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The Interaction of Curvature and Topology

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In this snapshot we will outline the mathematical notion of *curvature* by means of *comparison geometry*. We will then try to address questions as the ways in which curvature might influence the topology of a space, and vice versa.

1 Introduction

At its origin, geometry is the study of distances and angles in the 2-dimensional Euclidean plane and has been around in the form of a mathematical theory as we know it today since the time when Euclid (the famous Greek mathematician and philosopher who lived between the fourth and third Centuries BC) wrote the vastly influential "Elements" in circa 300 BC.[1] The concepts expressed in his treatise are directly inspired by everyday observations – for example, by lengths, curves or slopes that we can encounter in our surroundings. More recently, and essentially supported by the development of *differential calculus*, curvature started to be understood as a fundamental geometric entity in the 19th century. This new way of looking at curvature was initiated by the work of Nikolai Iwanowitsch Lobatschewski (1792–1856) and János Bolyai (1802–1860), who discovered that the *parallel postulate* is independent of the other axioms of Euclid's geometry by constructing a geometry satisfying all but the parallel postulate (nowadays called *hyperbolic geometry*, to which we will come back

[1] Clearly, other cultures were independently aware of geometric concepts, which are surely an inevitable ingredient for larger construction projects.

shortly).[2] Afterwards, it was the work of Georg Friedrich Bernhard Riemann (1826–1866) that started the conceptual analysis of geometries in mathematics in terms of structures on spaces; beginning with surfaces and reaching far into mathematical ground where the human intuition breaks down.

2 Metrics and curvature bounds

In the broad area of pure mathematics, one is often interested in abstract notions that are characterized by a minimal number of requirements based on which a set of concepts can be formulated.

2.1 Metric spaces

For the concepts of distance, angle and curvature the nowadays favourable notion of this kind is a that of a *length space*.[3] A length space is a metric space with an additional property that ensures that distances can be understood in terms of paths between points in the space.

First, let us recall that a metric space is a *tuple*[4] (X, d), where X is a set and the *metric* $d\colon X \times X \to [0, \infty)$ is a map satisfying

(i) $d(x, x) = 0$ for all $x \in X$,
(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$,
(iii) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

The letter "d" stands for "distance", of course, which is measured between two points by the metric. For example, every interval $[a, b]$ is a metric space, where $d(x, y) = |y - x|$ for $x, y \in [a, b]$.

Crucially, a metric space yields a notion of continuity for maps as follows. We call $f\colon (X, d_X) \to (Y, d_Y)$ *continuous*, if for all points $x \in X$ and all $\varepsilon > 0$ there exists a $\delta = \delta(x, \varepsilon) > 0$ such that $d_X(x, y) < \delta$ implies $d_Y(f(x), f(y)) < \varepsilon$ for very $y \in X$. A continuous map $\gamma\colon [0, 1] \to (X, d)$ from the interval $[0, 1]$ into X is called a *path in X* if the *length* $L(\gamma)$ defined by

$$L(\gamma) := \sup \left\{ \sum_{i=1}^{\#\{t_i\}} d(\gamma(t_{i-1}), \gamma(t_i)) \mid \{t_i\} \text{ partition of } [0, 1] \right\}$$

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[2] Euclidean geometry is based on five postulates, of which the parallel postulate states that "Given any straight line and a point not on it, there exists one and only one straight line which passes through that point and does not intersect the first line, no matter how far they are extended."

[3] It is a mathematical object that enables us to speak about such concepts within it, for example we can tell how far two points are apart or what we mean by the angle between two paths.

[4] A tuple is a finite ordered list (sequence) of elements.

exists. Here the supremum is taken over all partitions $\{t_i\}$ of the interval $[0,1]$, which are characterised by a finite set of numbers $0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = 1$, and the sum runs over the elements in this set.[5] The idea behind this definition is to approximate the actual path by using line segments between finer and finer partitions, as illustrated in Figure 1.

