Snapshots of modern mathematics
from Oberwolfach

№ 9/2016

Polyhedra and commensurability

Rafael Guglielmetti • Matthieu Jacquemet

This snapshot introduces the notion of commensurability of polyhedra. At its bottom, this concept can be developed from constructions with paper, scissors, and glue. Starting with an elementary example, we formalize it subsequently. Finally, we discuss intriguing connections with other fields of mathematics.

1 A warming-up example

Ernest goes camping with some friends. His tent has the shape of a pyramid based on a square. The square's sides have a length of 1.5 m and the tent is 1.5 m in height:

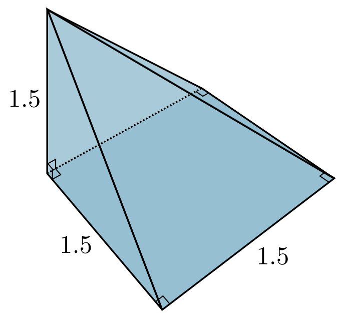

Figure 1: Ernest's tent.

Ernest could enjoy a carefree time on the camping site if it was not for the forest being full of mosquitoes. But Ernest expected this and has taken with him mosquito fogger. He chanced that two such bug bombs should be sufficient for his trip. He reads the label where it says that one bomb is enough for a volume of 1.2 m^3 for one night. Ernest does not want to be bitten, but neither does he want to use up the two bombs if not necessary. What should he do? The problem is of course to compute the volume of the pyramid-shaped tent. Even without knowing the volume formula for a pyramid by heart, it is not difficult to determine the volume of the tent. Namely, by gluing together three copies of the pyramid in a smart way, one obtains a cube with sidelength 1.5 m:

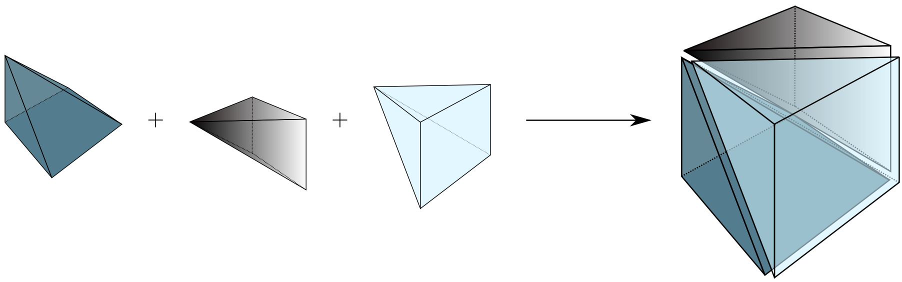

Figure 2: Three copies of Ernest's pyramid give a cube.

Hence, the volume (in the sequel, always per m^3) of the pyramid is one third of the volume of the cube, that is, written in formulas, one has

$$\mathrm{vol}(\mathrm{pyramid}) = \frac{1}{3} \cdot \mathrm{vol}(\mathrm{cube}) = \frac{1}{3} \cdot \left(\frac{3}{2}\right)^3 = \frac{9}{8} = 1.125.$$

The solution to Ernest's problem is now clear: even if Ernest uses only one bomb, he will be safe from being bitten for the whole night!

Ernest's example shows how quantitative geometric data (here: volume) can be deduced from qualitative ones (here: arrangement of copies of a pyramid). This is our first contact with the subject of this paper: the concept of *commensurability of polyhedra.*

2 Do-it-yourself with polyhedra

A *polyhedron* in dimension 3 is a solid shape bounded by polygons, called *faces* (this notion can be extended to arbitrary dimension). All faces are bounded by straight *edges*, each of them connecting two *vertices*. Any edge is shared by exactly two faces, and a vertex is shared by at least three faces and edges. For example, Ernest's pyramid has 5 faces, 8 edges and 5 vertices.

If two polyhedra P and Q have at least one identical face, then, by translating, rotating and/or reflecting one of the polyhedra, say Q, one can move Q such that the identical faces coincide. We call this operation and the resulting polyhedron the *gluing* of P and Q along the prescribed identical face. Notice that P and Q need not be identical. However, they might have more than one identical face. In this case, one has to specify along which face P and Q are glued.

