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Configuration spaces and braid groups

Rita Jiménez Rolland • Miguel A. Xicoténcatl

In this snapshot we introduce *configuration spaces* and explain how a mathematician studies their 'shape'. This will lead us to consider paths of configurations and braid groups, and to explore how algebraic properties of these groups determine features of the spaces.

1 What is algebraic topology?

1.1 Topological spaces

Topology[1] is a branch of mathematics that formalizes how to describe shapes that appear in nature (or are defined abstractly!) and gives us mathematical tools to verify our intuition about them. The objects of study are called *topological spaces.*

Basic examples of topological spaces are curves, surfaces and their higher dimensional analogues, known as *manifolds.* An n-dimensional manifold M is a topological space that locally[2] looks like the Euclidean space $\mathbb{R}^n$. Thus 1-dimensional manifolds are just curves (like the circle or the line), and 2-dimensional manifolds are surfaces (like balloons or doughnuts or pretzels).

In the painting in Figure 1, we can identify examples of "natural" topological spaces. A typical example of an n-manifold is the unit sphere $S^n = \{x \in \mathbb{R}^{n+1} \mid$

[1] For interesting answers to the question "what is topology?" we recommend the series *Defining topology through interviews* by R. Boyld [6].

[2] Locally means if we "zoom in" to a small enough region.

Figure 1: Topological spaces that appear in nature.

$\|x\| = 1\}$ in $\mathbb{R}^{n+1}$, the case $n = 2$ being the familiar two-dimensional sphere S^2 in $\mathbb{R}^3$.

1.2 Equivalence of topological spaces

We will consider that two topological spaces X and Y are equivalent if they can be deformed into each other through *continuous* deformations: twisting, stretching, contracting, etc. Tearing, however, is not allowed.

We formalize this as follows:

- Two continuous functions $f, g : X \to Y$ are said to be *homotopic* if there is a continuous mapping $H : X \times [0, 1] \to Y$ (a homotopy) such that $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$ for all $x \in X$. In that case we write $f \simeq g$.
- The spaces X and Y are *homotopy equivalent* if there are continuous functions $f : X \to Y$ and $g : Y \to X$ (called *homotopy equivalences*) such that:

$$g \circ f \simeq id_X \qquad \text{and} \qquad f \circ g \simeq id_Y .$$

Here we use the notation id_X for the identity function on the space X.

If there is a *homeomorphism* $f : X \to Y$, which is a continuous bijection with a continuous inverse, the spaces X and Y are said to be *homeomorphic*.

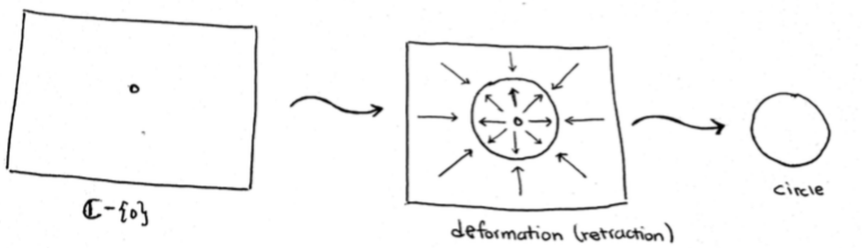


Figure 2: The 'punctured plane' $\mathbb{C} \setminus \{0\}$ can be continuously deformed (without tearing) into the circle S^1.

In particular, two homeomorphic spaces are homotopy equivalent (the converse is not true).

One of the fundamental questions in topology is: *what is the classification of topological spaces up to homotopy equivalence?* In other words, we are interested in arranging topological spaces into classes where each class contains all topological spaces that are homotopic to each other. For example, the plane (you can think of it as $\mathbb{R}^2$ or $\mathbb{C}$) can be continuously shrunk down to a point and the 'punctured plane' $\mathbb{C} \setminus \{0\}$ is homotopy equivalent to a circle S^1 (see Figure 2). The plane with two punctures $\mathbb{C} \setminus \{0, 1\}$ can be continuously retracted to the space $S^1 \vee S^1$, which is homeomorphic the 'figure-eight' space.[3] However, how do we know whether $\mathbb{C} \setminus \{0, 1\}$ is equivalent or not to $\mathbb{C}$?

1.3 Algebraic topology

Algebraic topology uses tools from abstract algebra to study and classify topological spaces. The classical method of algebraic topology consists in the construction of *algebraic invariants*[4] of topological spaces allowing us to translate geometrical problems into algebraic ones, possibly easier to analize.