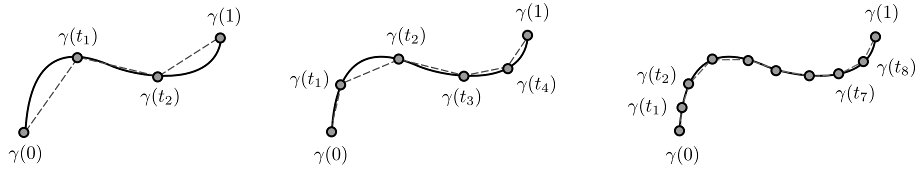

Figure 1: Improving the approximation of the curved path by finer partitions $\{t_i\}$. The dashed lines depict the distance between points in X.

A metric space (X, d) is called a *complete length space*, if it satisfies the following conditions:

(iv) $d(x, y) = \inf\{L(\gamma) \mid \gamma \colon [0,1] \to (X, d) \text{ a path with } \gamma(0) = x,\ \gamma(1) = y\}$ for all $x, y \in X$.

(v) For all $x, y \in X$ there exists a path $\gamma_{x,y} \colon [0,1] \to (X, d)$ with $\gamma_{x,y}(0) = x,\ \gamma_{x,y}(1) = y$ and $d(x, y) = L(\gamma_{x,y})$.

The first property ensures that the distance in X is intrinsically connected to the length it induces on paths, while the second property prevents the existence of situations where the distance is not actually realized by a path but by a limiting sequence not converging to a path in X, see Figure 2

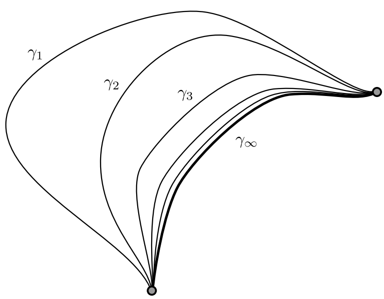

Figure 2: Limiting sequence of paths $\{\gamma_i\}$.

[5] If the supremum does not exist, γ is usually called a *non-rectifiable path*. In this text, we will not be concerned with these objects.

In this definition the path $\gamma_{x,y}$, which we will refer to as a *minimizing path from x to y*, is not necessarily unique.

2.2 Examples of length spaces

It is in Euclidean space $\mathbb{E}^n$, which is the length space $\mathbb{R}^n$ equipped with the metric $d(x, y) = |y - x|$, where $|(v_1, \ldots, v_n)^\top| = (\sum_{i=1}^n v_i^2)^{\frac{1}{2}}$ is the usual norm of a vector.

The minimizing path between two points is in fact unique in Euclidean space. This first example of a length space is merely the n-dimensional vector space $\mathbb{R}^n$ equipped with the metric given by the length of the difference of two vectors, where we interpret the vectors in $\mathbb{R}^n$ as the points of this space. More precisely, let $x = (x_1, \ldots, x_n)^\top$ and $y = (y_1, \ldots, y_n)^\top$ be vectors in $\mathbb{R}^n$. Then $y - x = (y_1 - x_1, \ldots, y_n - x_n)^\top$ is a vector of length $|y - x| = (\sum_{i=1}^n (y_i - x_i)^2)^{\frac{1}{2}}$ and we define the Euclidean distance between the points represented by x and y as $d(x, y) := |y - x|$.

Another interesting length space is given by the surface of a ball $\mathbb{S}^2(r)$, which is called a 2-sphere of radius r and is defined as the set $\mathbb{S}^2(r) := \{v \in \mathbb{R}^3 \mid |v| = r\}$ equip with the metric defined as follows. If we fix two distinct points x and y on the sphere then the lines through the points and the coordinate center $0 \in \mathbb{R}^3$ intersect forming an angle $\alpha_{x,y} \in (0, \pi]$. Now define $d(x, y) := \alpha_{x,y} \cdot r$, see Figure 3.

