Snapshots of modern mathematics
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Swallowtail on the shore

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Platonic solids, Felix Klein, H.S.M. Coxeter and a flap of a swallowtail: The five Platonic solids tetrahedron, cube, octahedron, icosahedron and dodecahedron have always attracted much curiosity from mathematicians, not only for their sheer beauty but also because of their many symmetry properties. In this snapshot we will start from these symmetries, move on to groups, singularities, and finally find the connection between a tetrahedron and a "swallowtail". Our running example is the tetrahedron, but every construction can be carried out with any other of the Platonic solids.

1 Set the stage: Symmetry

Look at the regular tetrahedron in three-dimensional space $\mathbb{R}^3$, see Fig. 1. Let us call it T for later reference. How can we transform T without changing its shape and position in space?

We are only allowed to perform *symmetries*, i.e., transformations in $\mathbb{R}^3$ that preserve the shape and position in space of T: essentially, we may "reorder" the vertices of T through rotating or reflecting T repeatedly. Each rotation is uniquely defined by its rotation axis and a fixed angle, a reflection is given by its

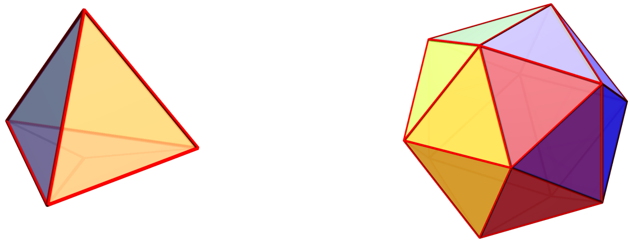

Figure 1: The Tetrahedron T (left) and the Icosahedron I (right).

mirror plane, i.e., the plane where any point is reflected on.[1] Only considering rotations, we obtain for the tetrahedron 12 rotational symmetries in total:

For rotations fixing the tetrahedron one has two different kinds of rotation axes: on the one hand, going perpendicular through the center of a face and meeting the opposite vertex (there are four of them, one for each face; see Fig. 2) and on the other hand axes connecting the midpoints of two vis-á-vis edges (three of them). For the "face-vertex" axes one can rotate two times by 120 degree about the axes and for the "vis-á-vis-edges" axes once about 180 degrees before reaching the initial position. So one gets $2 \cdot 4 + 1 \cdot 3 + 1 \cdot 1 = 12$ different rotations, where $1 \cdot 1$ comes from the identity rotation that does nothing.

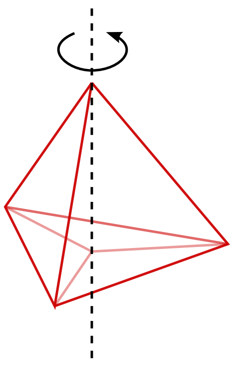

Figure 2: An example for a rotation axis going through one vertex and meeting the opposite face perpendicularly

[1] Note that the rotation axis is exactly the set of points, which is not moved when performing the rotation. Similarly, the mirror plane is the set of fixed points of a reflection. In general, we can write each rotation as the composition of two reflections. Also, by a theorem of Leonard Euler (1707-1783), a composition of two rotations is again a rotation.

These rotational symmetries form a group,[2] the so-called tetrahedral group $\mathcal{T}$. Also allowing reflections, one gets additional symmetries: 6 reflections in planes perpendicular to the edges and 6 reflections in a plane followed by a 90 degree rotation about an axis perpendicular to this plane. In total we get a group with 24 elements, the so-called full symmetry group $\mathbb{T}$ of the tetrahedron. If we had started for example with an icosahedron I instead (see Fig. 1), its rotational symmetry group consisted of 60 rotations and the full icosahedral group of 120.

The full symmetry group $\mathbb{T}$ can also be described with the help of a *fundamental triangle* on the two-dimensional sphere S (the ball in $\mathbb{R}^3$ around the origin with radius 1): Just by applying reflections in $\mathbb{T}$, any point in S can be transported to a point in the red spherical triangle on S, whose angles[3] are $\frac{\pi}{2}$, $\frac{\pi}{3}$, $\frac{\pi}{3}$ (see Fig. 3).

