Snapshots of modern mathematics
from Oberwolfach

№ 19/2019

The Mathematics of Fluids and Solids

Barbara Kaltenbacher • Igor Kukavica
Irena Lasiecka • Roberto Triggiani
Amjad Tuffaha • Justin Webster

Fluid-structure interaction is a rich and active field of mathematics that studies the interaction between fluids and solid objects. In this short article, we give a glimpse into this exciting field, as well as a sample of the most significant questions that mathematicians try to answer.

1 The Mathematics of Fluids

In 1901, the Wright brothers successfully flew the first airplane. At the heart of their success was a simple formula that they used to compute the lift force as a function of velocity. The Bernoulli formula

$$p + \frac{1}{2}\rho v^2 = C, \tag{1}$$

where p is the air pressure, ρ is the density, v is the fluid velocity and C is an appropriate constant, describes a physical law known as *Bernoulli's principle*,[1] which postulates that the lift on the airfoil is proportional to the square of the difference in air flow velocity on the top and bottom of the wing, induced by the geometry of an airfoil. The successful use of this simple formula was

[1] After Swiss mathematician Daniel Bernoulli (1700–1782).

another triumph for mathematics, setting in motion the development of the field of aerodynamics in which mathematical modelling and computation play a central role.

Fluid dynamics, the mathematical theory that describes how fluids flow, goes back to 1757, when Leonhard Euler (1701–1783) proposed the equation

$$v_t + (v \cdot \nabla)v + \nabla p = 0, \tag{2}$$

that describes the general motion of a fluid. Here, v is the fluid velocity and p is the pressure. Here, ∇ is the vector defined as $\nabla := (\partial/\partial_x, \partial/\partial_y, \partial/\partial_z)$. The equation, which was named after him, governs the velocity and pressure of the fluid, both depending on time and space. Euler used the principle of conservation of momentum to derive the equation but the equation still failed to account for internal forces within the fluid, which we refer to as *viscosity*. Such forces provide a friction mechanism slowing down the motion, and this is what distinguishes the motion of different fluids such as honey and water. Thus, the Euler equation describes the motion of an ideal fluid which has no viscosity. The Bernoulli equation was derived by Euler from the Euler equations under the assumptions of low viscosity and steady flow.

The question of how to incorporate the internal forces in the fluid into the equation remained a standing problem until it was resolved by Adhémar Jean Claude Barré de Saint-Venant (1797–1886). Ironically[2], the modified equations were named after French engineer Claude Louis Marie Henri Navier (1785–1836) who, at the time, was the chief designer of bridges in France, and after Sir George Gabriel Stokes (1819–1903), the famous English physicist. Therefore, they came to be known as the Navier-Stokes equations.

Phenomena involving fluid flow are very interesting and complex and have intrigued people since antiquity. Today, research on the mathematics of fluids is very active and is relevant to the study of many phenomena whether it is the atmosphere, the ocean or blood flow in the arteries.

2 The Mathematics of Solids

Mathematics is indispensable for the description of how different solids deform under the application of external forces. This is the object of study of a whole field within mechanical engineering and material science, known as solid mechanics. Take for example the vibrations of the drum membrane or the vibrations generated by plucking the strings of a guitar. Such elastic motions are described by *partial differential equations* of the same kind used to study

[2] It often happens that mathematical discoveries are not named after the first people who produced them, for a variety of reasons.

wave propagation in a medium. Partial differential equations are indispensable for the study of the natural world, be it heat transfer, wave propagation, fluid flow, or many other phenomena. These are equations involving an unknown function of more than one variable (typically space and time), and they relate the function to its derivatives. A reference for this topic is Snapshot 006/2018 Fast Solvers for Highly Oscillatory Problems by Alex Barnett.

Interesting solid mechanics arise from the study of beams and plates, which are used to model airplane wings, bridges, and many other structures. Mathematics can describe these dynamics very accurately and is successfully used to study deformations that such structures undergo and to provide predictions about potentially new phenomena and design strategies for novel applications. The mathematical description of all these phenomena in various domains – or areas of applicability – involves partial differential equations.

3 The Mathematics of Fluid-Structure interactions

3.1 Motivating the field of Fluid-Structure interactions

A more intriguing and intricate subject is the interaction of fluids with various deformable structures, such as aircraft wings, submarines, beams, plates, bridge structures, blood vessels, pipes etc. An accurate description of such interaction dynamics is instrumental in understanding various physical phenomena. The field of fluid-structure interaction constitutes a well-developed research area within mechanical and aerospace engineering as well as applied mathematics. This phenomenon, which is manifested in a wide range of different mechanical and natural processes, occurs when the flow of a fluid causes deformation of a solid structure, which in turn interferes with the flow.

