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From the dollar game to the Riemann-Roch Theorem

Sara Lamboglia • Martin Ulirsch[1]

What is the dollar game? What can you do to win it? Can you always win it? In this snapshot you will find answers to these questions as well as several of the mathematical surprises that lurk in the background, including a new perspective on a century-old theorem.

1 The dollar game

A group of friends, Anna, Emma, Karl, Max and Otto, start to play the following cooperative game. Each participant is assigned a certain number of dollars; a negative number corresponds to a debt. So, for example, Karl and Emma both get two dollars, Otto gets none, while Anna and Max are assigned to have a debt of one dollar. Then every participant is assigned a group of other players they are allowed to lend money to; if one of them can lend money to a friend than they can also borrow money in return. The goal of the game is for all players to be without debt. The only allowed move for a player is to lend one dollar to every person they are allowed to lend money to.

We picture the starting configuration of the game in Figure 1. Here two names are connected by a segment if they are allowed to lend money to each other. Mathematically, the resulting figure is called a *graph*, the names are its

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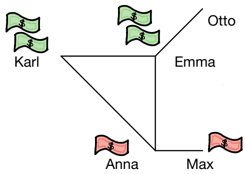

Figure 1: The starting configuration of the game.

vertices and the segments between them are called *edges*. We draw the amount of dollars each person has, and we indicate debts with red dollars. For example, in Figure 1 we can see that Karl is allowed to lend to or borrow money from Emma and Anna, but Otto only to and from Emma.

Karl starts the game by lending one dollar to Emma and one to Anna. In the first graph shown in Figure 2, we can see that in the new configuration Emma is the only one with any money, since Karl has lost all his money and Anna paid back her debt. However the game is not finished since we still have Max with a debt of one dollar. In order to achieve the goal of the game they decide that Emma needs to make the second move by lending money to her group. In this way, Anna will get one dollar and make the next move.

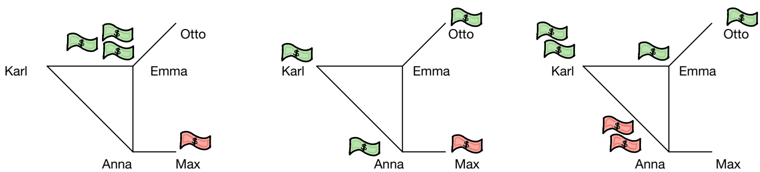

Figure 2: The first three moves of the game (left to right).

Now Max is out of debt but Anna is two dollars in debt. The aim now is to pay out Anna's debt. So Karl lends her one dollar and so does Emma. Finally Otto makes the final move leading the group to the victory! The game ends with Karl and Otto having one dollar each and the rest of the players with no money (Figure 3).

After the first game, they decide to change the rules: Max and Emma are now allowed to lend each other money. They start a new game with the same configuration of dollars as in Figure 1. What will happen now? Can they still win the game? In Figure 4 we can see the graph has changed since one edge has been added connecting Emma and Max. They start the game trying the

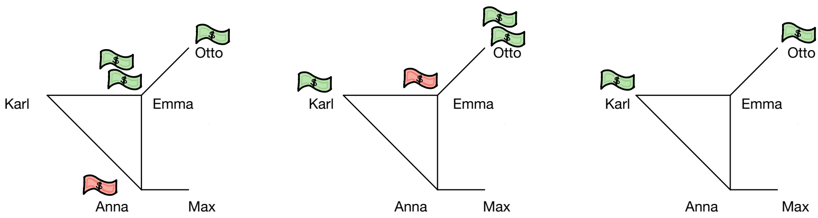

Figure 3: The last three moves of the first game.

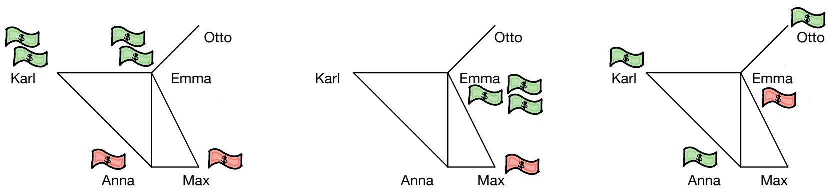

Figure 4: The starting configuration, showing one extra edge, and first two moves of the second game (left to right).

same moves as before (see Figure 4). But after the first two moves Otto notices he can finish the game by lending a dollar to Emma.

Now, in a third round, we add two further edges, one between Otto and Max and another between Karl and Otto (see Figure 5). The players try and try, but none of their efforts works out.

From these three games we notice that the shape of the graph significantly changes the outcome of the game. After adding one edge between Emma and Max, the game was still winnable and the number of moves to win the game even decreased. Adding two further edges made it an impossible game.

