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# Curriculum development in university mathematics: where mathematicians and education collide

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This snapshot looks at educational aspects of the design of curricula in mathematics. In particular, we examine choices textbook authors have made when introducing the concept of the completness of the real numbers. Can significant choices really be made? Do these choices have an effect on how people learn, and, if so, can we understand what they are?

## 1 Introduction

Theory in mathematics education is rather different from theory in mathematics. One of the goals of educational theory is to try to find patterns of behaviours amongst the complex ways people behave and so understand the obstacles these people have in learning mathematics. A theory in mathematics is crystalline, clean, and austere. A theory in education is more organic and often empirical. A particular individual may act as a mathematician or as an educational researcher, or indeed both. Whereas for a mathematician the subject matter itself is most prominent, for the educator it is the student learning. However, it doesn't make sense to dwell on specifying this distinction. Instead, this snapshot looks at an aspect of educational work where both minds meet: the design of curricula.

Designing a curriculum involves carefully specifying the topics and the order in which they will be taught. It also takes account of what we expect the students to know at the start and how people learn. This is investigated both in general and more specifically in so-called *local instruction theories*. They are called local since they deal with the teaching and learning of a particular topic within mathematics, in contrast to what we could call a *global* instruction theory, studying mathematics education in general. Selections need to be made at a range of levels of granularity, from the big ordering choices, to the individual examples.

A curriculum also says something about what we think mathematics *is*. Is mathematics the collection of definitions, theorems, proofs, and examples, or does it also include the process of going about solving problems in that particular area? That is an additional philosophical question and many areas of mathematics, in pure, in applied mathematics, and in statistics, have important if subtle differences in the ways problems are posed, investigated, and solved. Curricula sometimes go beyond just specifying topics to include these processes.

Generally speaking, there are choices to be made and these have an important effect on what we believe the nature of the subject is and how people learn.

## 2 Curriculum studies: analysis and the completeness of the real numbers

To illustrate these choices I have chosen the completeness of the real numbers. What is the essential difference between the real numbers and the rational numbers? Rational numbers are those which can be written as a *ratio* or fraction $\frac{p}{q}$, where $p$ and $q$ are integers and $q \neq 0$. The first task is to convince you that not all numbers are rational.

Take, for example, the graph of $y = 2^x$, shown in Figure 1. This graph looks continuous and smooth. It appears to cut the horizontal line $y = 3$ just once. What value of $x$ gives $2^x = 3$? Could this be a rational number? Well, if $x = \frac{p}{q}$ then

$$2^x = 3 \Leftrightarrow 2^{\frac{p}{q}} = 3 \Leftrightarrow 2^p = 3^q.$$

Can we find whole numbers $p$ and $q$ so that $2^p = 3^q$? For any $p$, $2^p$ is even, but for any $q$, $3^q$ is odd. Therefore they can never be equal! The number $x$ cannot be a rational number.

Those people who have already studied real analysis will probably, by now, be jumping up and down and raising all sorts of objections to this argument. For example, I've assumed that $2^x$ is a *continuous curve*. How do we know that there isn't a "gap" (however small) through which the line $y = 3$ slips, missing the points on the curve? A central question in real analysis is precisely to pin down what we mean by completeness and continuity. There is a fundamental difference between the real numbers and the rational numbers and we need an

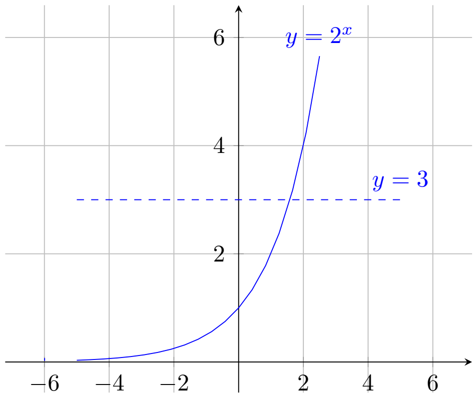


Figure 1: The graph of $y = 2^x$ and the line $y = 3$.

axiom (i.e. a basic starting assumption) to capture this difference. What should we take as our “completeness axiom” to separate rational from real numbers?

In real analysis there are five really large topics: real sequences (seq), convergence of series (ser), continuity (C), differentiability (D), and integration theory (I). These are all theoretical topics, rather than more practical calculus methods. Other topics may also be included, such as sequences of functions. Table 1 contains a small selection of real analysis books and lists the order in which topics are taught, and what each book states as the completeness axiom. Notice there is really a very significant variety amongst the books.

The first author in Table 1, [11], puts continuous functions first. He then talks about differentiation before moving to sequences and series, leaving integration until last. Along the way he introduces three “propositions”, which are used in key places to prove major theorems.

1. Existence of a least upper bound.
2. The intermediate value property for a continuous function.
3. Continuous functions on a closed bounded interval are bounded.