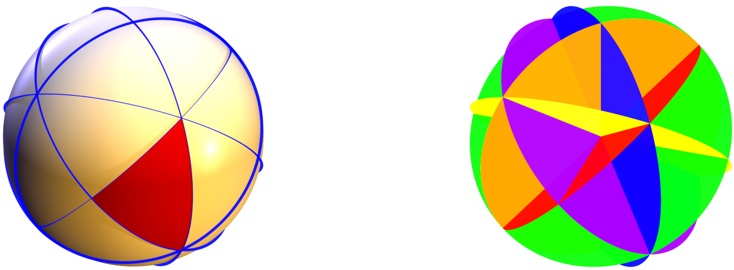

Figure 3: The fundamental triangle of $\mathbb{T}$ (left) and the six mirrors of $\mathbb{T}$, forming the Hyperplane arrangement A_3 (right).

One can also find the fundamental regions for the other Platonic solids. One striking result is that for each finite reflection group (see later for the explanation of this term), there is a fundamental triangle with angles $(\frac{\pi}{p}, \frac{\pi}{q}, \frac{\pi}{r})$, where p, q, r are natural numbers that satisfy the following so-called diophantine

[2] The concept of a group is basic in mathematics (in short: a group is a set where "addition" and "subtraction" makes sense). A *group* is a set of elements G together with an operation $\circ$ (variably denoted, most frequently: addition, multiplication or composition) satisfying three properties: (1) there exists an identity element $e_G \in G$, (2) for each g exists an inverse g^{-1} such that $g \circ g^{-1} = e_G$ and (3) for any $g, h, f \in G$: $(g \circ h) \circ f = g \circ (h \circ f)$. For our tetrahedron T, the rotational symmetry group consists of the 12 rotations and $\circ$ is the composition of two of them, the inverse of a rotation is the same rotation backwards and the identity e_G means not moving T at all.

[3] For convenience, we rather write here the angles in radians, π corresponds to 180 degrees, $\frac{\pi}{2}$ to 90 degrees, $\frac{\pi}{3}$ to 60 degrees and so on.

inequality:

$$\frac{1}{p}+\frac{1}{q}+\frac{1}{r}>1.$$

Thinking a bit more about this image, one sees that the only possibilities for (p,q,r) are:

- $(1,n,n)$, $n \geq 1$ - degenerated triangles (so-called *di-gons*), corresponding to the symmetries of an n-gon in the plane;
- $(2,2,n)$ - the symmetries of a n-dihedron[4];
- $(2,3,3)$ - symmetries of the tetrahedron;
- $(2,3,4)$ - symmetries of the cube/octahedron and finally
- $(2,3,5)$ - symmetries of the dodecahedron/icosahedron.

And these are in fact the only symmetry groups of $\mathbb{R}^3$ generated by reflections.

2 Snap: Singularities!

The German mathematician Felix Klein (1849–1925) studied finite rotation groups[5] in $\mathbb{R}^3$. The symmetry group of the sphere S is the set of *all* rotations of $\mathbb{R}^3$ about *any* axis through the origin, and thus an infinite group. This group is denoted by $SO_3(\mathbb{R})$ and one says that $\mathcal{T}$ is a finite subgroup of $SO_3(\mathbb{R})$. Klein found that the only finite rotation subgroups of $SO_3(\mathbb{R})$ are the (rotation) symmetry groups of the five Platonic solids, the symmetry group of a 2-dimensional n-gon and the symmetry group of an n-dihedron (where $n \geq 2$!), see [2].
It is no coincidence that this is the same list as before, when we considered fundamental triangles on the sphere. By covering the sphere into fundamental triangles (as in fig. 3) with angles $(\frac{\pi}{p},\frac{\pi}{q},\frac{\pi}{r})$, one can reconstruct the rotations and the corresponding platonic solid (remember that the only possibilities for (p,q,r) are $(1,n,n)$ for $n \geq 1$, $(2,3,3)$, $(2,3,4)$ and $(2,3,5)$): there is an axis through each vertex of the triangle and its antipode (the point diametrically opposite on the sphere) and the angles of a rotation correspond to twice the angle of the triangle. Relating these finite rotations in $\mathbb{R}^3$ with complex rotations in $\mathbb{C}^2$ (the so-called double cover of $SU_2(\mathbb{C}) \to SO_3(\mathbb{R})$), Klein found a relation of *singular algebraic surfaces*[6] with Platonic solids: to any finite subgroup of

[4] A *n-dihedron* in $\mathbb{R}^3$ consists of two regular n-gons lying above each other such that the edges coincide — think of a box whose top and bottom are regular n–gons.

