Snapshots of modern mathematics
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The Willmore Conjecture

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The Willmore problem studies which *torus* has the least amount of *bending energy*. We explain how to think of a torus as a donut-shaped surface and how the intuitive notion of bending has been studied by mathematics over time.

1 Smooth surfaces

When you think of a *geometric object*, you probably think of a triangle, a circle or a tetrahedron, because these objects are very popular in high school geometry. Modern *differential geometry* is a branch of mathematics which can study objects of almost any shape. In particular, geometers today are interested in *smooth surfaces*. These are objects that can be described locally by two smooth coordinate functions. The simplest example of such a surface is the flat Cartesian *plane*, which we denote by $\mathbb{R}^2$, shown in Figure 2. Its coordinates are usually called x (horizontal) and y (vertical).

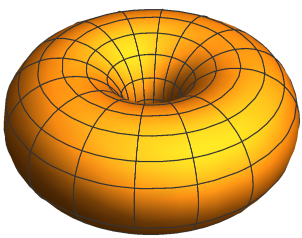


Figure 1: The Clifford torus $\mathbb{T}^2$.

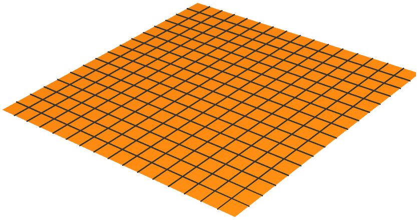


Figure 2: The Cartesian plane $\mathbb{R}^2$.

The surface of the Earth, which is approximately a round sphere as in Figure 3, is an example of a curved smooth surface. Its coordinates are traditionally called *latitude* and *longitude*.

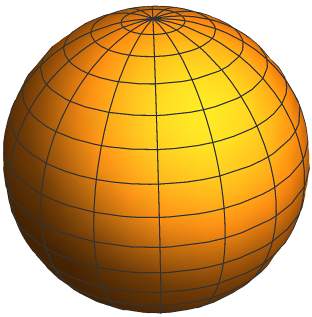


Figure 3: The round sphere $\mathbb{S}^2$.

There are of course many more examples. We will mainly be interested in *closed orientable* surfaces. A surface is called *closed* if it is without any boundary and does not extend indefinitely in any direction. It is called *orientable* if it has a well-defined inside and an outside.

Of the examples above, the Cartesian plane is not closed, because it extends indefinitely in all directions. However, the sphere from Figure 3 is a closed orientable surface: It does not extend indefinitely in any direction and it clearly has in inside and an outside. The *Clifford torus* in Figure 1 is another good example. In fact it will be the star of this snapshot.

2 Embeddings of tori

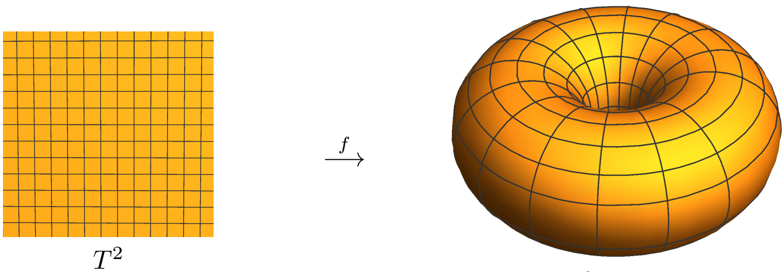
A surface is something, which is regarded as a two dimensional object, because it has two coordinate functions (latitude and longitude in case of the sphere).

However, we often think of it as a subset of the three dimensional Euclidean space $\mathbb{R}^3$ as evident from Figures 1 and 3. In general, geometers distinguish between an abstract surface Σ and how it is *embedded* in $\mathbb{R}^3$. An *embedding* is a special type of map $f : \Sigma \to \mathbb{R}^3$, which can be thought of as a parametrization of its image $f(\Sigma)$. These images are precisely what you can see in Figure 3 and Figure 1. In the former $f(\Sigma)$ is $\mathbb{S}^2$ and in the later $f(\Sigma) = \mathbb{T}^2$.