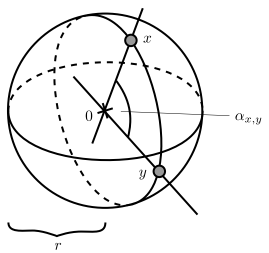

Figure 3: The 2-sphere as a length space.

This is precisely the length of a circle segment through x, y and centered at 0, which shows that $\mathbb{S}^2(r)$ is in fact a complete length space.

An interesting property of a length space (X, d) is its *diameter* $\mathrm{diam}(X)$, defined by

$$\mathrm{diam}(X) := \sup_{x,y \in X} \{d(x, y) \in \mathbb{R}_{\geq 0}\} \cup \{\infty\}. \tag{1}$$

It is not difficult to convince oneself that the diameter of Euclidean space is infinite, while it is well known that the sphere $\mathbb{S}^2(r)$ has a diameter of πr. Two length spaces with distinct diameters cannot be the same, and we conclude that $\mathbb{S}^2(r)$ and $\mathbb{E}^2$ must be fundamentally different.

The final example of a length space is slightly more complicated. Consider the open ball $B^2(r) := \{v \in \mathbb{R}^2 \mid |v| < r\}$, that is, the solid ball of radius r without its boundary circle, and let x and y be points in $B^2(r)$. Now we draw a circle through x and y that intersects the boundary circle in right angles in the points a and b, see Figure 4. The labels are chosen in a way that if we move around the circle counterclockwise, we read: a, x, y, b. It could happen that the points in fact lie on a straight line, which intersects the boundary circle in right angles. We will resolve this issue by employing an old trick in geometry – we simply call a straight line a circle of infinite radius.

Now we define the *hyperbolic distance*

$$d(x, y) := \log \frac{|y-a||x-b|}{|x-a||y-b|}. \tag{2}$$

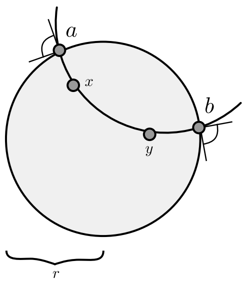

Figure 4: The distance in hyperbolic space.

The open ball $B^2(r)$ equipped with this metric is called *hyperbolic plane of scale* r and is denoted by $\mathbb{H}^2(r)$. The minimizing path between two points is exactly the circle segment connecting x and y from the definition above. From the definition given above, it can be calculated that the minimizing path between any two points x and y in $H^2(r)$ is precisely the segment of the circle through the two points that meets the boundary at right angles.

Note that points on the boundary, that is, points described by vectors of absolute value r, are not contained in $\mathbb{H}^2(r)$. If we move x along the arc closer and closer to a (as shown in Figure 4), we see that the expression (2) becomes increasingly large and we conclude that $\operatorname{diam} \mathbb{H}^2(r) = \infty$. But what distinguishes the geometries $\mathbb{H}^2(r)$ and $\mathbb{E}^2$ if the diameter does not? The answer should not be surprising to anyone who has read the text so far: They can be distinguished by their curvature!

Early on in the development of differential geometry the spaces $\mathbb{E}^2, \mathbb{S}^2(r)$ and $\mathbb{H}^2(r)$ were extensively studied and it turns out that the metrics we have defined are forerunners of a stronger mathematical structure called a *Riemannian metric*. The Killing-Hopf theorem in Riemannian geometry states that these three types of spaces are in a certain sense the most important spaces of constant curvature.