On the other hand, if a plane intersects the interior of a polyhedron P, it determines two polyhedra P_1 and P_2 such that P is the gluing of P_1 and P_2 along the face they have in common in the plane. Such an operation is called the *cutting* of P with respect to the prescribed plane.

For example, the cube depicted in Figure 2 is obtained by gluing three copies of Ernest's pyramid, and Ernest's pyramid is obtained as one of the congruent pieces when cutting this cube correspondingly.

The cutting of a polyhedron into congruent pieces is not always unique. For instance, the cube of Figure 2 can also be cut into six similar pyramidal pieces:

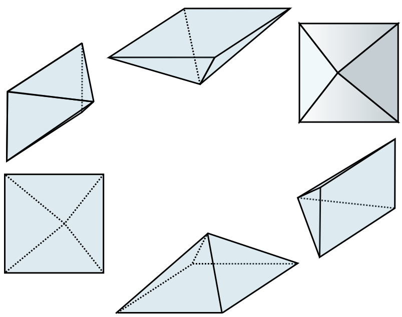

Figure 3: Dissection of the cube into six pyramids.

Ernest's pyramid and the new, small pyramid from Figure 3 can both be used in order to build the same polyhedron, that is, the cube. The two pyramids are therefore said to be *commensurable*. In general, two polyhedra P and Q are called *commensurable* if it is possible to glue a certain number of copies of P, say k, together to obtain a new polyhedron that can be cut into a certain number of copies of Q, say l. The numbers k and l are positive integers, and they allow us to relate the volumes of P and Q. Indeed, we have

$$k \cdot \mathrm{vol}(P) = l \cdot \mathrm{vol}(Q), \text{ that is, } \mathrm{vol}(Q) = \frac{k}{l} \cdot \mathrm{vol}(P). \qquad (\star)$$

In our example, the volume of the small pyramid is then given by

$$\mathrm{vol}(\text{small pyramid}) = \frac{3}{6} \cdot \mathrm{vol}(\text{Ernest's pyramid}) = \frac{9}{16} = 0.5625.$$

The relation ($\star$) is a *necessary condition* for commensurability of P and Q, but it is not a *sufficient condition.*

In other words, if $\frac{\mathrm{vol}(P)}{\mathrm{vol}(Q)}$ is not a rational number, that is, $\frac{\mathrm{vol}(P)}{\mathrm{vol}(Q)} \notin \mathbb{Q}$, then P and Q cannot be commensurable; on the other hand, they could be commensurable, but one cannot be sure just by looking at their volume.

3 Through the looking-glass

In the above, we have considered a very particular setting: the 3-dimensional Euclidean space, that is, our everyday geometric space $\mathbb{E}^3$. However, the notion of commensurability does neither depend on the dimension nor the type of the abstract geometric space in which one studies (generalized) polyhedra. The only thing we need is a general notion of polyhedron (qualitative), and a general notion of volume (quantitative). Thus, commensurability can be defined in any kind of geometric space with arbitrary dimension n, where n is a positive integer. Such a space could be the 7-dimensional *spherical space* $\mathbb{S}^7$, or the 21-dimensional *hyperbolic space* $\mathbb{H}^{21}$ (both exist!) – whether or not our human minds can figure it! These spaces all have in common to be *spaces of constant curvature*[1] ($\mathbb{E}^n$ has zero, $\mathbb{S}^n$ positive, and $\mathbb{H}^n$ negative curvature) and are the most important model geometric spaces (see the textbook [6], for example).