One of these invariants is the so called *fundamental group*, introduced in 1985 by Jules Henri Poincaré (1854–1912) in his attempt to classify surfaces and manifolds up to homeomorphism. Namely, if we start with a space X and a point $\mathbf{x_0} \in X$, in order to identify algebraically certain 'holes' in the space one can consider paths in X that start and end at the point $\mathbf{x_0}$, in other words, continuous functions $\gamma : [0, 1] \to X$ such that $\gamma(0) = \gamma(1) = \mathbf{x_0}$. We call these

[3] $S^1 \vee S^1$ denotes the *wedge sum* of two circles: a topological space obtained by gluing together two circles at a single point.

[4] An *algebraic invariant* of a space is a quantity or an algebraic object, such as a group, that does not change under homeomorphism or homotopy equivalence.

paths *based loops* and think of $\gamma(t)$ as the position in X at time t.

Two loops are equivalent if we can deform continuously one to another, keeping the point $\mathbf{x_0}$ fixed, and we say that the loops are *homotopic*. We can think that the curves that describe the paths are made of a rubber string that can be stretched and deformed without tearing (see for example Figure 3). Moreover, two loops γ_1 and γ_2 can be combined to get a new loop $\gamma_1 * \gamma_2$ by concatenation. First, trace the path γ_1 from $\mathbf{x_0}$ back to $\mathbf{x_0}$ and then trace the loop γ_2 going in both cases "twice as fast" in the time $t \in [0, 1]$. Notice that the resulting path $\gamma_1 * \gamma_2$ is a loop based at $\mathbf{x_0}$ and moreover $\gamma_1 * \gamma_2(1/2)=\mathbf{x_0}$.

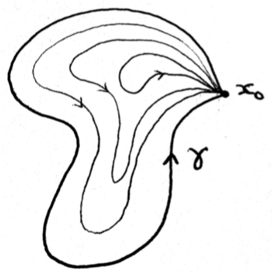


Figure 3: All based loops in the plane are homotopic to the constant loop.

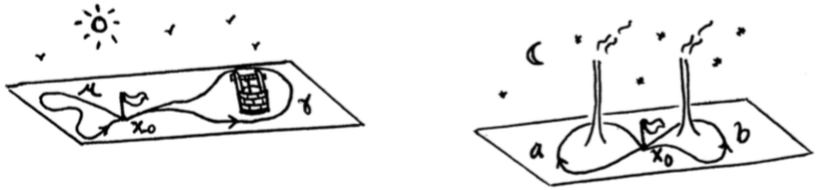
Definition 1. The *fundamental group* $\pi_1(X, \mathbf{x_0})$ is the set of homotopy classes of loops in X based at $\mathbf{x_0}$ with the operation of concatenation of loops.

This concatenation is well-defined on homotopy classes of loops and gives $\pi_1(X, \mathbf{x_0})$ the structure of a group[5]: the operation is associative, it has an identity element (what happens when you concatenate with the constant loop?) and there are inverses (the loops that go backwards!). Fundamental groups help us when classifying topological spaces: if two spaces X and Y are homotopy equivalent, their corresponding fundamental groups should be isomorphic groups.

Example 1. Let us provide a list of examples of fundamental groups.

- In the plane $\mathbb{C}$ all loops can be deformed to the constant loop since there are no holes in the plane (see Figure 3). The fundamental group is thus the trivial group.

[5] For the basics of group theory we refer the reader to the Snapshot 005/2016 *Symmetry and characters of finite groups* by Eugenio Gianelli and Jay Taylor or the Snapshot 003/2018 *Computing with symmetries* by Colva M. Roney-Dougal.



(a) (b)

Figure 4: (a) In $\mathbb{C} \setminus \{0\}$, the loop γ "captures" the hole in the plane and, therefore, is not homotopic to the loop μ. (b) The based loops a and b in $\mathbb{C} \setminus \{0, 1\}$ around the punctures generate the fundamental group, each of these loops detects one of the holes of the space.

