854. Synchronization of a chaotic gyroscopic system under settling time constraints

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Abstract. A simple and easy-to-implement method that guarantees the effective active synchronization of a chaotic gyroscopic system within a specified settling time limit is presented. A closed-form expression is given for the determination of the appropriate synchronizing control signal. The method is successfully validated through simulations for various initial conditions of the gyroscopic system.

Keywords: chaotic gyroscopic system, chaos synchronization, nonlinear control.

Introduction

Gyroscopic systems pervade in modern systems such as aerospace systems, electromechanical devices and telecommunication systems. Furthermore, since they are used for precise angular measurements in critical systems, they require accurate control to ascertain the validity of their measurements. Therefore, they deserve rigorous processing mainly when their dynamics are subject to chaos. Some nonlinear dynamical systems may exhibit chaotic behavior due to specific values of their parameters. Besides, in some applications such as secure signal transmission, it may be necessary to control a regular or chaotic signal to make it track the motion of another chaotic system. Basically, the principle of active control of chaotic systems consists in perturbing the dynamics of a given chaotic system by adding control terms to its dynamic model so as to force its overall dynamics to be identical to that of the same or another system departing from totally different initial conditions. The tracking control process of chaotic systems is known as chaos synchronization. Chaos synchronization deals with driving a chaotic system (called the driven system) to track the dynamics of the same or another chaotic system (called the main system) with different initial conditions. The difficulty is that chaotic systems are known to behave in significantly different manner even for relatively close initial conditions. Therefore, the problem of chaos synchronization deserves a special treatment that mostly requires special tools borrowed from control theory. Chaos synchronization has received a high interest in nonlinear science during the last two decades [1-10]. This is due to the fact that many mechanical, electrical and natural systems are prone to vibration phenomena that are in general governed by highly nonlinear dynamics and which may in some conditions degenerate to chaos [11-13]. The present paper deals with the active synchronization of a chaotic gyroscopic system [7]. Existing methods in the research literature [7-9] on the synchronization of such a system mainly deal with a mere asymptotic synchronization without any guarantee that the synchronization will effectively occur within a specified time limit. Furthermore, the methods in use for synchronizing chaotic systems [1-10] require either linearizing the nonlinear model of the system, or finding Lyapunov function candidates that are usually cumbersome to achieve for time-varying systems. The present paper proposes a nonlinear control method that guarantees the synchronization of a chaotic gyroscopic system within a pre-specified settling time. The paper is organized as follows: the following section describes the model of the chaotic gyroscopic system that is subject to the synchronization problem and states the problem to be solved in the paper; then the next section deals with a nonlinear control law to ascertain an actual synchronization process under settling time constraints; numerical simulation results based on different initial conditions of the main and the driven system are analyzed in a subsequent section to demonstrate the effectiveness of the proposed method.

Problem Statement

Before stating the problem to be solved, let us explain formally what the word synchronization means when it comes to dynamical systems [4].

Consider a controlled system described by the following differential equation:

$$\dot{x} = f(t, x, u) \tag{1}$$

where t represents time, and x and u are respectively the state vector and the control vector.

Let $s(t_0, x_0, u; t)$ be the solution of the controlled differential equation (1) with initial state x_0 at initial time t_0 for a given control trajectory u. Let now consider the state trajectory $t \mapsto x_{ref}(t)$ of another system taken as reference. Then, the synchronization problem between the system described by equation (1) and the reference system consists in finding an appropriate control function $u = \hat{u} : t \mapsto \hat{u}(t)$ such that the solution trajectory $s(t_0, x_0, \hat{u}; t)$ be as close as possible to the reference trajectory, that is:

$$\lim_{t \to +\infty} \left\| s(t_0, x_0, \hat{u}; t) - x_{ref}(t) \right\| = 0 \tag{2}$$

The condition described by equation (2) shall be met even when the initial reference signal $x_{ref}(t_0)$ differs from the initial state x_0 of the driven system described by equation (1). That condition simply means that for a given tolerance $\varepsilon > 0$, there exists a time-instant $\tau > t_0$ such that for any $t \geq \tau$ the following inequality holds:

$$\left\| s(t_0, x_0, \hat{u}; t) - x_{ref}(t) \right\| \leq \varepsilon \tag{3}$$

where $\|.\|$ represents an appropriate norm on the state space. The inequality (3) means that the driven system behaves practically exactly as the reference system from time τ, thus the two systems are synchronized.

Now, let us state the main problem to be solved in the present paper. The normalized equation of a single axis gyroscopic system is established by Chen [7] as:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= g(x_1) - c_1 x_2 - c_2 x_2^3 + \beta \sin x_1 + f \sin \omega t \cdot \sin x_1 \end{aligned} \tag{4}$$

where $g(x_1) = -\alpha^2 (1 - \cos x_1)^2 / \sin^3 x_1$, t is the time, the system being chaotic if the parameters have the following values: $\alpha = 10$, $\beta = 1$, $c_1 = 0.5$, $c_2 = 0.05$, $\omega = 2$ and $f = 35.5$.

The problem to be solved is to synchronize such two chaotic gyroscopic systems departing from different initial states so that the motion of the driven system be synchronized with that of the main system from a specified settling time T_s.

Proposed Method

Let equation (4) be the model of the main system, then consider the driven system with the following model:

$$\begin{aligned} \dot{y}_1 &= y_2 \\ \dot{y}_2 &= g(y_1) - c_1 y_2 - c_2 y_2^3 + \beta \sin y_1 + f \sin \omega t \cdot \sin y_1 + u \end{aligned} \tag{5}$$

where function $g(.)$ is defined accordingly as in equation (4), the parameters being identical to those in equation (4) and u is a time-varying perturbation that has been added to the gyroscopic model to act as a control variable for the driven system. One may observe that assuming $u = 0$ makes the driven system have the same model as the main system.