In this kind of abstract settings, the notion of volume can be difficult to handle. For example, we do not even know the volume of very simple objects in the 7-dimensional hyperbolic space. However, commensurability can still help us

[1] Curvature is an involved concept and there is no need to discuss it exhaustively at this point. We nevertheless drop some short comments in case you are not already familiar with curvature: You will have no difficulties to think of a curved line or a curved surface embedded in our everyday Euclidean space (an example would be of course the 2-dimensional sphere, the surface of a ball). In mathematics, the concept of curvature is abstracted to objects in arbitrary dimensions and even objects which can not be embedded into a surrounding space. These may then be *intrinsically curved*; this notion of curvature does not depend on the embedding into a surrounding space. In such intrinsically curved spaces, some of the rules from our everyday flat space typically do not work anymore. Take for instance the sum of angles of a triangle. In flat space, the angles sum up to 180°, whereas in a curved space, the sum of angles will deviate from this value! In spherical spaces $\mathbb{S}^n$, the angle sum in any triangle is bigger than 180° (positive curvature), in hyperbolic spaces $\mathbb{H}^n$, the angle sum in any triangle is smaller than 180° (negative curvature).
Furthermore, we say that a space has *constant curvature* if the curvature is the same at every point in the space.

to have a better understanding of volumes in these abstract spaces, just as before with Ernest's tent: if we can show that two polyhedra are commensurable, then we are sure that their volumes are related by a rational factor as the relation ($\star$) expresses. Hence, volume can be both a tool and a goal.

A typical task mathematicians set themselves is to *classify* polyhedra up to commensurability, meaning, their aim is to find out which polyhedra are mutually commensurable and sort them by their respective *class of commensurability.*

Commensurability of polyhedra is at the junction of several fields of mathematics, and is related to very different methods and questions. Mathematicians are currently still working on some of them. Here are a few examples:

- **Hilbert's Third Problem**: In 1900, the mathematician David Hilbert (1862–1943) presented a list of 23 problems to the mathematical community. He thought that solving these problems would greatly improve our understanding of mathematics. The third problem from Hilbert's list can be stated as follows: given any two polyhedra of equal volume (in the Euclidean space $\mathbb{E}^3$), is it always possible to cut the first one into finitely many polyhedra that can be moved around and glued together to build the second one? Hilbert's student Max Dehn (1878–1952) showed that the answer to this question is "no" in general, by introducing a new notion nowadays called the *Dehn invariant.* With this tool, it is possible to show that even if they have the same volume, the cube with side 1 and the regular *tetrahedron*[2] (with angle $\arccos\frac{1}{3} \approx 73°$) with side $\sqrt[3]{\frac{\sqrt{2}}{12}}$ are not commensurable.

Let us pause here for a moment. Maybe you are already familiar with the concept of a *group* in mathematics; then you might want to skip the following paragraph. Otherwise, this short digression hopefully can help you to gain a better understanding of what follows.[3]

Groups in mathematics are a very basic and fundamental structure and you will find them again in all sorts of mathematical branches. By a *group* we mean a set (denoted by G, for example) together with an operation, which we denote by $*$. The operation $*$ combines any two elements from G mapping them to another element from G. If g_1 and g_2 are elements of G, we typically write this as

$$g_1 * g_2 = h \tag{1}$$

and call h the *product of* g_1 *and* g_2. There are further restrictions on $*$, the set G and its elements: There always has to be a *neutral element*, denoted by e, which sloppily

[2] A tetrahedron is simply a pyramid with triangular base.

[3] Besides, the Snapshot 5/2015 *Symmetry and characters of finite groups* written by Eugenio Gianelli and Jay Taylor can be worth reading in this context if you like to learn more about the intriguing field of groups. The authors introduce the concept of group and symmetry in far more detail than can be done here.

speaking does not change anything under the operation $*$, that is, for any element g from G we always have $e * g = g * e = g$. Also, for any g in G there exists an *inverse element* g^{-1} which "neutralizes" g if we combine them: $g^{-1} * g = g * g^{-1} = e$. Furthermore, the operation $*$ has to be *associative*, a rule we know from the addition (as well as multiplication) of real numbers: for any g, h, and k we have $g*(h*k) = (g*h)*k$.

This last fact already gives us a hint to a first example of the many various and different ways in which groups manifest themselves: When we think of the common addition "+" to be a group operation on $G = \mathbb{R}$, the real numbers form a group $(\mathbb{R}, +)$. Can you guess which number is the neutral element of this group?