2.3 Model spaces of constant curvature

From the examples explained so far it is not yet clear what curvature is. It surely cannot be related to the shape of the minimizing paths – these were lines in Euclidean space and circle segments in both the spherical and the hyperbolic cases. The fundamental difference between these three spaces only becomes clear after examining *triangles*, by which we refer to three distinct points which are pairwise interconnected by minimizing paths - the *sides*. Triangles in Euclidean space are well-known to have a sum of angles of π, while a triangle on the sphere looks "thicker" and a triangle in the hyperbolic plane looks "thinner" (as shown in Figure 5). We will henceforth think of curvature as a real number κ, which determines to which degree triangles in a space look different from a triangle in Euclidean space. It is positive, if triangles look "thicker" and negative, if triangles look "thinner". After examining the Riemannian structure of these spaces it turns out that $\mathbb{H}^2(r)$ has curvature $-\frac{1}{r^2}$, $\mathbb{E}^2$ has no curvature and $\mathbb{S}^2(r)$ has curvature $\frac{1}{r^2}$.

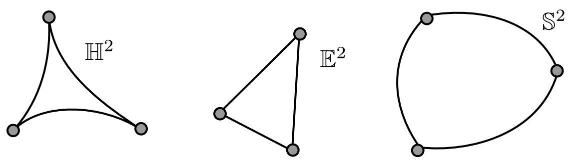

Figure 5: Triangles in the hyperbolic plane, the Euclidean plane and the sphere. Keep in mind, that the sides are minimizing paths in the respective spaces.

This motivates the definition of these spaces as *model spaces of constant curvature* and we introduce the notation:

$$M_\kappa := \begin{cases} \mathbb{S}^2(\frac{1}{\sqrt{\kappa}}) & \kappa > 0 \\ \mathbb{E}^2 & \kappa = 0 \\ \mathbb{H}^2(\frac{1}{\sqrt{-\kappa}}) & \kappa < 0 \end{cases} .$$

Here, constant curvature refers to the fact that it does not matter where and how in the space we consider a triangle with fixed side lengths. The sum of angles will not be affected by these choices.

2.4 Extending the notion of curvature

The way to extend this notion of curvature to arbitrary length spaces is by comparison with these model spaces. If we have a triangle $\Delta = \Delta(x, y, z)$ in a length space (X, d) whose sides are minimizing paths in X, then we can draw a triangle $\overline{\Delta}$ with the same side lengths in a model space M_κ. We call $\overline{\Delta}$ a comparison triangle for Δ.[6] For every point p on a side of the triangle Δ we can find a comparison point $\overline{p}$, which has the same distance to the end points of the side it is located on as in the original triangle.

Note that curvature for length spaces is only defined in this comparison sense, which implies that all triangles in a space we have in this way in comparison to triangles in M_κ. Moreover, this κ does not have to be optimal.

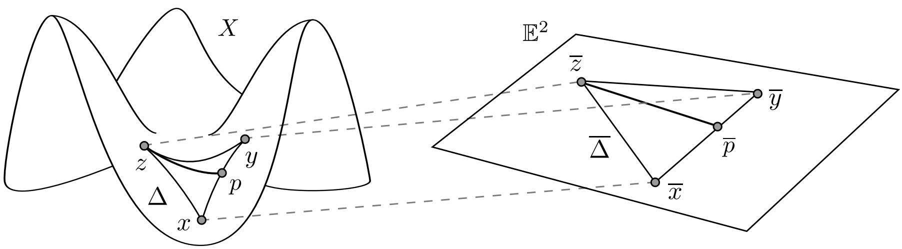

Figure 6: Triangle in a space and comparison triangle with point p and comparison point $\overline{p}$. We compare the distances $d_X(z, p)$ and $d_{\mathbb{E}^2}(\overline{z}, \overline{p}) = |\overline{p} - \overline{z}|$.

We say that X has *curvature at least* κ, if the distance between every point on a side and the opposite vertex is greater or equal to the corresponding distance for the comparison point. Analogously, we say that X has *curvature at most* κ, if this distance is always smaller than the distance in the comparison triangle.

[6] It does not matter where in our model space we draw the comparison triangle, since the curvature is everywhere the same. Moreover, we note that it might not always be possible to draw a comparison triangle, because the finite diameter of S^2 prevents us from considering larger side lengths.