- The loop γ around the puncture in $\mathbb{C} \setminus \{0\}$ "detects" the hole in the plane, since it cannot be deformed to the constant loop without crossing the puncture; see Figure 4(a). Each time that we go around we get essentially a different loop. In this case, the fundamental group is isomorphic to $\mathbb{Z}$ and is generated by γ. Notice that the circle S^1 also has fundamental group $\mathbb{Z}$ since it is homotopy equivalent to $\mathbb{C} \setminus \{0\}$.
- The space $\mathbb{C} \setminus \{0, 1\}$ has fundamental group $F[a, b]$, a free group in two generators a and b, each one corresponding to a loop around one of the punctures as in $\mathbb{C} \setminus \{0, 1\}$: the circles in the 'figure-eight' space represented in Figure 4(b). The elements of this group are given by words on the letters a and b (and their formal inverses a^{-1} and b^{-1}) and the operation is given by concatenation. This group is not abelian since $ab \neq ba$.
- Let S^2 be the unit sphere in $\mathbb{R}^3$ and $\mathbb{R}\mathrm{P}^2$ be the projective plane, namely the space obtained form S^2 after identifying antipodal points. See for example [17]. Notice that a meridian in S^2, going from the north to the south pole, becomes a loop after projecting to $\mathbb{R}\mathrm{P}^2$. This suggests that not every loop in $\mathbb{R}\mathrm{P}^2$ comes from a loop in S^2, and indeed one can show that $\pi_1(S^2)$ is the trivial group, but $\pi_1(\mathbb{R}\mathrm{P}^2) = \mathbb{Z}/2$. Therefore S^2 and $\mathbb{R}\mathrm{P}^2$ cannot be homotopy equivalent.
- The 3-dimensional sphere S^3, the group $SO(3)$ of rotations in $\mathbb{R}^3$, the product $S^2 \times S^1$ of a 2-sphere and a circle, and the 3-dimensional torus $S^1 \times S^1 \times S^1$, are all examples of 3-dimensional manifolds. However, no two of them are homotopy equivalent, since their fundamental groups are 0, $\mathbb{Z}/2$, $\mathbb{Z}$ and $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$, respectively.

Observe that by computing fundamental groups we can distinguish different topological spaces. For instance, we just saw that the fundamental group distinguishes $\mathbb{C}$ from $\mathbb{C} \setminus \{0\}$ from $\mathbb{C} \setminus \{0, 1\}$ from $\mathbb{R}\mathrm{P}^2$. Be careful! There are spaces that are not homotopy equivalente but have the same fundamental group. That is the case of the sphere S^2 and the plane $\mathbb{C}$, which both have trivial fundamental groups.

2 Configuration spaces

Very frequently we are faced with the problem of avoiding collisions between objects of the same kind, like cars moving around a surface or airplanes flying through the air. A formal way to treat this problem using topology is by introducing configuration spaces.

Definition 2. Given a manifold M (or more generally a topological space) and $k \geq 1$, we define the *configuration space* of k distinct *ordered* points in M as the topological space $\mathrm{Conf}_k(M) = \{(x_1, x_2, \ldots, x_k) \in M^k \mid x_i \neq x_j \text{ if } i \neq j\}$.

Figure 5: A configuration of 5 vehicles in the plane corresponds to a point $(z_1, z_2, z_3, z_4, z_5)$ in $\mathrm{Conf}_5(\mathbb{C})$.

In classical mechanics, the configuration space of a physical system is the space of possible positions of the system. In the field of topology, configuration spaces were studied in 1962 by Ralph Hartzler Fox (1913–1973) and Lee Neuwirth [13] in connection with *Artin's braid group* [3]. They showed that this group is nothing but the fundamental group of the configuration space of distinct points in the plane, as will be observed in Section 3.[6] Since then, their topology has been intensively studied mainly because of their relation to braid groups [5], spaces of functions [9] and more recently to diverse problems in robotics [1, 14].

In some sense, the labelling of the points in any specific ordered configuration is "artificial". Therefore we say that two configurations $(x_1, \ldots, x_k)$ and $(y_1, \ldots, y_k)$ are equivalent if there is a permutation σ (that is an element of the symmetric group $\mathcal{S}_k$ on k letters) such that $y_i = x_{\sigma(i)}$ for all $i = 1, \ldots, k$.

[6] As Hans Heinrich Wilhelm Magnus (1907–1990) pointed out in [15], braid groups and their interpretation as fundamental groups of configuration spaces were already implicit in the work of Adolf Hurwitz (1859–1919) on monodromy from 1891.

Definition 3. The set of equivalence classes $\mathrm{UConf}_k(M) := \mathrm{Conf}_k(M)/\mathcal{S}_k$ is the *unordered configuration space*. The space $\mathrm{UConf}_k(M)$ can be thought as the space of subsets $\{x_1, \ldots, x_k\}$ of M of cardinality k.

Example 2. Some examples of configuration spaces are the following.