We could be more minimalistic and look only at a small subset of the real numbers, the integer numbers $\mathbb{Z}$. Indeed, $(\mathbb{Z}, +)$ already forms a (discrete) group. Here, *discrete* means that the elements of $\mathbb{Z}$ are "separated" in the sense that they have a fixed minimal distance from each other. And there are more groups to be found within the common numbers and operations.

Imagine an equilateral triangle in the plane. Rotations by an angle of 0°, 60°, or 120° around the geometric center of the triangle respect the symmetry of the triangle and keep the picture the same. These three rotations together form a (discrete and finite) group with three elements. They are part of the whole *symmetry group* of the triangle which also includes reflections with respect to its symmetry axes. In contrast, the symmetry group of a circle will contain far more elements, rotations are possible about any abitrary angle. The symmetry group of the circle is continuous (and infinite!) – as are the real numbers.

The concept of such symmetry groups can be extended to arbitrary polygons in the plane and even polyhedra in more dimensions!

- **Group theory**: The classification up to commensurability of polyhedra is related to the classification of certain algebraic structures. For example, in group theory one might want to look specifically at not all but a selection of specific elements H of a group G. If this selection of group elements itself forms a group of its own with the group operation from G then H is called a subgroup of G. As two polyhedra may be commensurable or not, there is also a notion of commensurability in group theory; different subgroups may be *commensurable in a group theoretic sense* or not.[4]
  Miraculously, commensurability of groups can be related with commensurability of polyhedra! Namely, let $\mathbb{X}^n \in \{\mathbb{E}^n, \mathbb{S}^n, \mathbb{H}^n\}$ be one of the model geometric spaces. Think of a transformation of $\mathbb{X}^n$ that gives any point in $\mathbb{X}^n$ a new place but at the same time preserves the original distance between any two points. Such a transformation is also called an *isometry*. Isometry transformations may be composed by applying them successively. The result is a new isometry of course! In this sense, isometries form a

[4] The precise definition is: subgroups H_1 and H_2 of a group G are said to be *commensurable* if the intersection $H_1 \cap H_2$ is of finite index (not necessarily the same) in both H_1 and H_2, that is, if the number of sets $\{aH_1 \cap H_2 \mid a \in H_1\}$ and $\{bH_1 \cap H_2 \mid b \in H_2\}$ are both finite.

group. To any discrete (sub)group of isometries of $\mathbb{X}^n$, one can associate at least one *fundamental polyhedron*. And in fact, also the two notions of commensurability are related to each other: if two such discrete groups are commensurable, then their associated fundamental polyhedra are commensurable. The converse, however, is not true in general, that is, there are commensurable fundamental polyhedra whose associated discrete groups are not commensurable.

- **Algebra**: Commensurability of polyhedra is often also related to other algebraic objects. This is especially the case if we look at polyhedra associated to *arithmetic* discrete groups of isometries.[5] These groups, and accordingly the polyhedra, can be shown to come from a *quadratic form* (which furthermore satisfies some nice properties). A quadratic form is a polynomial (in possibly several variables) where all terms have degree 2.[6] In this setting, one can associate to the polyhedron a complete list of *invariants*, that is, characteristic features which will be the same for all polyhedra from one and the same commensurability class. These then allow us do decide about (non-)commensurability only by comparing these invariants: for the experts, these are a certain number field, together with a Clifford algebra, and a quaternion algebra over it.
- **Coxeter groups**[7]: If the angle between any two intersecting facets of a polyhedron is of the form $\frac{\pi}{k}$, $k \in \{2, 3, \ldots, \infty\}$, then the polyhedron (and the discrete group generated by the reflections in its facets) is called a *Coxeter polyhedron* (a special type of a Coxeter group).