3.2 Problems within Fluid-Structure interactions

During the last decade, we have witnessed an increased interest in fluid-structure interaction modelling and control in various domains.

This type of interaction is precisely the source of two important forces in aerodynamics, lift and drag, without which no airplane can fly [1, 2]. For example, the dynamics that arise because of the interactions between two different systems, are precisely what causes bending in the wing of an aircraft and the blades of a helicopter, and thus they have serious implications in aerodynamic design and control. In addition, understanding these dynamics is crucial for correct modelling and analysis of oil reservoirs and pipes. Moreover, modelling and simulation of blood flow in the arteries is instrumental in many biomedical applications such as the design of artificial valves.

A problem of particular interest to engineers and mathematicians concerns the stability of deformable structures in a fluid flow. For instance, the prevention of extreme vibration is a primary goal in the design of various objects from bridges and buildings to aircrafts and motor vehicles. Such instabilities, whether they are caused by the flutter (sustained vibrations) or resonances (growing vibrations usually induced by a periodic force like wind gust), can cause major structural damage and even failure if sustained for a long time. Consider for instance the famous Tacoma Narrows bridge incident [7], in which the bridge underwent large oscillations that led eventually to the collapse of the bridge (see Figure 1). In the aerospace industry, a new airplane must be subjected to a special test called flutter test, to determine whether it can endure the flying conditions up to its maximum speed and maximum altitude. In contrast, for some other applications such as energy harvesting from wind, such instabilities are desirable and the design goals are concerned with their generation rather than their prevention [5].

For mathematicians, the phenomenon of fluid-structure interaction provides an abundance of interesting mathematical problems. There are many different models used to describe fluid-structure interaction depending on the particular phenomenon or application in question. To describe a fluid flow, we typically use the Navier-Stokes equations

$$v_t - \nu\Delta v + (v \cdot \nabla)v + \nabla p = 0 \tag{3}$$

or the Euler equations

$$v_t - (v \cdot \nabla)v + \nabla p = 0, \tag{4}$$

where ν represents the fluid viscosity and $\Delta := (\partial^2/\partial x^2, \partial^2/\partial y^2, \partial^2/\partial z^2)$, and their variations depending on the type of fluid and the flow regime. To describe structural motion, the equations used depend on the type of structure, but are typically reminiscent of the equations used to study wave propagation, such as hyperbolic equations.

The interaction between these different phenomena is typically described by a set of conditions known as *boundary conditions*. In the theory of differential equations, in order to uniquely determine the solutions, a set of constraints (known as boundary conditions) must be given on the initial values of the solutions and their derivatives, that is, on the values that the functions and their derivatives take within specific domains of the variable space.

3.3 Open questions

Some of the questions that mathematicians aim to answer involve existence of solutions to these equations, their properties and long-term behavior (the behavior of the solutions after a lot of time has elapsed). Other important questions pertain to the possibility of imposing controls on the structure to achieve particular objectives such as preventing extreme vibrations or inducing them. While these questions are mathematical in nature, they are directly related to control engineering and will help to understand further structural instabilities. On the other hand, recent collaborative work by mathematicians and physicians has resulted in simulating the interaction between blood flow and arterial walls treated with vascular prostheses called *stents* [6]. A stent is a metallic mesh-like tube used to prop open a clogged artery, and the purpose of these mathematically based simulations is to assess the stability and performance of different stents and ultimately decide which stent is most appropriate.

On the fluid dynamics side, there are also many standing problems regarding the correct modelling of flows in the region in the vicinity of a structure, known as the "boundary layer". This is related to another shortcoming of the Euler equation, which was observed by the famous physicist and mathematician Jean-Baptiste le Rond d'Alambert (1717–1783), in what is known as d'Alembert's paradox. D'Alambert used the equations to demonstrate that the equations failed to account for drag forces experienced by an object moving in airflow, even when the viscosity is very small. This is a paradox because empirical evidence shows that there is drag while the equations do not account for this drag. Ludwig Prandtl (1875–1953) proposed one solution to this paradox by introducing boundary layer equations to model the flow near the boundary,such as the surface of the object, while retaining the Euler equations as a good model for the flow away from the boundary. Such questions about the correct modelling and size of the boundary layer constitute a whole field of very active mathematical and scientific research. Other questions also relate to the possible development of *shocks*[3] due to flow separation when the flow meets a structure such as an airfoil, and how to mathematically capture these effects. These questions are mathematically very challenging, and they are also the subject of whole fields of experimental and computational research within fluid and aerodynamics [8, 3].