- For any topological space M and $k = 1$, the space $\mathrm{Conf}_1(M)$ is just M. For $k = 2$, the space $\mathrm{Conf}_2(M)$ is the set of points in $M \times M$ that do not lie in the diagonal $\Delta = \{(x, x) \mid x \in M\}$, that is:

$$\mathrm{Conf}_2(M) = \{(x, y) \in M \times M \mid x \neq y\} = (M \times M) \setminus \Delta.$$

- When M is the real line $\mathbb{R}$ and $k \geq 2$ the configuration space $\mathrm{Conf}_k(\mathbb{R})$ is *disconnected*, which means that it has several pieces. For instance, for $k = 2$ the set Δ is the diagonal line $y = x$ in the the Cartesian plane $\mathbb{R}^2$ with coordinates (x, y). The configuration space $\mathrm{Conf}_2(\mathbb{R})$ consists of the two halves in the plane that are left after removing the diagonal line $y = x$.
- The particular case when M is the complex plane $\mathbb{C}$ is already interesting and it is related to the theory of *hyperplane arrangements*. A hyperplane is a subspace whose dimension is one less than that of its ambient space. For $1 \leq i < j \leq k$, let $H_{i,j} = \{(z_1, z_2, \ldots, z_k) \in \mathbb{C}^k \mid z_i = z_j\}$. Then the space $\mathrm{Conf}_k(\mathbb{C})$ is nothing but the complement of the union of all these hyperplanes $H_{i,j}$ in $\mathbb{C}^k$:

$$\mathrm{Conf}_k(\mathbb{C}) = \mathbb{C}^k \setminus \left(\bigcup_{i<j} H_{i,j} \right).$$

  The theory of hyperplane arrangements studies geometrical, topological and combinatorial properties of the set that remains when a finite set of hyperplanes is removed from the ambient space. This theory is interesting in its own right as it combines ideas from combinatorics, algebraic topology and algebraic geometry. A nice introduction to the subject can be found in [11].
- The space $\mathrm{Conf}_2(\mathbb{C})$ is the complement of $\Delta = \{(z, z) \mid z \in \mathbb{C}\}$ (note that Δ coincides with $H_{1,2}$) which is a *c*omplex line in $\mathbb{C}^2$. In other words, $\mathrm{Conf}_2(\mathbb{C})$ is the complement of a 2-dimensional plane in real 4-dimensional space. By subtracting one dimension, this space can be deformed onto the complement of a line in $\mathbb{R}^3$ and eventually, onto the complement of a point in $\mathbb{R}^2$ (a punctured plane!), which is homotopy equivalent to S^1. Another way to see this is to notice the space $\mathrm{Conf}_2(\mathbb{C})$ is homeomorphic to $\mathbb{C} \times (\mathbb{C} \setminus \{0\})$, one such homeomorphism is given by $(z, w) \mapsto (z, w - z)$.
- If $M = G$ is a topological group (a topological space with a continuous operation that makes it into a group), the space $\mathrm{Conf}_k(G)$ is homeomorphic

to the product $G \times \mathrm{Conf}_{k-1}(G \setminus \{e\})$ where e is the identity element if G. A homeomorphism

$$\mathrm{Conf}_k(G) \xrightarrow{\approx} G \times \mathrm{Conf}_{k-1}(G \setminus \{e\})$$

is given by the formula

$$(g_1, g_2, \dots, g_k) \longmapsto (g_1) \times (g_1^{-1} \cdot g_2, \dots, g_1^{-1} \cdot g_k).$$

- We can consider $\mathbb{C}$ as a topological group with the sum of complex numbers. In this case:

$$\mathrm{Conf}_3(\mathbb{C}) \approx \mathbb{C} \times \mathrm{Conf}_2(\mathbb{C} \setminus \{0\}).$$

Notice that $\mathbb{C} \setminus \{0\}$ is again a topological group under the product of complex numbers and the identity element is 1. Thus we have:

$$\mathrm{Conf}_3(\mathbb{C}) \approx \mathbb{C} \times (\mathbb{C} \setminus \{0\}) \times (\mathbb{C} \setminus \{0, 1\}),$$

which is in turn homotopy equivalent to $S^1 \times (S^1 \vee S^1)$, the product of the unit circle with a 'figure-eight' space. Thus, the fundamental group of $\mathrm{Conf}_3(\mathbb{C})$ is $\mathbb{Z} \times F[a, b]$ which is not even abelian!

What can we say about the shape of other configuration spaces? That is in general a hard question: the space M where the configurations lie may be already a complicated space and one would expect that dealing with larger configurations (say $k = 1,000,000$) will make $\mathrm{Conf}_k(M)$ harder to visualize and understand.