  An *n-simplex* is the geometric shape constructed from $n+1$ vertices in general position in a space of dimension n: a 2-simplex is a triangle, a 3-simplex is a tetrahedron, etc. An *n-orthoscheme* is a very particular type of n-simplex: all but n of its angles are right angles (i.e. of measure $\frac{\pi}{2}$). For example, a 2-orthoscheme is a right-angled triangle. A Coxeter n-orthoscheme whose non-right angles are respectively $\frac{\pi}{k_1}, ..., \frac{\pi}{k_n}$ can be represented by the symbol $[k_1, ..., k_n]$ collecting the respective denominators. Figure 4 shows a representation of a 3-dimensional hyperbolic Coxeter orthoscheme $[k, l, m]$.

  Hyperbolic Coxeter simplices exist only up to dimension 9. Their volumes and commensurability classes have been determined by Johnson, Kellerhals, Ratcliffe and Tschantz [2].

---

[5] For the experts: A typical example is the *modular group* $\mathrm{PSL}(2, \mathbb{Z})$, a discrete subgroup of the so-called Möbius tranformations, isometries of the 3-dimensional hyperbolic space $\mathbb{H}^3$.

[6] For example, $q(x_1, x_2) = x_1^2 + 3x_1x_2 + 2x_2^2$ is a quadratic form in two variables.

[7] In Section 3 of the Snapshot 7/2014 *Swallowtail on the shore* by Ragnar-Olaf Buchweitz and Eleonore Faber you find further material on reflection groups, so-called geometric Coxeter groups, which also served as a starting point in investigating Coxeter groups in general.

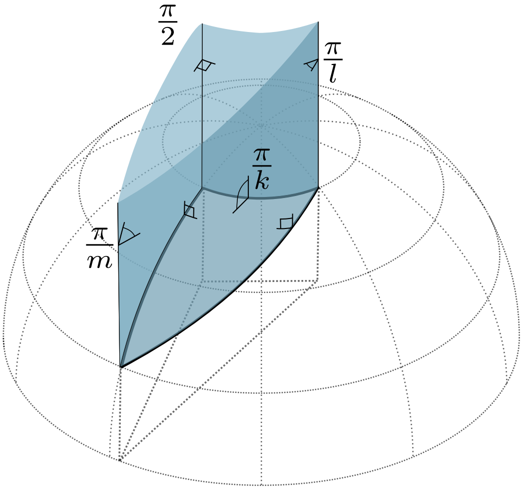

Figure 4: The hyperbolic Coxeter 3-orthoscheme $[k, l, m]$.

In a joint work with Ruth Kellerhals [1, 3], we determine the commensurability classes of all Coxeter groups coming from Coxeter pyramids with $n+2$ facets in the hyperbolic space $\mathbb{H}^n$ (such polyhedra exist up to dimension 17!). With help of some of the methods mentioned above, such as algebraic invariants and cutting/gluing procedures, we can categorize all of them according to their class of commensurability.

- **Number theory**: The volume of polyhedra in the hyperbolic space of dimension 3, that is in $\mathbb{H}^3$, can be computed with help of the *Lobachevsky function* $\Lambda : \mathbb{R} \to \mathbb{R}$ given by

$$\Lambda(x) := \int_0^x \log|2\sin t|dt.$$

This function has certain symmetry properties and is related to other objects number theorists study, such as *polylogarithms* and *Clausen functions* (see [5] for example). It is still an open question whether the ratio $\lambda := \frac{\Lambda(\pi/3)}{\Lambda(\pi/4)}$ is a rational number or not.

A conjecture of Chowla, Milnor and others states that the quotient λ is not rational. If the conjecture is true, then any two polyhedra with respective volumes $\alpha \cdot \Lambda(\pi/4)$ and $\beta \cdot \Lambda(\pi/3)$, $\alpha, \beta \in \mathbb{Q}$, are not commensurable (remember our commensurability condition $(\star)$!). This would allow us to prove that the hyperbolic Coxeter orthoschemes $[3, 4, 4]$ and $[3, 3, 6]$, with volume $\frac{1}{6} \cdot \Lambda(\frac{\pi}{4})$ and $\frac{1}{8} \cdot \Lambda(\frac{\pi}{3})$, respectively, are not commensurable (in Section 4, we will present the idea of an alternative proof of this fact, see also [2]).

