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Reflections on hyperbolic space

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In school, we learn that the interior angles of any triangle sum up to π. However, there exist spaces different from the usual Euclidean space in which this is not true. One of these spaces is the "hyperbolic space", which has another geometry than the classical Euclidean geometry. In this snapshot, we consider the geometry of hyperbolic polytopes, for example polygons, how they tile hyperbolic space, and how reflections along the faces of polytopes give rise to important mathematical structures. The classification of these structures is an open area of research.

1 What is hyperbolic space?

Hyperbolicity is an interesting concept in geometry. A hyperbolic space (such as a hyperbolic surface in the 2-dimensional case) is defined by the property that it has negative curvature. To get a sense of what this means in terms of geometry, compare this to the more familiar Euclidean space, which has zero curvature everywhere. For a surface, curvature describes, loosely speaking, the deviation from being a flat plane. On a surface with negative curvature, the sum of interior angles of a triangle on the surface is less than π radians. On the contrary, on a surface with positive curvature, the sum is greater than π radians.

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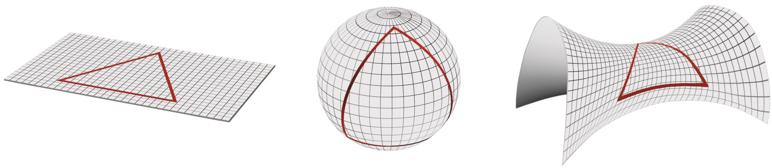

Figure 1: Triangles on surfaces with different curvatures. A flat plane has zero curvature, a sphere has positive curvature, and a saddle has negative curvature.

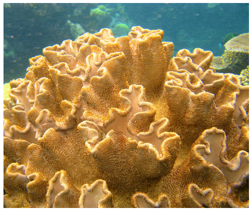

Figure 2: The negative curvature in the folds of the filter feeding coral reef plays an important role in maximizing its surface area.

In Figure 1, we see examples for the different curvatures and in Figure 2, we see an object of negative curvature in reality.

Studying Euclidean space, and its associated Euclidean geometry, begins with a set of grounding axioms to describe points, lines, planes, and shapes as we experience them right around us. Hyperbolic space has different axioms to describe these objects.

Therefore the change in curvature between Euclidean and hyperbolic space results from an important geometric difference between the two spaces. This difference can be detected by comparing the properties of parallel lines. In Euclidean geometry, we have the parallel postulate:

> Consider a line ℓ and a point P not on ℓ. In the plane containing both ℓ and P, there is **exactly one** line passing through P that does not intersect ℓ.

This line is the unique parallel line to ℓ which contains the point P.

In hyperbolic space, which exhibits hyperbolic geometry, the parallel postulate is replaced with the following:

> Consider a line ℓ and a point P not on ℓ. In the plane containing both ℓ and P, there are **many** lines passing through P that do not intersect ℓ.

Therefore the concept of unique parallel lines, as we know it from everyday life, no longer works in hyperbolic space. We now denote the 2-dimensional hyperbolic space by $\mathbb{H}^2$, which is also called a *hyperbolic plane*. In a hyperbolic plane, the negative curvature means that every point is a saddle point, which says that all tangents are parallel to the surface, but the point itself is not a local extremum. The appearance of a saddle point can be visualized by its embedding in Euclidean space. An *embedding* is one instance of a mathematical structure contained in another instance. More precisely, embedding the saddle point in Euclidean space means that there exists an injective and structure-preserving map from hyperbolic space to Euclidean space. So the saddle point can be drawn in Euclidean space as seen in Figure 1.

A theorem of the influential German mathematician David Hilbert (1862–1943) from 1901 [8] states that a hyperbolic plane cannot be entirely embedded into 3-dimensional Euclidean space – so we cannot draw a hyperbolic plane in Euclidean space – but partial embeddings of spaces with negative curvature into spaces with zero curvature can be visualized. So the visualizations of hyperbolic space seen in Figure 1 and Figure 2 are just partial embeddings of $\mathbb{H}^2$ in Euclidean space. This is convenient because our eyes and brains are most accustomed to interpreting visual input in Euclidean space, but inconvenient because the true geometry of hyperbolic space is lost in these embeddings.