[5] A finite group is a group containing only finitely many elements, such as our groups $\mathcal{T}$ and $\mathbb{T}$.

[6] An algebraic surface is the zero set of a polynomial in three variables. A singular point on a surface is a “special” point, where the surface has a cusp or branches out. Mathematically speaking, this means that one cannot find a unique tangent plane to a singular point.

$SO_3(\mathbb{R})$, one can relate a complex subgroup of $SU_2(\mathbb{C})$, which acts[7] on complex polynomials in two variables.
In the case of the tetrahedron, one looks at so-called invariant polynomials of the complex group $\bar{\mathcal{T}}$ in $SU_2(\mathbb{C})$ corresponding to $\mathcal{T}$. These are all polynomials which are not changed under the group action, meaning that

$$f(x_1, x_2) = f(\varphi^{-1}(x_1, x_2))$$

for all φ in $\bar{\mathcal{T}}$). Finally, one finds 3 polynomials u, v, w in 2 variables with complex coefficients such that any other invariant polynomial is a polynomial in u, v, w. These three invariants satisfy the equation $u^2 + v^3 + w^4 = 0$. Consider $\mathbb{R}^3$ as the set of all points (x, y, z), then

$$X = \{(x, y, z) \in \mathbb{R}^3 \text{ such that } x^2 + y^3 + z^4 = 0\}$$

is a singular algebraic surface, which has a singular point at the origin, see Fig. 4. In the last century, the connections of Platonic solids to singular algebraic surfaces were studied further, for example via resolution of singularities and in the theory of Lie groups.

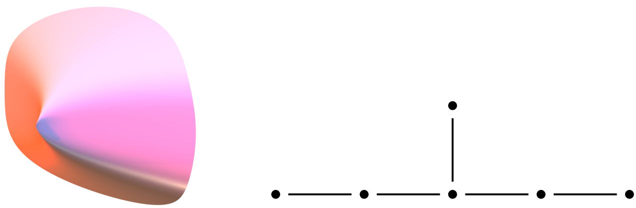

Figure 4: The algebraic surface $x^2 + y^3 + z^4 = 0$ (left), which is called an E_6-singularity. This name comes from its so-called resolution graph (right), which is the E_6 Dynkin diagram.

3 Snap: Reflections!

The Canadian mathematician H.S.M. Coxeter (1907–2003) looked at groups consisting of reflections in hyperplanes, called *reflection groups*, see [1]. One

[7] A transformation φ in the group sends a point $P = (p_1, p_2, p_3) \in \mathbb{R}^3$ to a point $\varphi(p_1, p_2, p_3) \in \mathbb{R}^3$. In this context one says that $SO_3(\mathbb{R})$ (or any of its subgroups) *acts* on a polynomial $f(x_1, x_2, x_3)$ via $f(\varphi^{-1}(x_1, x_2, x_3))$. Similarly for complex rotations in the plane on polynomials in two variables.

says that reflections generate[8] the group. In the case of our tetrahedron T, one finds six reflection planes (let us call them mirrors), going through the opposite edge midpoints of T, see Fig. 3. Three of these mirrors are enough to generate the group, one can obtain any of the rotations. But even more: *Any* other symmetry transforming a tetrahedron into itself can be written as a composition of these three reflections.
This can be seen by a more general observation, which starts one dimension lower, i.e., one looks at reflections fixing an equilateral triangle. In this case, we can easily see that there are exactly 3 mirrors (note: we are in dimension 2, so we have mirror *lines* instead of *planes*) generating the group, namely the three perpendicular lines through the opposite vertex of an edge (see Fig. 6). In this setting, one only needs two of the mirror-lines to obtain all possible symmetries: these two lines are called the generators of the group. Now one can associate a so-called *Coxeter diagram* to each reflection group: each generator is visualized by a dot, two dots are connected when the angle between the corresponding mirrors is $\frac{\pi}{3}$ and they are not connected if there is an angle of $\frac{\pi}{2}$ between them. For the equilateral triangle we thus obtain the so-called A_2-diagram: $\bullet$ ——— $\bullet$.