4 Some details: worked out examples

This section gives you a first (and at the same time very sketchy) impression how we work with Coxeter groups. For all details we would like to refer to the list of references at the very end of this snapshot.

4.1 Comparing two 3-dimensional hyperbolic Coxeter orthoschemes

Imagine we want to show that the two 3-dimensional hyperbolic Coxeter orthoschemes $P_1 = [3, 4, 4]$ and $P_2 = [3, 3, 6]$ (see Figure 4) are not commensurable (as polyhedra). This task is equivalent to showing that the Coxeter group Γ_1 generated by the reflections in the facets of P_1 is not commensurable to the Coxeter group Γ_2 generated by the reflections in the facets of P_2.

Luckily for us, these two groups are examples of what we call a *non-cocompact arithmetic hyperbolic Coxeter group*, which are particularly "easy" to work with (for Coxeter group experts).

We have already noted that an arithmetic group is always associated to a quadratic form (see Section 3) which can help us to decide about commensurability. Using a method described in [7], we obtain

$$q_1(x_0, x_1, x_2) = -2x_0^2 + x_1^2 + 3x_2^2 + 6x_3^2 \tag{2}$$

$$q_2(x_0, x_1, x_2) = -6x_0^2 + x_1^2 + 3x_2^2 + 6x_3^2 \tag{3}$$

for the quadratic forms associated to the arithmetic groups Γ_1 and Γ_2 respectively. Obviously, in this representation q_1 and q_2 do not contain any "mixed" terms of variables such as x_1x_2. Indeed, it is possible to transform any quadratic form into such a *diagonal* form without destroying any of its characteristics needed for our purposes. The *signature* of these forms coming from the non-cocompact arithmetic hyperbolic Coxeter groups in 3 dimensons is $(3, 1)$, which in the diagonal represenation can be easily read off from the sign of the coefficients: three are of the same type and one bears the opposite sign. Moreover, this diagonal form makes it easier for us to work with.

The crucial point is that if the two Coxeter groups Γ_1 and Γ_2 are commensurable, the corresponding forms will share a couple of properties (and vice versa), so characteristic that one speaks of *similar* quadratic forms. Similarity of quadratic forms is usually defined totally independently from commensurability of groups. However, miraculously, the notions coincide in our context, meaning that the corresponding Coxeter groups are commensurable if and only if the quadratic forms are similar to each other.

In general, it is not easy to decide whether two diagonal forms are similar or not. In our particular setting where we in addition have the quadratic forms in diagonal representation, we are lucky to have a criterion which can tell us that the forms *cannot* be similar: for the given signature $(3, 1)$ of the forms it is a necessary condition for similarity that the products of coefficients of the two forms q_1 and q_2 do not differ by more than a factor of a square of a rational number.[8]

Checking these products for the given forms q_1 and q_2, we see this is clearly not the case. Hence, the forms are not similar and the groups Γ_1 and Γ_2 are not commensurable.

The good thing when working with arithmetic hyperbolic Coxeter groups is that the question of their commensurability is not really more difficult in higher dimensions. However, the two following properties have an impact on the difficulty of the computations:

- the rank of the group (or equivalently the number of faces of the associated polyhedron);
- the compactness of the polyhedron: if it is compact, then the field of definition of the quadratic form is a number field and the comparison of the invariants is then more complicated.

Details about this procedure can be found in the article of Machlachlan [4].

4.2 Preamble: the graph of a Coxeter polyhedron

Drawing a picture of a polyhedron can be painful (either because of the dimension or because the polyhedron has a lot of facets). A particularly nice way to avoid this in the case of a Coxeter polyhedron is to use its *Coxeter graph*.[9]

Since it is our main geometric space, suppose that our favorite Coxeter polyhedron $\mathcal{P}$ lives in the hyperbolic space $\mathbb{H}^n$. Then, $\mathcal{P}$ is bounded by $N \geq n+1$ hyperbolic hyperplanes, say $H_1, ..., H_N$. Hyperplanes are a generalization of the concept of ordinary planes in 3-dimensional space; they are *subspaces* of one dimension less than the ambient geometric space.