In what follows we explore more about the shape of $\mathrm{Conf}_k(\mathbb{C})$, the configuration spaces of points in $\mathbb{C}$. So far we have seen what happens when we have configurations with few points by giving concrete homotopy equivalences. When an algebraic topologist is not able to come up with explicit homeomorphisms or homotopy equivalences to spaces that are already known and understood, the strategy is to study algebraic invariants like the fundamental group.

3 Braid groups

In this section we introduce braid groups and explore their connections with the topology of configurations spaces of points in the plane.

3.1 Loops of configurations and braids

In order to study the fundamental group of the configuration space $\mathrm{Conf}_k(\mathbb{C})$ we would like to get a sense of what the based loops of configurations in the plane look like. As before we take a starting point $\mathbf{x}_0 \in \mathrm{Conf}_k(\mathbb{C})$, which is an ordered configuration $(p_1, p_2, \ldots, p_k)$ of points $p_i \in \mathbb{C}$. Given a based loop $\gamma : [0,1] \to \mathrm{Conf}_k(\mathbb{C})$, at each time $t \in [0,1]$ we have that $\gamma(t)$ is a new ordered configuration of k points in the plane. We can think of it as the trajectories

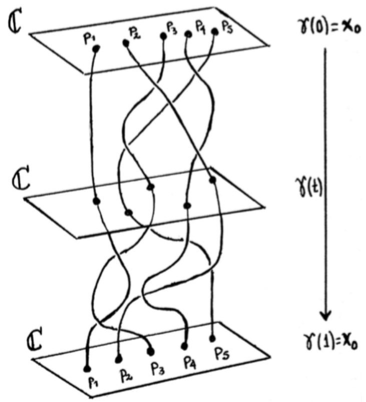


Figure 6: A loop of configurations in $\mathrm{Conf}_5(\mathbb{C})$.

$\gamma_i : [0,1] \to \mathbb{C}$ of k particles moving in the plane $\mathbb{C}$ that are not allowed to collide, starting and ending at the ordered configuration $\mathbf{x}_0 = (p_1, p_2 \ldots, p_k)$:

$$\gamma(t) = (\gamma_1(t), \gamma_2(t), \ldots, \gamma_k(t)) \text{ for } t \in [0,1], \text{ with } \gamma(0) = \gamma(1) = \mathbf{x}_0.$$

Hence, one can visualize (see Figure 6) a loop of configurations in $\mathrm{Conf}_k(\mathbb{C})$ as k strands going around each other: *a braid!*

Definition 4. The fundamental group of the ordered configuration space $\mathrm{Conf}_k(\mathbb{C})$, often denoted as P_n, is called the *pure braid group.* Braid groups can be defined abstractly, independently of configuration spaces. Consider the planes $z = 0$ and $z = 1$ in $\mathbb{R}^3$, which will be denoted by P and Q, respectively. Choose k distinct points $p_1, p_2, \ldots, p_k \in P$ and let $q_1, q_2, \ldots, q_k \in Q$ be the corresponding orthogonal projections in Q. By an arc in $\mathbb{R}^3$ we will understand a continuous function $\mathcal{A} : [0,1] \to \mathbb{R}^3$ which is a homeomorphism onto its image (no self-intersections are allowed).

Definition 5. A *(geometric) braid* β on k strands is a collection $\mathcal{A}_1, \ldots, \mathcal{A}_k$ of disjoint arcs (the strands) in $\mathbb{R}^3$ such that:

1. There exists a permutation $\sigma \in \mathcal{S}_k$ such that $\mathcal{A}_i$ connects the point p_i with the point $q_{\sigma(i)}$.

2. Every arc $\mathcal{A}_i$ intersects all parallel planes between P and Q in exactly one point.

We will say that two braids β_1 and β_2 are equivalent if either one can be deformed continuously into the other, that is, if the the corresponding systems of arcs are *isotopic*.

Definition 6. The set of equivalence classes of braids on k strands is denoted by B_k and is equipped with a natural product which makes it a group (see Figure 7) which is known as *Artin's braid group*.

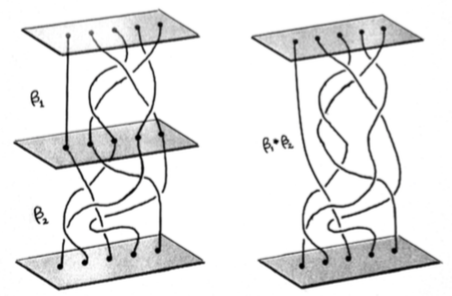


Figure 7: The product $\beta_1 * \beta_2$ of two geometric braids β_1 and β_2 corresponds to the concatenation of two loops of configurations.