In the following we want to give an answer to the question:

Question 1: For which starting configurations is the game winnable?

In order to play the dollar game yourself, you can check out

https://thedollargame.io

for a free online version. Alternative introductions to the dollar game on an elementary level can be found in Numberphile's video

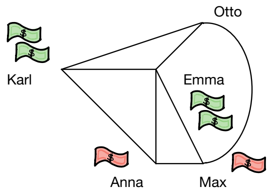

Figure 5: An unwinnable game.

https://youtu.be/U33dsEcKgeQ

and in Matt Baker's math blog [1]. For a more detailed discussion we suggest the books [4] by Corry and Perkinson and [5] by Klivans.

2 Two criteria for winnability

In this section we explain two criteria to determine whether a particular dollar game can be won. First we introduce the abstract tools needed.

We have already seen that graphs can be used to represent the different players and their connections. In order to formalise the initial configuration of dollars, we use a sum of the vertices, called a *divisor*. Let us explain by example: in the first game the divisor would be

$$2 \cdot \text{Karl} + 2 \cdot \text{Emma} + 0 \cdot \text{Otto} - 1 \cdot \text{Anna} - 1 \cdot \text{Max},$$

or, for short, $2\text{K} + 2\text{E} - \text{A} - \text{M}$. In general we give the following definition:

Definition 2.1 *Let G be a graph with n vertices $v_1, \dots, v_n$. A **divisor** D on G is a sum of the form $D = \sum_{i=1}^{n} a_i v_i$ where $a_i \in \mathbb{Z}$ is an integer. The **degree** of D is $\deg D = \sum_{i=1}^{n} a_i$.*

Each move corresponds to a change of divisor. In Figure 2 the new divisor after the first move is $\text{K}+\text{O}+\text{A}-\text{M}$. The goal of the game can be reformulated in the following way: start from D and only perform lending moves until you find a divisor D' on G whose coefficients are all non-negative. Since the total amount of money does not change after a lending move, we observe that $\deg D = \deg D'$ where D' is the divisor obtained after some lending moves on the graph, that is, the degree never changes. This leads to the first criterion for a game to be winnable.

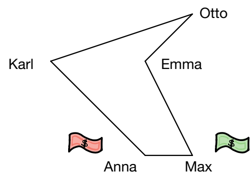

Figure 6: Example of a game to which neither the first nor second criterion apply.

First criterion: If $\deg D < 0$ the game is not winnable.

Using this criterion we can immediately spot some of the unwinnable games, however if $\deg D \geq 0$ we do not know if the game is winnable. For example consider the graph in Figure 6. The degree of the starting divisor D is zero, so the first condition cannot help us.

Definition 2.2 *Let G be a graph and denote by $|E|$ and $|V|$ respectively the number of edges and of vertices of G. The number*

$$g = |E| - |V| + 1$$

*is called the **genus** of the graph G.*

It is possible to prove that g is the number of "closed loops" in the graph. For example the genus of the graph in Figure 1 is 1 while in Figure 4 there are graphs of genus 2. The genus allows us to give a second criterion to determine whether a game is winnable or not.

Second criterion: If $\deg D \geq g(G)$ the game is winnable.

We will later see that this criterion is a special case of a deeper result. Using it we can see that the first two examples of Section 1 satisfy $\deg D \geq g(G)$. So, we could have decided that these example games were winnable without playing them. The game shown in Figure 6 has

$$\deg D = 0 < 1 = g(G),$$

so the second criterion does not help us decide whether or not this game is winnable either. (It turns out that the game in Figure 6 is not winnable. For

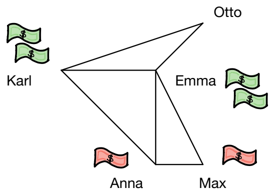

Figure 7: Graph and starting divisor for which $\deg D < g(G)$

example, if Max lends to his neighbors, Anna will be out of debt, Emma will have one dollar, and Max will be in debt one dollar. So the whole configuration just moves one step counterclockwise around the graph and Anna faces the exact same problem Max was facing earlier.) Note that there might be games where the inequality does not hold but the game is still winnable. See for example the graph and associated divisor in Figure 7. This game is winnable in 3 moves; can you find them?

3 The game is getting harder—the rank of a divisor

We now add a further complication to our game and ask the following question:

Question 2: Can you still win if k dollars are arbitrarily removed from the game?

Let D be a divisor that encodes our starting configuration of dollars. Removing k dollars means to consider the difference $D - D'$ as a starting divisor, where $D' = \sum_{i=1}^{n} a_i v_i$ is any divisor of degree k which has $a_i \geq 0$ for all $i = 1, \ldots, n$. Such a divisor is said to be *effective*.