The famous physicist Eugene Paul Wigner (1902–1995) once said that "The miracle of the appropriateness of the language of mathematics for the formulation of the laws of physics is a wonderful gift which we neither understand nor deserve. We should be grateful for it and hope that it will remain valid in future research and that it will extend, for better or for worse, to our pleasure, even though perhaps also to our bafflement, to wide branches of learning."
Regardless of this mystery, the beauty of the mathematics describing the natural world will continue to dazzle and amaze us for time to come.

[3] A shock occurs when the local speed flow abruptly exceeds the speed of sound.

Figure 1: Collapse of the Tacoma Bridge [4].

Acknowledgments

The authors thank the MFO for hosting them, and the organizers of the Workshop Mathematical Theory of Evolutionary Fluid-Flow Structure Interactions, held at the Mathematisches Forschungsinstitut Oberwolfach from the 20th to the 26th of November 2016, to which this snapshot is related.

Image credits

Fig. 1 Author of the image is Barney Elliott [4].

References

[1] J. D. Anderson, *Fundamentals of aerodynamics*, 4th ed., McGraw-Hill, 2007.

[2] H. J. Bungartz and M. Schäfer, *Fluid-structure interaction: Modelling, Simulation, Optimization*, Springer-Verlag, 2006.

[3] P. Constantin, *Some Open Problems and Research Directions in the Mathematical Study of Fluid Dynamics*, Mathematics Unlimited – 2001 and Beyond, Springer, 2001.

[4] Barney Elliott, *Image of The Tacoma Narrows Bridge Collapsing in 7 November 1940*, https://upload.wikimedia.org/wikipedia/commons/4/4a/Tacoma-narrows-bridge-collapse.jpg.

[5] J. Webster, *The invisible power of flutter*, https://theconversation.com/the-invisible-power-of-flutter-from-plane-crashes-to-snoring-to-free-energy-91796.

[6] ———, *UH Mathematician's Work to Become Basis for Open-Heart Surgery Alternative*, https://bionews-tx.com/news/2014/08/uh-mathematicians-work-to-become-basis-for-open-heart-surgery-alternative.

[7] Wikipedia, *Tacoma narrows bridge (1940) — Wikipedia, the free encyclopedia*, 2019, https://en.wikipedia.org/wiki/Tacoma_Narrows_Bridge_(1940), visited on August 3, 2019.

[8] V. I. Yuodovich, *Eleven great problems of mathematical hydrodynamics*, Moscow Mathematical Journal **3** (2003), no. 2, 711–737.

Barbara Kaltenbacher *is a professor of mathematics at the University of Klagenfurt, Klagenfurt, Austria.*

Igor Kukavica *is a professor of mathematics at the University of Southern California, Los Angeles, CA, USA.*

Irena Lasiecka *is a professor of mathematics at the University of Memphis, Memphis, TN, USA.*

Roberto Triggiani *is a professor of mathematics at the University of Memphis, Memphis, TN, USA.*

Amjad Tuffaha *is a professor of mathematics at the American University of Sharjah, Sharjah, United Arab Emirates.*

Justin Webster *is a professor of mathematics at the University of Maryland, Balitomore County, MD, USA.*

Mathematical subjects
Analysis

Connections to other fields
Physics

DOI
10.14760/SNAP-2019-019-EN

Snapshots of modern mathematics from Oberwolfach provide exciting insights into current mathematical research. They are written by participants in the scientific program of the Mathematisches Forschungsinstitut Oberwolfach (MFO). The snapshot project is designed to promote the understanding and appreciation of modern mathematics and mathematical research in the interested public worldwide. All snapshots are published in cooperation with the IMAGINARY platform and can be found on www.imaginary.org/snapshots and on www.mfo.de/snapshots.

ISSN 2626-1995

Mathematisches Forschungsinstitut
Oberwolfach gGmbH
Schwarzwaldstr. 9 – 11
77709 Oberwolfach
Germany

Director
Gerhard Huisken

Mathematisches Forschungsinstitut Oberwolfach

Member of Leibniz Leibniz Association

IMAGINARY open mathematics