3 Recognizing topology

3.1 The Euler characteristic

Topology describes a certain quality of a space, which captures its global structure. It does not detect changes such as bending, as long as we do not cut or rip apart. For example, the 2-sphere and the surface of a tetrahedron are topologically equivalent, as we will see later.

One of the oldest invariant quantities that can be defined on this global structure is the *Euler characteristic*. Let us give an intuitive definition of this important quantity.

Think of a length space X, which is the result of gluing together triangles along their edges without letting them intersect, and in such a way that the construct closes up, that is, it forms a closed surface. Then, we can count the number of triangles we needed – call this the number of faces F. Furthermore, we count the number of edges E and the number of vertices V after the gluing procedure in our construct. The Euler characteristic of this object is then defined as

$$\chi(X) := F - E + V \tag{3}$$

Quite surprisingly, the Euler characteristic does not depend on the way we choose to glue the triangles, if we happen to build the same construct.[7] It measures an entity called the *genus* of the surface given by X, which we can say in a very naive way counts the number of holes that are present in the space (as shown in Figure 7).

3.2 Connection between curvature and the Euler characteristic

Now comes the tricky part. In a length space like the one we described above, we can "flatten" all the trianglesin such a way that they become like those found in the Euclidean space, and this can be done without changing the topology of the space. Think of this as a procedure that "concentrates" all the metric information on the vertices (as shown in Figure 8).[8]

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[7] A proper definition of this notion requires a few technical assumptions, which I avoid here to keep the presentation simple. The reader not familiar with this fact is invited to carry this out for a few triangular subdivions of the 2-sphere. A triangulation requires to give the space the structure of a simplicial complex (up to a homeomorphism). For all of these the number $\chi(S^2(r))$ turns out to be 2 and is clearly independent of the radius we choose.

[8] The more advanced reader will immediately realize that this does not always work in terms of the pictures that we drew embedded in $\mathbb{R}^3$. Nevertheless, this is a valid operation in a more abstract sense, and we can make it work even for three dimensional figures, if we choose our triangles small enough.

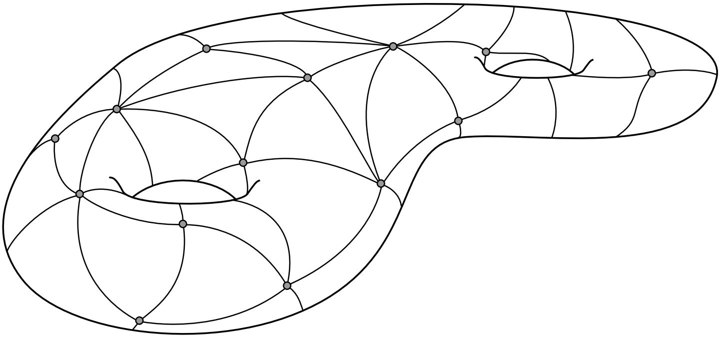

Figure 7: Example of a surface with genus 2 glued from triangles.

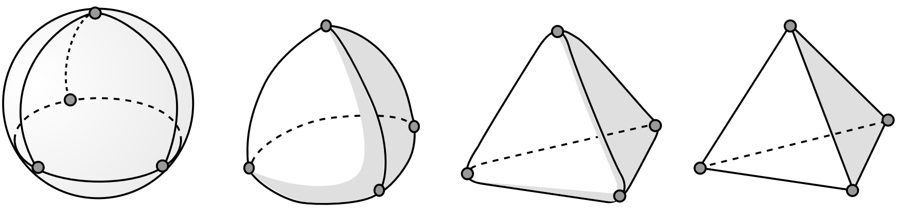

Figure 8: Flattening a triangular subdivision of the sphere.

That means curvature is only an interesting property if the sides of our triangles (of minimizing paths) we use to measure it enclose at least one vertex. Let $\{v_1, \ldots, v_V\}$ denote the set of vertices in X. Consider a small triangle around every vertex v_i in X and denote by α_i the sum of the angles. As we have seen above $\alpha_i < \pi$ would correspond to negative curvature, $\alpha_i = \pi$ would mean that at v_i we have glued triangles in a plane, while $\alpha_i > \pi$ corresponds to positive curvature.