One practical way to visualize the complete hyperbolic plane equipped with all of its geometric properties is by the Poincaré disk model, named after the French mathematician Henri Poincaré (1854–1912). In this model, every point of $\mathbb{H}^2$ is represented by a point in the unit disk, and straight lines in $\mathbb{H}^2$ are given by curved lines in the unit disk that intersect orthogonally with the circle that is the boundary of the disk, see Figure 3.

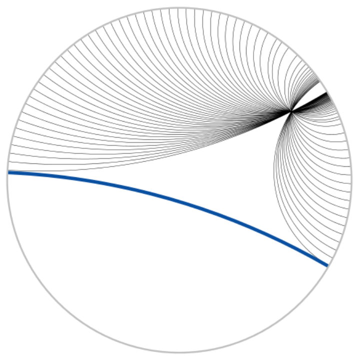

Figure 3: In this Poincaré disk model of $\mathbb{H}^2$, the thin black lines all pass through a common point and are all parallel to the bold blue line.

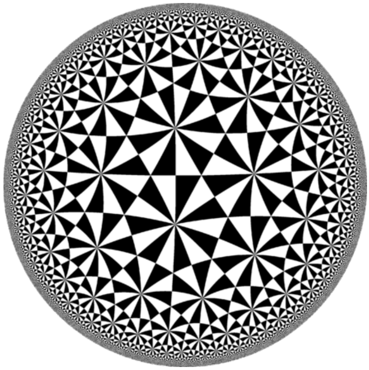

Figure 4: A tiling of the hyperbolic plane by triangles with angle measurements $\frac{\pi}{2}$, $\frac{\pi}{3}$, and $\frac{\pi}{7}$.

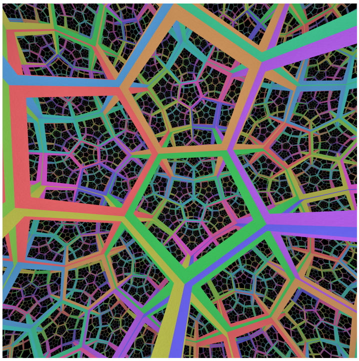

Figure 5: A tiling of $\mathbb{H}^3$ by right-angled dodecahedra.

In general, the curvature condition of hyperbolic space means that polytopes in hyperbolic space do not satisfy the familiar geometric constraints of Euclidean space. For instance in $\mathbb{H}^2$, the sum of the interior angles of a triangle is such a constraint that is different in Euclidean space. We will explore this further. Using the Poincaré disk model, we can visualize polygons on the hyperbolic plane. For example, let us consider a tiling of $\mathbb{H}^2$ by hyperbolic triangles, as seen in Figure 4. All of the triangles in Figure 4 share the common angle measurements $\frac{\pi}{2}$, $\frac{\pi}{3}$, and $\frac{\pi}{7}$ whose sum is $\frac{41\pi}{42}$, hence smaller than π. The defining property of *tilings* is that they consist of geometric shapes such as triangles which cover the whole disk without overlapping and with no gaps.

Similarly, we may consider tilings of higher dimensional hyperbolic space by polytopes. Polytopes are a generalisation of the 2-dimensional polygons. For an example in dimension $n = 3$, Figure 5 shows a tiling by right-angled dodecahedra, 3-dimensional polytopes with 12 faces.[2] For dimensions greater than 3, these hyperbolic tilings become very hard to visualize. The main focus of the remainder of this snapshot will be the mathematical underpinning of higher dimensional tilings of hyperbolic space.

2 Reflections in hyperbolic space

Consider now the n-dimensional hyperbolic space, denoted by $\mathbb{H}^n$. An *isometry* of $\mathbb{H}^n$ is a map from $\mathbb{H}^n$ to itself that preserves distances[3]. An example of an isometry of the hyperbolic plane $\mathbb{H}^2$ is the reflection across a line. The set of all isometries, which we denote by $\mathrm{Isom}(\mathbb{H}^n)$, forms a mathematical structure called a group. Roughly speaking, a group is a set in which it is possible to combine any two elements to obtain a new one, such as composing two maps to get a new map. We will not say much more here concerning the precise definition of a group. What we would like to explore more deeply, is the relationship between elements of $\mathrm{Isom}(\mathbb{H}^n)$ and the tilings described in the previous section.