So if what you can see on Figure 1 is the image of the embedding, you might wonder what the abstract surface and the map f look like in this case. The *abstract torus* is denoted by T^2 instead of $\mathbb{T}^2$. You can think of T^2 as the playground for the video game *Asteroids*: the square, except that when the little spaceship flies upwards and hits the top edge, it reappears at the bottom, and when it flies off the left edge it reappears at the right (see the left-hand side of Figure 4). The abstract torus T^2 has the advantage that it is much easier to do calculations on it. The embedding $f : T^2 \to \mathbb{T}^2$ depicted in Figure 1 is given by

$$f : T^2 \to \mathbb{R}^3, \qquad (x, y) \mapsto \begin{pmatrix} \cos(x)(\cos(y) + \sqrt{2}) \\ \sin(x)(\cos(y) + \sqrt{2}) \\ \sin(y) \end{pmatrix}. \tag{1}$$

The relationship between the abstract torus T^2 and the embedded torus $\mathbb{T}^2$ via f is visualized in Figure 4.



T^2 $\mathbb{T}^2$

Figure 4: The embedding $f : T^2 \to \mathbb{T}^2$.

An animation of this map can be found at this link.

Other embeddings of the abstract torus T^2 are given by other maps. For instance, if you choose $f : T^2 \to \mathbb{R}^3$ to be

$$(x, y) \mapsto \begin{pmatrix} \cos(x)\cos(y) + 3\cos(x)(1.5 + \sin(1.5x)/2) \\ \sin(x)\cos(y) + 3\sin(x)(1.5 + \sin(1.5x)/2) \\ \sin(y) + 2\cos(1.5) \end{pmatrix}, \tag{2}$$

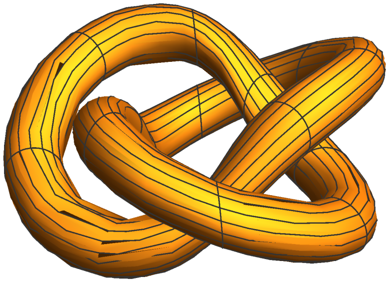


Figure 5: The trefoil knot.

you get a torus in the shape of the famous *trefoil knot* (see Figure 5).

There are many more examples of embedded tori $f : T^2 \to \mathbb{R}^3$. In fact the space of all embedded tori is infinite-dimensional. In some sense all tori are of the same shape — they are shaped like donuts, albeit differently twisted and knotted. One way to understand the very large space of embedded tori is to develop ways to measure the complexity of the surface, so we can focus on understanding what "simple" surfaces look like first.

3 Bending energy

One intuitive feature of surfaces is that some are *bent* more than others. It is kind of obvious that the complicated torus from Figure 5 is bent more than the plane. It also makes sense to describe the complicated torus in Figure 5 as more highly bent than $\mathbb{T}^2$, and $\mathbb{T}^2$ as more highly bent than $\mathbb{S}^2$, but you might consider this less obvious. It took quite a while for the notion of bending to be described on a formal mathematical level. In 1965 Thomas Willmore (see Figure 6) systematically collected ideas about the phenomenon and developed the modern differential geometric theory of bending, which we still use today. This theory associates to every embedding $f : T^2 \to \mathbb{R}^3$ a quantity called *bending energy* (or *Willmore energy*) denoted by $\mathcal{W}(f)$. For the experts, this energy is defined by

$$\mathcal{W}(f) := \int_\Sigma H^2, \tag{3}$$

where H denotes the *mean curvature*, a geometric quantity which is related to how the area of Σ can be decreased or increased by perturbing it slightly (see Snapshot $\mathcal{N}^o$ 2/2015 "Minimizing Energy", by Christine Breiner). Actually this definition makes sense for any other closed surface, too. In case this definition looks daunting to you, you can rest assured that this bending energy behaves exactly like you would expect: The plane from Figure 2 has bending energy zero — it is not bent at all. The bending energy of the complicated trefoil torus from Figure 5 is much larger (≈ 60.5) than the one of the Clifford torus in Figure 1 (≈ 19.7).

An interesting fact (for the experts) about the bending energy is that it is *scale-invariant*: If you *scale* or inflate the surface by any factor, its bending energy does not change. Think of the sphere $\mathbb{S}^2$ as an example: If you increase the radius, the surface becomes flatter, so its mean curvature H is decreased. But at the same time its area gets larger, which increases the value of the integral in (3). One can show that these two phenomena balance each other out on any surface. In fact, the bending energy stays the same even when you apply a more general kind of transformation, called a *conformal change*, which preserves angles, but might distort lengths.