For example consider the graph and the divisor $-K + 3E + M$ in Figure 8. The game can be won in one move (Emma lending two dollars) and the same holds if we subtract $D' = E + A + M$ from D, that is, we remove three dollars in total, one from Emma, one from Anna, and one from Max, so that our new starting divisor is $(-K + 3E + M) - (E + A + M) = -K + 2E - A$.

The choice of vertices from which we remove the dollars is crucial. In fact we cannot remove those three dollars only from Emma since then we would get an unwinnable game (can you see why?). In order to refine our question we introduce a new mathematical concept, the *rank of a divisor*.

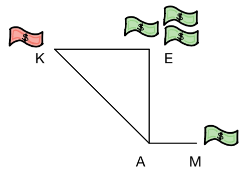

Figure 8: Graph with starting divisor $-K + 3E + M$.

Definition 3.1 *Let G be a graph and D a divisor on it. The **rank** $r(D)$ of D is the maximum k such that for all effective divisors D' of degree k the game with starting divisor $D - D'$ is winnable. If D is not winnable, we define $r(D)$ to be -1.*

In this definition we use D' to probe the starting configuration D. If the starting configuration D is winnable, we ask whether it is still winnable if we remove a dollar at an arbitrary vertex (that is, if we consider $D - D'$ as a starting configuration for an effective divisor D' of degree 1). If the answer is yes, we remove another dollar at an arbitrary vertex (so now we consider $D - D'$ for an effective divisor D' of degree 2) and check whether the game with starting configuration $D - D'$ is still winnable. We continue until we hit an effective "probing" divisor D' such that $D - D'$ is not winnable. As we have already seen in our first criterion, the game starting at $D - D'$ cannot be won if $\deg(D - D') < 0$. That is why this process has to end in finite time.

In our example in Figure 8 we see that $r(D) \leq 3$ since (as already noted) for $D' = 3E$ the game starting with $D - D'$ is unwinnable. However for any effective divisor D' of degree 2 the game with starting configuration $D - D'$ is still winnable. You can verify this by playing the game for all 10 effective divisors of degree 2, although it might take some time.

We can see already from this example that computing the rank of a divisor, hence answering Question 2, is not easy. To help us in this we introduce a new version of a classical result in algebraic geometry called the Riemann-Roch Theorem. Before stating it we need one last definition.

Definition 3.2 *Let G be a graph. The **valence** of a vertex v_i in G is the number of edges emanating from v_i; it is denoted by $\mathrm{val}(v_i)$. The **canonical divisor** K_G of the graph G is the divisor*

$$K_G = \sum_{i=1}^{n} (\mathrm{val}(v_i) - 2) v_i \ .$$

In our graph in Figure 4 the canonical divisor is $-O + 2E + A$, while for the graph in Figure 8 it is $A - M$.

Riemann-Roch Theorem 3.3 (M. Baker and S. Norine [3]) *For a divisor D on a finite graph G the following equality holds:*

$$r(D) - r(K_G - D) = \deg D - g + 1.$$

This theorem explains the second criterion we gave in Section 2. In fact, if $\deg D \geq g$ then we can rearrange this to obtain $\deg D - g + 1 \geq 1$ and using the formula in the theorem we get $r(D) - r(K_G - D) \geq 1$. Now from the definition of rank we see that the rank of a divisor is always at least equal to -1, hence $r(D) \geq 1 + r(K_G - D) \geq 0$ and the game with starting divisor D is winnable.

4 The big picture

Theorem 3.3 is named after a famous theorem that goes back to work of Bernhard Riemann (1826–1866) and Gustav Roch (1839–1866). Their work, *a priori*, has nothing to do with the dollar game. Their main interest was the study of *Riemann surfaces*. In mathematically precise terms, a Riemann surface is a connected one-dimensional complex manifold. One can think of these objects as two-dimensional surfaces together with a collection of charts that allow us identify a small neighborhood of each point with a subset of the complex plane $\mathbb{C}$. Here "connected" means that we can move from any point to any other point without leaving the surface, intuitively, that it is all one piece.

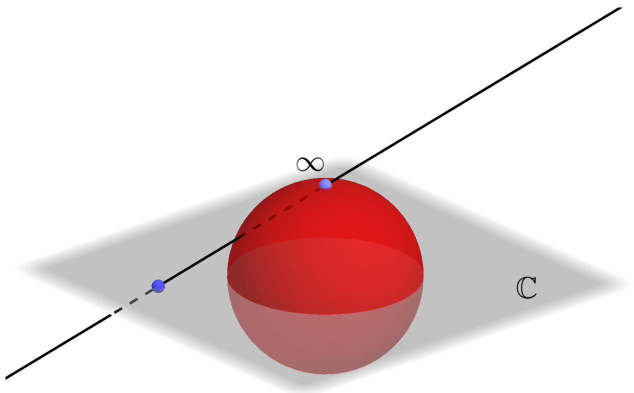

Figure 9: The Riemann sphere, the stereographic projection to the complex plane $\mathbb{C}$ and the point at infinity.