Consider the triangular tiling shown in Figure 4, and consider the set of isometries corresponding to reflections of the triangles across their edges. This set of isometries forms a subgroup of $\mathrm{Isom}(\mathbb{H}^n)$, called the *Hurwitz triangle group*. In dimension $n = 3$, we can consider the analogous construction by looking at the subgroups generated by reflections of 3-dimensional polytopes across their faces, like those seen in Figure 5.

A subgroup of $\mathrm{Isom}(\mathbb{H}^n)$, which contains reflections on the faces of an n-dimensional hyperbolic polytope is called a *hyperbolic reflection group*. The question of classifying hyperbolic reflection groups has been a driving force behind much research in an area called *geometric group theory* over the past century. The goal of classifying is to find properties to divide objects into classes. Hyperbolic reflection groups in dimension 2 were already completely understood in the late 1800s from work by Poincaré [11] and the German mathematician Walther von Dyck (1856–1934) [7]. Work in dimension 3 did not follow until nearly 100 years later with E. M. Andreev [3, 4] and a great part still remains to be understood in higher dimensions. For an overview of results in the classification of hyperbolic reflection groups, the reader is directed to the 2016 survey paper on this topic by Mikhail Belolipetsky [5].

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[2] This image is a screen shot captured from the computer game "Curved Spaces" by Jeff Weeks which includes a 3-dimensional hyperbolic space visualizer.

[3] Details about the distance in hyperbolic space and more concrete details and examples in hyperbolic geometry can be found in [2].

In the last section, we discuss an open question related to the classification of hyperbolic reflection groups in higher dimensions.

3 The classification problem

As it is often the case in mathematics, the problem of classifying hyperbolic reflection groups is best broken down into components, classifying families of such groups satisfying some additional conditions. One such condition is *arithmeticity*, which imbues the natural group theoretic structure of the reflection group with some number theoretic conditions.

For our purposes, the rather complex definition of arithmeticity can be best illustrated through one particular example of an "arithmetic subgroup of the simplest type", which arises in connection to quadratic forms. We want to think of quadratic forms as polynomials (in possibly several variables) where all terms are squares or products of exactly two variables. These polynomials can have coefficients in the field of rational numbers $\mathbb{Q}$ or, for example, in a *number field* which is a finite extension of $\mathbb{Q}$. Without introducing too many technical details, we note that an *arithmetic subgroup of the simplest type* is a subgroup of $\mathrm{Isom}(\mathbb{H}^n)$ that is equal to the set of integral isometries of a quadratic form (subject to some arithmetic constraints) over a number field.

It is a result of Ernest Vinberg [12] that a hyperbolic reflection group can only be arithmetic if it is an arithmetic subgroup of the simplest type. Therefore, to classify the arithmetic hyperbolic reflection groups it is sufficient to classify those which are arithmetic of simplest type. This is an active topic of research, determining which quadratic forms over which number fields give rise to arithmetic hyperbolic reflection groups, particularly in finding the possible number fields.

Moreover, a helpful tool to understand the classification problem is another important theorem of Vinberg [13], namely, that there are no arithmetic hyperbolic reflection groups of $\mathrm{Isom}(\mathbb{H}^n)$ for $n \geq 30$. So our research for arithmetic reflection groups needs only to extend up to 30-dimensional hyperbolic space. Of course, this is already far too large to visualize, but it should still be possible to explore the visualization of the mathematics computationally in these dimensions. It is still an open problem, to determine whether Vinberg's result is still true even if we have groups that are not arithmetic.

Instead of asking to find all arithmetic hyperbolic reflection groups, we can just ask for those which are *maximal*, that is, they are not properly contained in any other hyperbolic reflection group. In this case, we know that there are only finitely many groups satisfying this condition, a result first proven by Viacheslav Nikulin in 2007 [10]. Further advances in this area, specifically in determining the corresponding number field for an arithmetic hyperbolic reflection group, have been made in [1], [6], and [9], among many others.

Image credits

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Figure 5 Screenshot of “Curved Spaces”. Author: Jeff Weeks. Licensed under Freeware (GNU General Public License) http://www.geometrygames.org/CurvedSpaces, visited on July 30, 2019.

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