The *Gauß–Bonnet theorem* tells us that we can detect the Euler characteristic by examining the curvature; more precisely it states

$$\sum_{i=1}^{V}(\alpha_i - \pi) = 2\pi\chi(X).$$

For example, if we have no curvature at all, then X must have Euler characteristic zero. On the other hand if we know that the Euler charasteristic is zero, as we do for the *torus* – the surface of a doughnut, which has precisely one hole – then the sum on the left hand side has to vanish as well, that is, there have to be points, or areas, with positive curvature that compensate points, or areas,

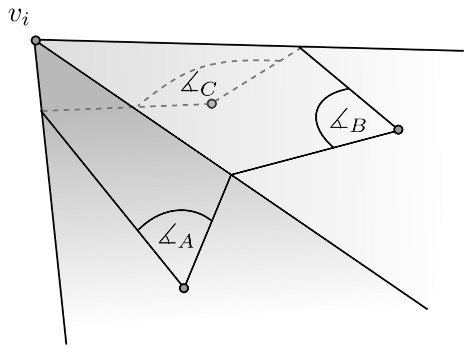

Figure 9: Small triangle around a vertex v_i, with a sum of all angles $\alpha_i = \measuredangle_A + \measuredangle_B + \measuredangle_C$.

with negative curvature. If we draw the standard picture to depict the torus this expected property of the space becomes immediately evident, see Figure 10.

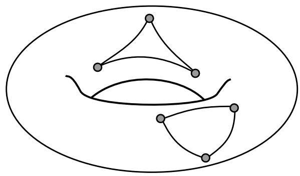

Figure 10: Negatively and positively curved triangles on a torus.

Now we want to pose a similar question for a more general complete length space X – we would like to "detect holes", which we can see as something contributing to the topological complexity of X. Recall that we constructed length spaces by gluing triangles. Next, we would like to consider 'gluings' from more general buildings blocks. Note that before we used points as vertices, line segments as edges and triangles as faces, and all of these entities have something in common – they arise if we consider 1, 2, or 3 points in $\mathbb{R}^n$ and then take the *convex hull*. Recall that a subset $U \subset \mathbb{R}^n$ is called *convex*, if any two points within U can be connected by a line segments which lies within U entirely. Taking the convex hull of a number of points is a procedure that determines the smallest convex subset in which all of these points are contained. The higher dimensional building blocks Δ^n for $n \in \mathbb{N}$, called *n-simplices*, are defined as the convex hulls of $n+1$ points in $\mathbb{R}^n$. For $n = 3$, this is a filled tetrahedron, while for $n \geq 4$ the objects are of course harder to imagine and to visualize.

We note that the boundary of an n-simplex is composed from $(n-1)$-simplices ($n+1$ of them, to be precise) and we can equip an n-simplex with the choice of labels for each corner point. Namely, we call Δ^n, together with a map $o\colon \{1,\ldots,n+1\} \to \Delta^n$ that sends each number $n \in \{1,\ldots,n+1\}$ to a distinct corner point, an *oriented simplex.*

Now think of a space X that is obtained by gluing in an "orderly fashion" from finitely many of these oriented building blocks, and consider the following device to keep track of the blocks we used. For every $n \in \mathbb{N}$ let $\sigma_n := \{\Delta_1^n, \ldots, \Delta_{k_n}^n\}$ be the set of n-simplices in X – recall that we allowed only finitely many, so most of the σ_n are just empty, say k_n many for every n. If we take σ_n as the basis of a real vector space $C_n := \langle \sigma_n \rangle$ – the vector space spanned by the σ_n – we can express every element $\eta \in C_n$ as formal sum $\eta = \sum_{i=1}^{k} \lambda_i \Delta_i^n$ for certain $\lambda_i \in \mathbb{R}$ depending on η. Moreover, we can keep track of the way we glued the simplices along their boundary by a *boundary map* $\partial_{n+1}\colon C_{n+1} \to C_n$, which is linear. An important observation here is that the boundary of the boundary is an empty set.