In the three dimensional space $\mathbb{R}^3$, the same procedure yields that the group $\mathbb{T}$ is generated by three mirror planes, where the angles of intersection are $\frac{\pi}{2}$ and $\frac{\pi}{3}$, as we saw above.

where each two of them intersect in an angle of $\frac{\pi}{3}$ degrees. So, one gets the Coxeter diagram A_3:

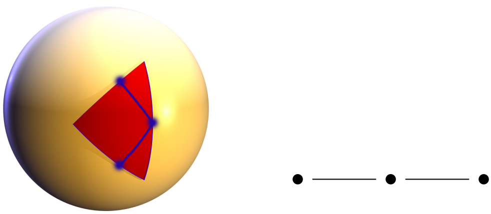

Figure 5: The fundamental triangle and the Coxeter diagram A_3.

Note that in this case, the tetrahedron corresponds to the diagram A_3, whereas in Klein's approach, the binary tetrahedral group was related to the E_6-diagram (see fig. 4) via its so-called resolution graph[9].

[8] This means that every element of the group can be obtained by composing the generating reflections. Most of the time one wants to find the minimal number of generators.

[9] These are instances of the so-called ADE-classification, which occurs in many areas of mathematics.

4 Snap: ...Swallowtail?!

The swallowtail is the so-called "discriminant of the A_3-arrangement", which is just the *discriminant* of a quartic equation in one variable t of the form

$$Q : t^4 + at^2 + bt + c = 0, \text{ for } a, b, c \in \mathbb{R},$$

i.e., the set of points (a, b, c) in $\mathbb{R}^3$ for which Q does not have four distinct solutions[10]. To understand the connection of the tetrahedron and the swallowtail, let us again descend one dimension and consider an equilateral triangle. By Coxeter's reasoning, the symmetry group of the triangle is the group A_2 generated by two mirrors, in total there are three mirrors. We can view them as the planes $x_1 = x_2$, $x_1 = x_3$ and $x_2 = x_3$ in $\mathbb{R}^3$, see Fig. 6.

Figure 6: Mirrors for A_2 (left), action of A_2 on a point P in the triangle (right).

The reflections in the three mirrors leave points on the diagonal (x, x, x) of $\mathbb{R}^3$ where they are. Said differently, reflections leave points $P = (x_1, x_2, x_3)$ with $x_1 = x_2 = x_3$ invariant, and hence also its perpendicular plane

$$H : x_1 + x_2 + x_3 = 0$$

Thus, the group A_2 acts on points in the plane H and we may consider our triangle lying in this plane. In this setting, the three mirrors correspond to the lines joining an edge of the triangle perpendicularly with the opposite vertex. Let us consider the effect of the group on a point in H: One point can be transported to at most six different points (one then says that these points all lie in the same *orbit*), see Fig. 6. It is clear that any point not lying on a mirror

[10] For a quadratic equation $t^2 + pt + q = 0$ the discriminant is $\frac{p^2}{4} - q$. So in this case, the discriminant is the expression under the root in the solution formula of the quadratic equation. For equations of higher degree, there does not exist a general solution formula but one can still look at discriminants.

has a six point orbit, whereas the mirror points lie in smaller orbits[11] (these are called *irregular orbits*).
The idea is now to relate the space of all orbits of points in the plane H to the space of roots of cubic polynomials such that the irregular orbits correspond to those cubic polynomials which have a multiple root. As in the case of a quadratic equation, a cubic equation has a multiple root whenever its discriminant vanishes. So let us set up this correspondence for A_2: consider the space of all orbits of A_2. A polynomial defined on our surrounding space $\mathbb{R}^3$ is invariant under the transformations of A_2 if its value is the same on all points in the same orbit. For example, take the function

$$f(x_1, x_2, x_3) = x_1 \cdot x_2 \cdot x_3.$$

Its evaluation at the point $p = (1, -1, 0)$ is just $f(1, -1, 0) = 0$. The orbit of p consists of the six points $(-1, 1, 0)$, $(0, -1, 1)$, $(0, 1, -1)$, $(1, 0, -1)$ and $(-1, 0, 1)$, and one can easily check that f evaluated at each of these six points equals 0. This means that the invariants can be viewed as functions on the space of orbits, meaning that for each orbit, f has a uniquely determined value. As "basic" invariants of A_2, one gets exactly three polynomials

$$\sigma_1 = x_1 + x_2 + x_3,\ \ \sigma_2 = x_1x_2 + x_2x_3 + x_3x_1,\ \ \sigma_3 = x_1x_2x_3,$$