Notice that a geometric braid can be regarded as a based loop in the unordered configuration space $\mathrm{UConf}_k(\mathbb{C})$ and two such braids are isotopic if and only if the corresponding loops are homotopic. Therefore, the group B_k is precisely the fundamental group of the space $\mathrm{UConf}_k(\mathbb{C})$.

The braid groups B_k were introduced explicitly by Emil Artin in 1925 [3]. It is a classical result of Artin's that B_k admits the following presentation in terms of generators and relations:

$$B_k \cong \left\langle \sigma_1, \sigma_2, \ldots, \sigma_{k-1} \;\middle|\; \begin{array}{c} \sigma_i\sigma_j = \sigma_j\sigma_i \quad \text{if} \quad |i-j| \geq 2 \\ \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \end{array} \right\rangle \tag{1}$$

That is to say, an element of B_k can be specified as a product of the σ_i and of their inverses, and two such products or 'words' define the same element if one can be transformed into the other using the relations among the σ_i. Geometrically, the generator σ_i can be interpreted as the elementary braid that interchanges the i-th and the $(i+1)$-th strands so that the i-th strand passes over the $(i+1)$-th one. Thus, for instance, the word $\sigma_1^2\sigma_3\sigma_4^{-3}\sigma_2^2\sigma_3\sigma_2^{-1}\sigma_4^{-1}$ represents the braid shown in Figure 8.

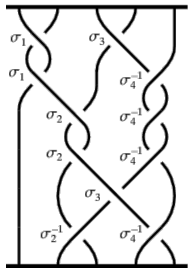


Figure 8: A braid in B_k written in terms of the generators σ_i.

By forgetting the crossings of the strands, each braid determines a permutation in the symmetric group $\mathcal{S}_k$ and clearly every permutation can be attained in this way. More precisely, there is a natural surjection

$$\pi : B_k \to \mathcal{S}_k \ \text{ given by } \ \sigma_i \mapsto (i \quad i+1),$$

at the level of generators, which sends σ_i to the transposition $(i \ \ i+1)$. Moreover, this function is compatible with the product in both groups in the sense that $\pi(\beta_1 \cdot \beta_2) = \pi(\beta_1) \cdot \pi(\beta_2)$,that is, π is a group homomorphism. For example, the function π sends the braid shown in Figure 8 to the permutation (2 3 4 5).

A braid $\beta \in B_k$ which induces the identity permutation in $\mathcal{S}_k$ is known as a *pure braid.* For instance, the squares of the generators $\sigma_1^2, \sigma_2^2, \dots, \sigma_{k-1}^2$ are pure braids. The set of all pure braids $P_k = \pi^{-1}(1)$ is a subgroup of B_k which is precisely the *pure braid group* from Definition 4 above. The pure braid σ_i^2 corresponds to the loop of configurations that makes the i-th point p_i in the based configuration $\mathbf{x}_0 = (p_1, p_2, \dots, p_k)$ loop around the $(i+1)$-th point p_{i+1} and fixes all the other p_j.

This topological reinterpretation of the groups P_k and B_k as the fundamental groups of the configuration spaces $\mathrm{Conf}_k(\mathbb{C})$ and $\mathrm{UConf}_k(\mathbb{C})$, respectively, was given in 1962 by Ralph Hartzler Fox and Lee Neuwirth [13].

3.2 Braid groups and platonic solids

It is not difficult to show that P_k is the normal[7] subgroup $\langle\langle \sigma_i^2 \rangle\rangle$ of B_k generated by the squares of the generators. Therefore, the following quotient group [8] is

[7] A subgroup H of a group G is *normal* in G if $gH = Hg$ for all $g \in G$. The notation $\langle\langle S \rangle\rangle$ is used to denote the normal subgroup generated by S.

[8] If H is a normal subgroup pf G the set $G/H = \{gH : g \in G\}$ is actually a group. In a quotient group we 'clump' together 'equivalent members' of a group, in such a way that we get a smaller group (in this case the symmetric group $\mathcal{S}_k$).

isomorphic to the symmetric group:

$$B_k/\langle\langle \sigma_i^2 \rangle\rangle \;=\; B_k/P_k \;\cong\; \mathcal{S}_k.$$

An interesting variation is obtained by replacing the pure braid group by

$$\langle\langle \sigma_i^n \mid i = 1, \ldots, k-1 \rangle\rangle,$$

the normal subgroup of B_k generated by the n-th powers of the generators. Thus, for $k, n \geq 2$ we define the quotient group

$$B_k(n) = B_k/\langle\langle \sigma_i^n \rangle\rangle.$$

For $n = 2$ and k arbitrary we have $B_k(2) \cong \mathcal{S}_k$. On the other hand, when $k = 2$ it is clear that $B_k \cong \mathbb{Z}$ (it is generated by σ_1 with no relations). It follows that the quotient $B_2(n)$ is isomorphic to $\mathbb{Z}/n$, the cyclic group of order n. Then, the following question arises naturally: for which other values of k and n is $B_k(n)$ a finite group? The answer is surprising and it was given by Harold Scott MacDonald Coxeter in 1957, see [10].