For obvious reasons we would like to ask, which part of the vector space C_n says something meaningful about the topology of X. The idea is to look at formal sums of simplices that do not have a boundary, that is, lie in the kernel of ∂_n, and that are not a boundary themselves, that is, are not in the image of ∂_{n+1}. Thus, these elements have to enclose something, which cannot be filled by higher dimensional simplices – a more general "hole".

A comment is in place here. The material presented above is a very short overview of objects that show up once we introduce simplicial homology. A proper definition is surely beyond the scope of this note and the entire material should be considered as an outlook on topological invariants for higher dimensional objects. I refer to the bibliography for the more interested reader.

To measure how many of these we have in X, we define the *Betti numbers* of X as $b_i(X) := \dim(\operatorname{Ker}\partial_n / \operatorname{Im}\partial_{n+1})$.

If X is a surface as in the beginning of this section, then $b_0(X) - b_1(X) + b_3(X)$ turns out to be precisely the Euler characteristic of X and thus we define the *Euler characteristic* for a more general length space X glued from building blocks as

$$\chi(X) := \sum_{i=1}^{l} (-1)^i b_i(X),$$

where l is the largest number such that $\sigma_l \neq \emptyset$.

The Euler characterstic and Betti numbers are powerful tools to distinguish two spaces in terms of their topology.

4 A glimpse at Riemannian geometry

As mentioned earlier, some of the examples of length spaces, such as $\mathbb{S}^2(r)$, $\mathbb{H}^2(r)$, $\mathbb{E}^n$ or the torus T^2, can be endowed with a much stronger structure encoding their metric. First of all, they are examples of *smooth manifolds of dimension* n, which are spaces locally homeomorphic to $\mathbb{R}^n$ (this means that the proximity of every point looks topologically like a ball in $\mathbb{R}^n$) and maps between them admit a notion of differentiability. Naively, we think of these spaces as glued from building blocks (for example, triangular surfaces in dimension 2) in a way which does not produce corners or sharp edges (see Figure 7). Secondly, they are equipped with a *Riemannian metric*, which enables us to consider certain derivatives of the metric that determine curvature by means of functions on M. Thereby, in a *Riemannian manifold*, which is an (underlying) smooth manifold equipped with a Riemannian metric, we can think of curvature as an actual value (not just as a bound on the shape of triangles).

Riemannian geometry was kicked off by a twist of fate, namely Johann Carl Friedrich Gauß 1777–1855) picking the topic for Riemann's habilitation from a list of suggestions. In his inaugural lecture entitled "Über die Hypothesen, welche der Geometrie zu Grunde liegen" (held on June 10, 1854) B. Riemann laid the foundation of this important area of mathematics, which was then further studied by geometers such as H. A. Lorentz, H. Minkowski and H. Poincaré, leading Albert Einstein (1879–1955) to choose a (pseudo-)Riemannian manifold to model space-time in his theory of general relativity published in 1915. Only afterwards, thanks to the work of Hermann Klaus Hugo Weyl (1885–1955) and Hassler Whitney (1907–1989), mathematicians came to understand a Riemannian manifold by means of the definition we briefly outlined above – this applies especially to the theory of Hausdorff topological spaces and differential structures, that had not been studied in a rigourous axiomatic treatment before. In contrast, the idea to study geometric entities such as curvature in terms of metric spaces is even younger and owes much to the work of Heinrich Hopf (1894–1971) and Willi Ludwig August Rinow (1907–1979) from 1932.