Figure 6: Thomas Willmore

4 Willmore's conjecture

After having a formal mathematical description of the bending energy, mathematicians started looking for the surfaces with smallest bending energy. We already said that the Cartesian plane has zero bending energy. Since one can see from (3) that no surface can have negative bending energy, it follows that the plane is a surface with smallest bending energy among all surfaces. In other words, this question is not so interesting, but recall that the plane is not a closed surface.

Willmore was interested in finding a surface of minimal bending energy among all closed oriented surfaces. He calculated the bending energy of the sphere from Figure 3 to be 4π (≈ 12.6) and proved that this is the absolute minimum among all closed oriented surfaces. Then he calculated the bending energy of the Clifford torus from Figure 1 to be $2\pi^2$ (≈ 19.7) and conjectured

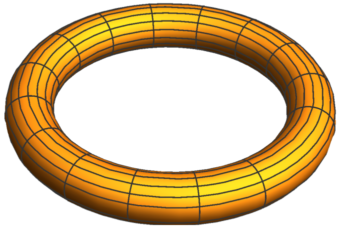


Figure 7: A torus which does not minimize bending energy.

that this is the minimum among all tori.

Intuitively, it seems quite clear that the torus in Figure 1 is much less bent than the torus in Figure 5. However, one can compute that even a slightly different torus such as the one in Figure 7 has more bending energy than $\mathbb{T}^2$ (≈ 59.2). Willmore came up with the following:

Willmore Conjecture. *The bending energy $\mathcal{W}(f)$ of any embedded torus $f : T^2 \to \mathbb{R}^3$ is greater than or equal to $2\pi^2$. In symbols:*

$$\mathcal{W}(f) \geq 2\pi^2.$$

In other words, he conjectured that among all embedded tori $f : T^2 \to \mathbb{R}^3$, the Clifford torus from Figure 1 has smallest bending energy. Willmore's Conjecture states that $\mathbb{T}^2$ is both qualitatively and quantitatively special — not only is it obtained by revolving one circle about another, the ratio of radii of the two circles is "just right" at $\sqrt{2}$.

Conversely, if $\mathcal{W}(f) = 2\pi^2$, then f has to be the Clifford torus (up to a conformal transformation).

Willmore's conjecture turned out to be very difficult to prove. Since Willmore posed the conjecture in his 1965 paper [4], the problem has continuously been under active research. Over the decades, more and more classes of tori were proven to have bending energy at least $2\pi^2$. An overview of the historical progress can be found in [3]. Finally, in 2014 the conjecture was proven by André Neves and Fernando Marques (Figure 8).

5 A word about the proof

Why do mathematicians bother for almost half a century to prove something that is intuitively clear? First of all, if one examines this huge infinite dimensional space of embeddings $f : T^2 \to \mathbb{R}^3$ and the precise technicalities of the definition of the bending energy in (3), the statement becomes much less obvious. And

Figure 8: André Arroja Neves and Fernando Codá Marques

although “Intuition is our most powerful tool” (Sir Michael Atiyah, see [1]), it is sometimes wrong. There are other examples in mathematics where a conjecture was formed from intuition and disproven later. You cannot be sure that a conjecture is true until you have a proof. The gap between our intuition and our ability to formally prove the Willmore Conjecture is one reason why it became so famous. In such a situation one always wonders if one has really understood the phenomenon — in that case the bending energy of surfaces — as well as one thinks. If the Willmore Conjecture had turned out to be wrong, this would have been a sensation just as big as its proof. Much of the research carried out on the Willmore Conjecture produced tools that have been useful in other fields, even if they were not powerful enough to give a full proof of the conjecture.

Another reason why mathematicians are interested in the proof is that a proof is much more than just a verification that a conjecture is true. It also gives an explanation why it is true. The proof given in [2] has almost a hundred pages and establishes a connection between surface theory, topology and geometric measure theory in a way that was unknown before. These new insights are precisely what mathematical researchers find interesting. Therefore, they are being discussed vividly for instance at Oberwolfach.

Image credits

Figures 1–5, 7 Courtesy the author.

Figure 6 Author: Konrad Jacobs. Archives of the Mathematisches Forschungsinstitut Oberwolfach, http://opc.mfo.de, 1979.

Figure 8 Archives of the Mathematisches Forschungsinstitut Oberwolfach, http://opc.mfo.de, 2014; and courtesy Fernando Codá Marques

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