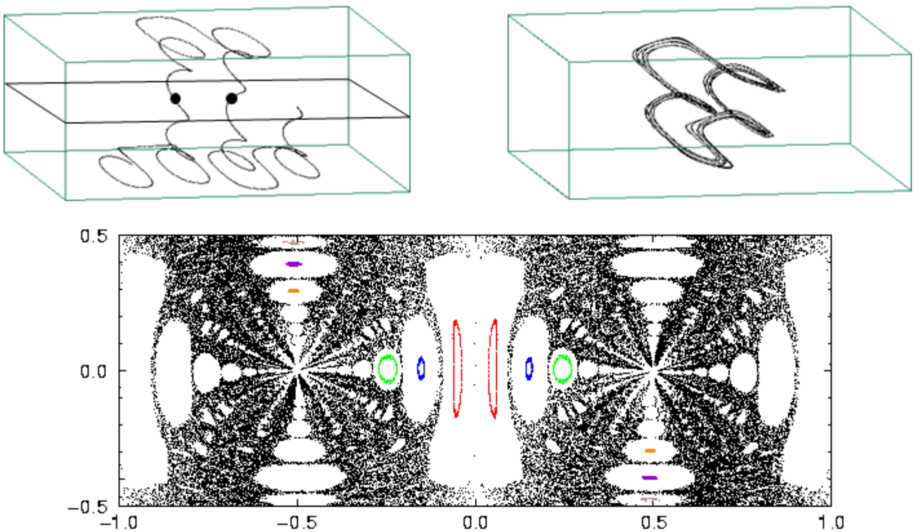

Figure 6: Sample trajectories simulated from the 3D nested vortex flow, along with a Poincaré section showing the coexistence of both ordered and chaotic mixing regions in the flow.

This coexistence of both ordered and chaotic trajectories is a common feature of chaotic mixing and, since impurities (substances in the flowing liquid or gas) can't cross between ordered and chaotic regions in the flow, has significant applications in chemical engineering, cellular- or embryonic-scale biological mixing, and even planetary-scale mixing. Examples of this kind of transport barrier include the Ozone Hole over Antarctica, the Great Red Spot of Jupiter, and long-lived vortices in the ocean including the Gulf Stream Rings, the Meddies and the Agulhas Rings.

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