These three propositions turn out to be closely related. Unusually for a real analysis textbook, there is a discussion of these three propositions. Real numbers are eventually defined using Dedekind cuts, and from this it can be proved that all three “propositions” hold for real numbers but fail for rational numbers. Contrast this approach with [3], who puts Dedekind’s axiom at the beginning, and works with sequences first. This ensures nothing is used before it is formally defined, but it is sometimes hard to understand why a definition is really necessary until it has been used. Many educators, such as [4], question the

| Book | Order of topics | Completeness Axiom |
|---|---|---|
| Quadling [11] | C, D, seq, ser, I | p. 9: If $E$ is any set of numbers which is bounded above, then of all possible upper bounds there is a least one. |
| Burkill [3] | seq, C, D, ser, I | p. 12: Dedekind's axiom. Suppose that the system of all real numbers is divided into two classes $L$, $R$ every member $l$ of $L$ being less than every member $r$ of $R$ (and neither class being empty). Then there is a dividing number $\xi$ with the properties that every number less than $\xi$ belongs to $L$ and every number greater than $\xi$ belongs to $R$. The number $\xi$ itself may belong to either $L$ or $R$. If it is in $L$, it is the greatest member of $L$; if it is in $R$, it is the least member of $R$. |
| Lang [10] | seq, C, D, ser, I | p. 27: Archimedean axiom. Every non-empty set of real numbers which is bounded from above has a supremum. Every non-empty set of real numbers which is bounded from below has a greatest lower bound. |
| Spivak [13] | C, D, I, seq, ser | p. 113: (P13) The least upper bound property. If $A$ is a set of real numbers, $A \neq \emptyset$, and $A$ is bounded above, then $A$ has a least upper bound. |
| Ball [2] | seq, ser, C, D, I | p. 23: A real number is a Dedekind section of the rationals; i.e. the set of real numbers is the set of all Dedekind sections. |
| Reade [12] | seq, ser, C, D, I | p. 11: Every non-empty set $E$ of real numbers which is bounded above has a supremum. |
| Hart [8] | seq, ser, C, D | p. 12: Every non-empty set of real numbers which is bounded above has a supremum. |
| Burn [5] | seq, ser, C, D, I | p. 72: Every infinite decimal is convergent. |

Table 1: Curriculum choices in analysis texts.

efficacy of *"requiring a definition in order to construct proofs when it is only the proofs that clarify which properties are needed in the definition"*. [13] has a similar ordering of topics, but chooses the first of [11]'s propositions as an *axiom*. He discusses the others as two of his "Three Hard Theorems" in Chapter 7 (the third is that a continuous function on a closed bounded interval achieves its bound somewhere).

Another popular choice is to take the least upper bound as an axiom and put sequences and series first. What is an axiom in [12], [8], and others is a theorem in [3]:

> **Theorem 1.8.** *If $S$ is a (not-empty) set of numbers which is bounded above, then of all the upper bounds there is a least one.* [3]

Notice the axiom in [10] has two parts. The second part is a theorem in some books, e.g. [8, Theorem 1.4.3] and an exercise in [12] (Exercise 7, on p. 16).

It is also intriguing to notice that in almost all these treatments integration theory comes last. Historically, some of the most important mathematical problems involve calculating areas and volumes, and these are essentially integration, see for example [6]. To what extent should a current curriculum respect the historical development of the subject?

## 3 The questions of educational research

It is difficult for someone new to a subject to appreciate the consequences of these choices and their relative merits. Indeed, most students find real analysis a challenge the first time they learn it! The point of this snapshot is to argue that choices have been made. One goal of educational research is to systematically investigate whether these choices are effective, although scientific controlled experiments on the curriculum level are very rare (see [1] for one example). As the author of our last book [5] says *"learning or growth in mathematics consists of a transition from experiences of the particular, through pattern recognition or problem solving, to perceptions of a generic"* [4].

Discussion of *why* these choices have been made are very rare. There are longstanding cultural reasons within mathematics why discussions about definitions, for example, are hidden particularly from students. One particular exception is [9]. In his criticism of the argument between Bernoulli and Leibniz about the "correct" definition of the logarithm of a negative number, Euler acknowledges this freely:

> *If at times this disagreement is not expressed strongly the reason is clearly that people do not want the certainties of pure mathematics in general to come under suspicion by revealing in public the difficulties and even contradictions that mathematicians find in this area.* [7]

When choosing what to take as a definition we seek the best class of correct theorems, while introducing the fewest problem cases. It is helpful to be able to *use* the definition simply, and essential not to contradict established results. In mathematical research definitions often come last. A mathematics educator also has to mediate these demands with cognitive processes, including pre-existing intuitive notions from the real world, and our understanding of the particular social teaching situation. Just as the traces of discovery in mathematics are obscured, so the reasons for particular curriculum orderings are brushed away. This essential conundrum lies at the heart of curriculum development in pure mathematics, and is one which can only be solved through a close collaboration between mathematicians and educators.

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*Mathematical subjects*
Analysis

*Connections to other fields*
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