---

[8] If you are already familiar with some linear algebra you might want to know some facts more precisely: given two quadratic forms q and q' on a K-vector space, we say they are isomorphic, $q' \cong q$, if and only if the matrices Q and Q' we obtain from the associated bilinear forms are connected via $Q = S^t Q' S$ with an invertible matrix S. Under such a transformation, the determinant of a matrix changes at most by the square of the transformation matrix's determinant. On the other hand, we speak of *similar* quadratic forms q and q', if $q \cong \lambda q'$ for some number $\lambda \in K$.

In general, it is much more difficult to show that two forms are similar than that they are ismorphic. But in our case this means, as long as n is an odd number, that it is worth to check whether the forms have the same determinant modulo a square of an element in K. For the special type of Coxeter groups we have here, K has to be the field of rational numbers $\mathbb{Q}$.

[9] A *graph* is a very general concept in mathematics (particularly in *discrete mathematics*) and the Coxeter graphs we encounter here are examples.

Since $\mathcal{P}$ is a Coxeter polyhedron, the angle between any two intersecting hyperplanes has the value $\frac{\pi}{k}$ for some integer number k. The Coxeter graph Γ of $\mathcal{P}$ can be constructed by using the following simple rules:

- The hyperplane H_i bounding $\mathcal{P}$ is represented by a *vertex* v_i (also called *node*) of Γ.
- If the hyperplanes H_i and H_j intersect at an angle $\angle(H_i, H_j) = \frac{\pi}{k_{ij}}$, $k_{ij} \in \{4, 5, \ldots, \infty\}$, then the corresponding vertices v_i and v_j of Γ are connected by an *edge* labelled by its *weight* k_{ij}.
- If the hyperplanes H_i and H_j intersect at an angle $\angle(H_i, H_j) = \frac{\pi}{3}$, then the corresponding vertices v_i and v_j of Γ are connected by an unlabelled edge.
- If the hyperplanes H_i and H_j intersect at an angle $\angle(H_i, H_j) = \frac{\pi}{2}$, then the corresponding vertices v_i and v_j of Γ are not connected.
- If the hyperplanes H_i and H_j do not intersect, then they have a common perpendicular of positive length, say $l_{ij} = d(H_i, H_j)$. Then, the corresponding vertices v_i and v_j of Γ are connected by a dotted edge (sometimes labelled with the length l_{ij}).

Applying these rules, it is straightforward to construct the Coxeter graph of the 3-orthoscheme of Figure 4. Can you draw it? – The solution is depicted below in Figure 5!

This special graph is an example for a *linear graph*, where you can step from one vertex to another following a line path through the whole graph. Indeed, all Coxeter n-orthoschemes are represented by linear graphs. Can you think of why this is so?

Altogether, this convention allows us to condense the combinatorial and geometric features of a Coxeter polyhedron in one quite simple picture.

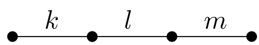

Figure 5: The graph of the Coxeter orthoscheme $[k, l, m]$.

4.3 Subgroups and gluings of polyhedra

Let $P \subset \overline{\mathbb{H}^n}$ be a hyperbolic Coxeter polyhedron whose graph $\Sigma = \Sigma_\times \cup \Delta$ is given in Figure 6. It is made of two parts, namely, on the one hand by the graph $\Sigma_\times$ which is spanned by the vertices $1, 2, 3, 4, 5$ and on the other hand by a graph, whose exact form is of no further importance here and which we therefore just denote by Δ. We can then write Σ as the *union* of $\Sigma_\times$ and Δ as $\Sigma = \Sigma_\times \cup \Delta$.

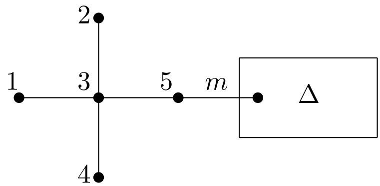

Figure 6: The graph $\Sigma = \Sigma_\times \cup \Delta$.