Theorem 1 (Coxeter). For $k, n \geq 3$, the group $B_k(n)$ is finite if and only if $(k, n) = (3, 3), (3, 4), (4, 3), (3, 5), (5, 3)$.

Figure 9: The five platonic solids.

In other words, the group $B_k(n)$ is a finite group if and only if (k, n) is the *type* of one of the five *platonic solids*. Recall that in a regular polyhedron all the faces are congruent regular polygons. Such a polyhedron is said to be of type (k, n) if k is the number of edges of each face and n is the number of faces meeting at each vertex. It is well known that there are only five convex regular polyhedra, also known as the platonic solids: the tetrahedron $(3,3)$, the cube $(4,3)$, the octahedron $(3,4)$, the dodecahedron $(5,3)$, and the icosahedron $(3,5)$.

The original proof by Coxeter uses groups of hyperbolic isometries. Nowadays there are proofs of this result that only use basic tools from the representation theory of finite groups, see [4].

Another relation between the groups $B_k(n)$ and the platonic solids is a nice formula for the order of these groups. Namely, if $B_k(n)$ is finite and f is the number of faces of the corresponding solid of type (k,n), then:

$$|B_k(n)| = \left(\frac{f}{2}\right)^{k-1} k! \quad \text{(see [16])}.$$

The following table gives the orders of all of the groups $B_k(n)$:

Group $B_k(n)$	order
$B_3(3)$	24
$B_3(4)$	96
$B_3(5)$	600
$B_4(3)$	648
$B_5(3)$	155520

3.3 Braid groups and the topology of configurations spaces

Since the groups P_k and B_k are the fundamental groups of the configuration spaces $\mathrm{Conf}_k(\mathbb{C})$ and $\mathrm{UConf}_k(\mathbb{C})$, the topology of these spaces determines the algebraic structure of the corresponding braid groups. For instance, we have already seen that

$$P_2 = \pi_1\left(\mathrm{Conf}_2(\mathbb{C}), \mathbf{x_0}\right) = \pi_1(S^1) = \mathbb{Z}, \qquad \text{and}$$
$$P_3 = \pi_1\left(\mathrm{Conf}_3(\mathbb{C}), \mathbf{x_0}\right) = \pi_1\left(S^1 \times (S^1 \vee S^1)\right) = \mathbb{Z} \times F[a,b].$$

Another example is the fact that the groups P_k and B_k are torsion-free, which means that there are no elements of finite order [9]. A proof of this result uses the fact that the configuration spaces of $\mathbb{C}$ are finite dimensional aspherical[10] manifolds. But the converse is also true, namely the structure of the groups P_k and B_k determines in some sense the topology of the configuration spaces. For example, it can be shown that the abelianzation[11] of P_k is isomorphic to

[9] An element g of a group G is of finite order if there exists a positive integer n such that g^n is the identity element of the group.

[10] An *aspherical* space is a topological space with all homotopy groups $\pi_n(X)$ equal to 0 when $n > 1$.

[11] The *abelianization* of a group it is obtained by adding relations that make every product commutative.

$\mathbb{Z}^{\frac{k(k-1)}{2}}$. Therefore, if $k \neq k'$ the groups P_k and $P_{k'}$ cannot be isomorphic and thus, the configuration spaces $\mathrm{Conf}_k(\mathbb{C})$ and $\mathrm{Conf}_{k'}(\mathbb{C})$ cannot be homeomorphic (or even have the same homotopy type).

4 More points = more complicated?

In the discussion above we have been actually dealing with families of configurations spaces $\{U\mathrm{Conf}_k(\mathbb{C})\}_{k\geq 1}$ and $\{\mathrm{Conf}_k(\mathbb{C})\}_{k\geq 1}$, parametrized by the number k of points in a configuration. A natural question to ask in this context is:

Does the 'shape' of the configuration spaces $U\mathrm{Conf}_k(\mathbb{C})$ and $\mathrm{Conf}_k(\mathbb{C})$ get more complicated as the number k of points in a configuration increases?