For Riemannian manifolds, there are several remarkable theorems that interconnect their curvature with the topology of the underlying smooth manifold. The first we would like to mention is a theorem of Jacques Salomon Hadamard (1865–1963) and Élie Joseph Cartan (1869–1951)

Hadamard-Cartan's Theorem. *Let M be a* simply-connected *complete Riemannian manifold with curvature at most* 0 *(in other words, M has non-positive curvature). Then M has the same topology as $\mathbb{R}^n$.*

If not familiar with simply-connectedness, think of it as an assumption that makes absolutely sure that there are no "one dimensional holes" – in particular it implies $b_1(M) = 0$. Therefore, this roughly says that there is only one simply-connected smooth manifold that allows us to endow it with a Riemannian metric with curvature at most 0. On the other hand, there is a "gigantic" number of complete Riemannian manifolds with non-positive curvature and arbitrarily large first Betti number.

This picture changes fundamentally if we are interested in *non-negative curvature*, that is, in curvature at least 0. In this regime "too much" curvature restricts the size of the space in terms of the diameter.[9] Even stronger, if the curvature in a Riemannian manifold is larger than a certain threshold, while the diameter is not too small as well, then the manifold must be the sphere. The latter statement is called the *Grove-Shiohama diameter sphere Theorem* (1977).

Somewhat complementary to the Hadamard-Cartan theorem for non-positive curvature, Mikhail Leonidovich Gromov proved the following surprising statement for manifolds of non-negative curvature

Gromov's Theorem. *Let M be a complete Riemannian manifold of dimension n with curvature at least 0. Then, the Betti numbers $b_i(M)$ cannot be larger than a certain constant $C(n)$, which only depends on the dimension n.*[10]

With this at hand, it becomes easy to construct manifolds that do not even allow us to find a single metric with non-negative curvature on them.

Although the above mentioned results are important theorems that provide great insight in the topic at hand and require quite a bit of mathematical tools and dexterity to state the proofs, geometers are still only scratching the surface of Riemannian geometry.

5 Where to proceed from here?

In case the discussion up until this point raised the appetite for modern geometry in the reader, there are numerous places to continue the investigation certainly depending on your mathematical background.

To start off, one could begin reading K. Jänich's "Topology" [6], which is a gentle introduction to the realm of topology and continue into geometry with C. Bär's "Elementary Differential Geometry" [1].

From there, one can advance deeper into Riemannian and differential geometry, e.g. with "An Introduction to Differentiable Manifolds and Riemannian Geometry" [3] by W. Boothby or "Riemannian Geometry and Geometric Analysis" [7] by J. Jost. Alternatively, the book "A Course in Metric Geometry"

[9] This is called the *Bonnet-Myers Theorem* in classical Riemannian geometry, though it actually requires curvature at least C for a positive constant C.

[10] It is a conjecture of Gromov that $C(n) = 2^n$, but the upper bounds found so far are off by several orders of magnitude.

[4] by D. Burago, Yu. Burago and S. Ivanov tends more towards the metric viewpoint that we have employed in beginning of the text.

Ultimately, the article “Sign and geometric meaning of curvature” [5] by M. L. Gromov paints a clear picture of modern global Riemannian geometry and anticipates the development of Alexandrov geometry by Yu. Burago, M. Gromov and G. Perelman. An introduction to the latter is currently being written by S. Alexander, V. Kapovitch and A. Petrunin.

A delightfully complete overview of current trends in Riemannian geometry is given in the book “A Panoramic View of Riemannian Geometry” [2], which was written by M. Berger – one of the most influential figures in geometry of the late 20th century and a member of the enigmatic group Arthur L. Besse.

Finally, even the very starting point – B. Riemann’s “Über die Hypothesen, welche der Geometrie zu Grunde liegen” – has an interesting viewpoint to offer and is available in several commented versions, e.g. by J. Jost [8]. Then maybe you will find a reason to light a candle on June 10, celebrating the day of birth of Riemannian geometry and tip your hat to C. F. Gauß for his neat choice.

Image credits

All images from the author.

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