The simplest Riemann surface is the *Riemann sphere*, as pictured in Figure 9. Technically, it is the one-dimensional complex projective line $\mathbb{P}^1(\mathbb{C})$. Topologically $\mathbb{P}^1(\mathbb{C})$ is a sphere. We may also think of $\mathbb{P}^1(\mathbb{C})$ as the complex plane $\mathbb{C}$ together with a point ∞ at infinity. In analogy with divisors on graphs, a *divisor* on $\mathbb{P}^1(\mathbb{C})$ is a finite formal sum $D = \sum_{i=1}^{n} a_i p_i$ of points p_i on $\mathbb{P}^1(\mathbb{C})$, where again the numbers a_i are integers and each p_i is a point on $\mathbb{P}^1(\mathbb{C})$.

We would now like to define what it means for two divisors to be *equivalent*. In the graph case, two divisors are equivalent if you can pass from one to the other by a finite number of lending/borrowing moves of the game. You can also define two divisors D and D' on a graph to be equivalent if there is a continuous piecewise linear function f with integral slopes on the graph such that

$$D - D' = \sum_{v \in V(G)} \mathrm{ord}_v(f) v,$$

where $\mathrm{ord}_v(f)$ is the sum of outgoing slopes at the vertex v. The way to understand this is to consider the unique piecewise linear function whose value on a given vertex v is 1 and whose value on all other vertices is zero. Then adding $\sum_{v \in V(G)} \mathrm{ord}_v(f)$ to a divisor D corresponds to a move of the game, since it will move val(v)-many dollars from v to adjacent vertices.

On the Riemann sphere, we say that two divisors D and D' are equivalent if there is a rational function $f(x) = \frac{g(x)}{h(x)}$, where both $g(x)$ and $h(x)$ are polynomials, such that

$$D - D' = \sum_{p \in \mathbb{P}^1(\mathbb{C})} \mathrm{ord}_p(f) p.$$

Here, to define $\mathrm{ord}_p(f)$, we first recall that every polynomial function $g(x)$ of degree n has exactly n complex zeros, which may be repeated. The number of times a particular zero appears in the factorisation of g is called the *order* of the zero. Then, $\mathrm{ord}_p(f)$ is defined to be the difference of the zero order of $g(x)$ at the point p and the zero order of $h(x)$ at the point $p \in \mathbb{C}$ and as $\deg(h) - \deg(g)$ at the point $p = \infty$. In other words, if p is a zero of the numerator of our rational function, $\mathrm{ord}_p(f)$ is a positive integer, if p is a zero of the denominator, $\mathrm{ord}_p(f)$ is a negative integer, and otherwise it is equal to 0. Thus, we obtain a finite sum also in this case. We may think of $\sum_{p \in \mathbb{P}^1(\mathbb{C})} \mathrm{ord}_p(f) p$ in analogy with a finite number of lending moves that are governed by the rational function $f(x)$.

It is a very surprising consequence of the work of Riemann and Roch that, as soon as these very geometric Riemann surfaces are compact, that is, as soon as they do not have any boundary or open ends, they always arise as what is called an *algebraic curve*. This means that they can be described as the set of zeroes of some finite collection of polynomials with coefficients in $\mathbb{C}$ in a (possibly rather high-dimensional) projective space $\mathbb{P}^r$. Such realizations as algebraic curves are associated to divisors on a Riemann surface. The classical Riemann-Roch-Theorem is concerned with relations between the possible dimensions r of the projective space associated to a divisor, which is also known as the *rank* of the divisor. It looks formally the same as Theorem 3.3 above. We refer the interested reader to [6] for an introduction to Riemann surfaces with a focus on the connection to algebraic curves.

The dollar game on a graph was originally discovered in the context of abelian sandpile models, that is, in the context of mathematically modelling the dynamics of avalanches of piles of sand (see, for instance, [4] for an entry into this topic). Only recently it has become clear that the connection between divisor theory on graphs and the classical theory on Riemann surfaces goes far beyond a mere analogy. One can use divisors on graphs to understand the structure of linear systems of divisors on Riemann surfaces and vice versa. A good entry point to this tantalizing new story, which is now considered a part of what is known as "tropical geometry", is the survey paper [2].

Image credits

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Sara Lamboglia *is Postdoc at Goethe Universität Frankfurt am Main.*

Martin Ulirsch *is a Junior Professor at Goethe Universität Frankfurt am Main.*

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Mathematisches Forschungsinstitut
Oberwolfach gGmbH
Schwarzwaldstr. 9 – 11
77709 Oberwolfach
Germany

Director
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