In a way the answer to this question is yes. The dimension of the configuration spaces $U\mathrm{Conf}_k(\mathbb{C})$ and $\mathrm{Conf}_k(\mathbb{C})$ is $2k$ and gets larger as the number k of points in a configuration increases. And so it happens with the number of generators and relations of the braid groups B_k in Artin's presentation (1) which depends on k.

But surprisingly there are features of the spaces $U\mathrm{Conf}_k(\mathbb{C})$ that do not get more complicated. The abelianization of B_k is the algebraic invariant $H_1(U\mathrm{Conf}_k(\mathbb{C}))$ (the first homology group) and it captures information about the number of certain 'holes' in the space $U\mathrm{Conf}_k(\mathbb{C})$. Vladimir I. Arnold and Fred Cohen proved that $H_1(U\mathrm{Conf}_k(\mathbb{C}))$ is a group that is eventually independent of k and, as a consequence, the number of those types of 'holes' in the spaces $U\mathrm{Conf}_k(\mathbb{C})$ does not increase as the number k gets larger. More generally, Arnold and Cohen showed ([2], [8]) that the ith homology group of the spaces $U\mathrm{Conf}_k(\mathbb{C})$ satisfies:

$$H_i(U\mathrm{Conf}_k(\mathbb{C})) \cong H_i(U\mathrm{Conf}_{k+1}(\mathbb{C})) \text{ when the number of points } k \geq 2i.$$

The family $\{U\mathrm{Conf}_k(\mathbb{C})\}_{k\geq 1}$ of unordered configuration spaces is said to satisfy *homological stability*: the algebraic invariants of the spaces (homology groups $H_i(U\mathrm{Conf}_k(\mathbb{C}))$) stabilize when the parameter k is large enough. Several natural families of unordered configuration spaces and moduli spaces have this type of behavior.

In contrast, known computations show that the family of ordered configuration spaces $\{\mathrm{Conf}_k(\mathbb{C})\}_{k\geq 1}$ does not satisfy this type of stability. For instance, the first homology $H_1(\mathrm{Conf}_k(\mathbb{C}))$ is the abelianization of P_k and it is isomorphic to $\mathbb{Z}^{\frac{k(k-1)}{2}}$, which clearly depends on the number of points k of the configuration. Thomas Church and Benson Farb first noticed [7] that this dependance on k could be explained if we take into account the natural symmetries that the spaces $\mathrm{Conf}_k(\mathbb{C})$ have: the points in an ordered configuration can be permuted to obtain a new ordered configuration in $\mathrm{Conf}_k(\mathbb{C})$.

In recent years it has been observed that the importation of representation theory into the study of homological stability makes it possible to extend classical theorems of homological stability to a much broader variety of examples. This new notion has been called *representation stability* and it is satisfied by the family $\{\mathrm{Conf}_k(\mathbb{C})\}_{k\geq 1}$: the algebraic invariants (homology or cohomology groups) do get more complicated as k increases, but they stabilize up to the symmetries of the spaces. Families of configuration spaces, hyperplane arrangements and related topological spaces equipped with symmetries, as well as several 'relatives' of the pure braid groups have been shown to have this behavoir. We refer the interested reader to the survey paper [12] by Benson Farb.

The study of these stability phenomena is a way to better understand shapes of natural families of spaces that appear in mathematics and it is an active area of research, as was witnessed in the Oberwolfach workshop that inspired this snapshot.

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Image credits

Fig. 1 By http://www.rijksmuseum.nl/collectie/SK-A-390, Public Domain, https://commons.wikimedia.org/w/index.php?curid=34250587. *Baker with pretzels* by Jan Steen (1658).

Fig. 7 From p. 49 in Frenkel, E. *Love and math. The heart of hidden reality.* Basic Books, New York, 2013.

Fig. 8 From http://mathworld.wolfram.com/Braid.html, visited on May 3rd, 2018.

Fig. 9 From https://en.wikipedia.org/wiki/Platonic_solid, visited on May 3rd, 2018.

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Rita Jiménez Rolland *is an assistant professor of mathematics at the Universidad Nacional Autónoma de México.*

Miguel A. Xicoténcatl *is a professor of mathematics at the Centro de Investigación y de Estudios Avanzados del IPN.*

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Mathematisches Forschungsinstitut
Oberwolfach gGmbH
Schwarzwaldstr. 9 – 11
77709 Oberwolfach
Germany

Director
Gerhard Huisken

Mathematisches Forschungsinstitut Oberwolfach

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