all other invariants are polynomials in $\sigma_1, \sigma_2, \sigma_3$. Conversely, for any value of $\sigma_1, \sigma_2, \sigma_3$ or their polynomial combination, there is one corresponding orbit. On the plane H, the invariant σ_1 is zero, and we are left with σ_2 and σ_3. They determine the polynomial

$$\begin{aligned}
&(t - x_1)(t - x_2)(t - x_3) \\
&\qquad = t^3 - t^2(x_1 + x_2 + x_3) + t(x_1x_2 + x_2x_3 + x_3x_1) - x_1x_2x_3 \\
&\qquad = t^3 + t\sigma_2 - \sigma_3
\end{aligned}$$

and as a result the set of roots $\{x_1, x_2, x_3\}$. In this way, an orbit of the group action of A_2 on H corresponds to the set of roots of a cubic polynomial $t^3 + \lambda_1 t + \lambda_2$. (In order for these notions to be well-defined, we need to consider $\lambda_1, \lambda_2 \in \mathbb{C}$ and solutions x_1, x_2 in $\mathbb{C}^2$.) As we are in H, one can substitute $x_3 = -x_1 - x_2$ and obtain the so called Vieta map

$$\sigma : \mathbb{C}^2 \to \mathbb{C}^2, (x_1, x_2) \mapsto (\lambda_1(x_1, x_2), \lambda_2(x_1, x_2)),$$

with $\lambda_1(x_1, x_2) = \sigma_2$ and $\lambda_2(x_1, x_2) = -\sigma_3$, sending a point in H to the space of its orbit. This correspondence works in the other direction as well. The

[11] How many points lie in the orbit of a mirror point that is not the center?

so-called *critical set*[12] of the Vieta mapping is the union of the mirrors in H, which corresponds to the polynomials having a multiple root. So the irregular orbits of A_2 are exactly the points

$$(\lambda_1, \lambda_2) = (\lambda_1(x_1, x_2), \lambda_2(x_1, x_2))$$

such that $t^3 + \lambda_1 t + \lambda_2 = 0$ has a multiple solution. Similar to the case of a quadratic equation, these multiple solutions are described by the vanishing of the discriminant, which in this case is $9\lambda_2^2 - \lambda_1^3$ (see Fig. 7).

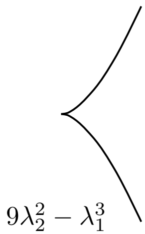

Figure 7: The *cusp*, discriminant of the polynomial $t^3 + \lambda_1 t + \lambda_2 = 0$.

The same ideas can be generalized and lead to discriminants of reflection arrangements. For our favorite example, the tetrahedral group A_3, the discriminant is that of the quartic polynomial

$$t^4 + \lambda_1 t^2 + \lambda_2 t + \lambda_3 = 0.$$

This discriminant can still be computed (as the determinant of a 3×3-matrix). If we rename $x = \lambda_1, y = \lambda_2, z = \lambda_3$, the equation is

$$16x^4 z - 4x^3 y^2 - 128x^2 z^2 + 144xy^2 z - 27y^4 + 256z^3 = 0,$$

and the set of real solutions is a singular surface in $\mathbb{R}^3$. Can you guess why this surface (see Fig. 8) is called the *Swallowtail*[13]?

[12] The critical set is the set of points $p = (p_1, p_2)$ such that the vectors of derivatives $v_1(p) = \left(\frac{\partial \lambda_1}{\partial x_1}(p), \frac{\partial \lambda_1}{\partial x_2}(p)\right)$ and $v_2(p) = \left(\frac{\partial \lambda_2}{\partial x_1}(p), \frac{\partial \lambda_2}{\partial x_2}(p)\right)$ of $\lambda_1(p)$ and $\lambda_2(p)$ are linearly dependent, that is, the vector $v_1(p)$ is a multiple of $v_2(p)$.

[13] An animation of the swallowtail can be found under
https://www.youtube.com/watch?v=MV2uVYqGiNc&list=UUNuhwFeHg6EIqQaVIs1KRRg

Figure 8: Swallowtail from above (left) and below (right).

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[2] Felix Klein, *Vorlesungen über das Ikosaeder und die Auflösung der Gleichungen vom fünften Grade*, (1993), Reprint of the 1884 original, Edited, with an introduction and commentary by Peter Slodowy.

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