For $i = 1, ..., 5$, let H_i be the hyperplane corresponding to the vertex i of $\Sigma_\times$. Let $H_{1,2}$ be the hyperplane bisecting the dihedral angle between H_1 and H_2. One can see that $H_{1,2}$ dissects the polyhedron P into two copies of a polyhedron P'. This polyhedron P' itself can be represented by a Coxeter graph, which we call Σ'. The graph Σ' can be written as $\Sigma' = \Sigma_< \cup \Delta$ and is depicted in Figure 7.

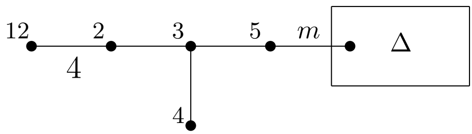

Figure 7: The graph $\Sigma' = \Sigma_< \cup \Delta$.

It is spanned by the vertices 12, 2, 3, 4 and 5 of Σ'. The graph Δ is connected to $\Sigma_<$ by a single edge of weight m emanating from the vertex 5 (as before in the original graph Σ).

Hence, techniques coming from geometry and graph theory can be combined in order to provide the following group theoretical observation: if Γ and Γ' are the Coxeter groups with graphs Σ and Σ' respectively, then Γ is an index 2 subgroup of Γ'. This interplay between different fields of mathematics – geometry, combinatorics, group theory, graph theory – coming from the different, yet closely related, objects described in this snapshot – Coxeter polyhedron, (geometric) Coxeter group, Coxeter graph – shows how useful it can be to have diversified approaches to a single problem. This illustrates why mathematicians particularly like to find 'hidden connections' between different fields of mathematics.

Acknowledgements

The authors would like to thank Sabiha Tokus and the editorial team for their many suggestions and improvements during the editing process, as well as their co-author Ruth Kellerhals for her comments and suggestions about an early version of the snapshot. The first author was fully and the second author was partially supported by the Swiss National Science Foundation, grant number 200020_144438/1 and fellowship number P2FRP2-161727, respectively.

References

[1] R. Guglielmetti, M. Jacquemet, and R. Kellerhals, *On commensurable hyperbolic Coxeter groups*, Preprint, 2015.

[2] N. W. Johnson, R. Kellerhals, J. G. Ratcliffe, and S. T. Tschantz, *Commensurability classes of hyperbolic Coxeter groups*, Linear algebra and its applications **345** (2002), no. 1, 119–147.

[3] R. Kellerhals, *Commensurability of hyperbolic Coxeter groups*, Oberwolfach reports **38** (2014), 29–32.

[4] C. Maclachlan, *Commensurability classes of discrete arithmetic hyperbolic groups*, Groups, Geometry, and Dynamics **5** (2011), 767–785.

[5] J. Milnor, *On polylogarithms, Hurwitz zeta functions, and the Kubert identities*, L'Enseignement Mathématique **29** (1983), no. 3-4, 281–322.

[6] J. G. Ratcliffe, *Foundations of hyperbolic manifolds*, second ed., Graduate Texts in Mathematics, vol. 149, Springer, New York, 2006.

[7] E. B. Vinberg, *Discrete groups generated by reflections in Lobachevskii spaces*, Matematicheskii Sbornik **114** (1967), no. 3, 471–488.

Rafael Guglielmetti *is a PhD student at Fribourg University.*

Matthieu Jacquemet *is a former PhD student from Fribourg University and now a postdoc at Vanderbilt University .*

Mathematical subjects
Algebra and Number Theory, Geometry and Topology

DOI
10.14760/SNAP-2016-009-EN

Snapshots of modern mathematics from Oberwolfach are written by participants in the scientific program of the Mathematisches Forschungsinstitut Oberwolfach (MFO). The snapshot project is designed to promote the understanding and appreciation of modern mathematics and mathematical research in the general public worldwide. It started as part of the project “Oberwolfach meets IMAGINARY” in 2013 with a grant by the Klaus Tschira Foundation. The project has also been supported by the Oberwolfach Foundation and the MFO. All snapshots can be found on www.imaginary.org/snapshots and on www.mfo.de/snapshots.