Basically, the problem to be solved is to control the state $y = (y_1\ y_2)^T$ of the driven system to make it equal to the state $x = (x_1\ x_2)^T$ of the main system across time, that is, the control shall aim at driving the synchronization error $e(t) = x(t) - y(t)$ to zero within a pre-specified time limit T_s. Meanwhile, since $\dot{x}_1 = x_2$ and $\dot{y}_1 = y_2$, equaling y_1 to x_1 across time necessarily forces y_2 to be then equal to x_2. Therefore, to synchronize the two systems, it is necessary and sufficient to drive y_1 to x_1.

From equations (4) and (5), we get:

$$\begin{aligned} \ddot{x}_1 &= h(t, x_1, x_2) \\ \ddot{y}_1 &= h(t, y_1, y_2) + u \end{aligned} \tag{6}$$

where function $h(.,.,.)$ is defined for any pair (v_1, v_2) and any time t as:

$$h(t, v_1, v_2) = g(v_1) - c_1 v_2 - c_2 v_2^3 + \beta \sin v_1 + f \sin \omega t \sin v_1 \tag{7}$$

Equations (6) imply that the tracking error $e_1 = x_1 - y_1$ has a second order dynamics since it is the difference of two different second order dynamic systems. However, to drive e_1 to zero, we would like it to follow a linear second order dynamics through an appropriate choice of control variable u. Therefore, since the reference value of the tracking error shall be zero, this requires an error dynamics which should be modeled as equation (8) below according to single-input single-output linear control theory [14]:

$$\ddot{e}_1 + 2\zeta\omega_n \dot{e}_1 + \omega_n^2 e_1 = 0 \tag{8}$$

where ω_n is the natural frequency, and ζ the damping ratio of the error dynamics.

Using equations (4-7) and the expression of the tracking error, we can write equation (8) explicitly as:

$$(h(t, x_1, x_2) - h(t, y_1, y_2) - u) + 2\zeta\omega_n (x_2 - y_2) + \omega_n^2 (x_1 - y_1) = 0 \tag{9}$$

Then, equation (9) is solved for the unknown control u to get the time-varying synchronizing control, which gives:

$$u(t) = h(t, x_1, x_2) - h(t, y_1, y_2) + 2\zeta\omega_n (x_2 - y_2) + \omega_n^2 (x_1 - y_1) \tag{10}$$

Equation (10) provides a way to compute the appropriate control for synchronizing the driven system with the main one. That control depends on the natural frequency (ω_n) and damping ratio (ζ). Therefore, the settling time, that is the necessary time to acceptable synchronization between the two systems will depend on these two parameters.

Let t_0 be the time from which the secondary system starts to track the main system, and assume that we want the two systems to fully synchronize at time $t_0 + T$ with an error that equals ε percent of the initial error $e_1(t_0) = x_1(t_0) - y_1(t_0)$, that is, for any time-length $\tau \geq T$ we would like to have $|e_1(t_0 + \tau)| \leq \varepsilon \cdot |e_1(t_0)|/100$. From second order systems dynamics in linear control theory [14], it is known that the minimum T_s of such time-lengths T is equal to the settling time for ε % error and is computed as:

$$T_s = (\zeta\omega_n)^{-1} \ln(100/\varepsilon) \tag{11}$$

where 'ln' is the natural logarithm.

Therefore, for a specified settling time T_s, equation (11) gives means to determine the appropriate values of parameters ω_n and ζ in equation (10) so that the driven system synchronizes with the main one at the given settling time T_s with a guaranteed static tracking error less than ε %. In that case, since there are two unknown parameters (ω_n and ζ) for one equation, the idea is to assume a value for one of the parameters (for instance ζ), and to compute the other parameter accordingly, for a specified settling time T_s and acceptable error level ε specified as well.

Numerical Simulations

Example 1

The behavior of the main system in the phase plane is depicted in Fig. 1(a), with initial conditions $(x_1^{(0)} = 1, x_2^{(0)} = -1)$ and for the driven system in Fig. 1(b) with different initial conditions $(y_1^{(0)} = -1, y_2^{(0)} = 2)$. The objective is to synchronize the driven system with the main system at required settling time $T_s = 5$ s with less than 2 % static error. We chose a damping ratio of the error dynamics as $\zeta = 0.8$, which, by equation (11), provides a corresponding value of the natural frequency as $\omega_n = 0.978$ rad/s.

The simulation results of the synchronization errors are displayed in Fig. 2(a), representing the synchronization error while tracking the angle, whereas Fig. 2(b) displays the error related to the synchronization of the angular rate. Both errors converge to zero, meaning that the driven system tracks accurately the main system. The practical settling time in both cases is equal to 4.94 seconds corresponding to the 2 % static error, which is consistent with the required $T_s = 5$ s settling time. Thus the specifications about the settling time and the static error are fulfilled in practice using the control law given in equation (10).

Example 2

The initial conditions for both the main and the driven systems are chosen different from those used in the first example. The behavior of the main system in the phase plane is depicted in Fig. 3(a), with initial conditions $(x_1^{(0)} = -0.5, x_2^{(0)} = 1)$ and for the driven system with initial conditions $(y_1^{(0)} = -1.5, y_2^{(0)} = 0.5)$ in Fig. 3(b). The objective is again to synchronize the driven system with the main system but here at required settling time $T_s = 7$ s with less than 1 % static error. The damping ratio of the error dynamics is set to be $\zeta = 0.7$, which, by equation (11), provides a corresponding value of the natural frequency as $\omega_n = 0.9398$ rad/s.

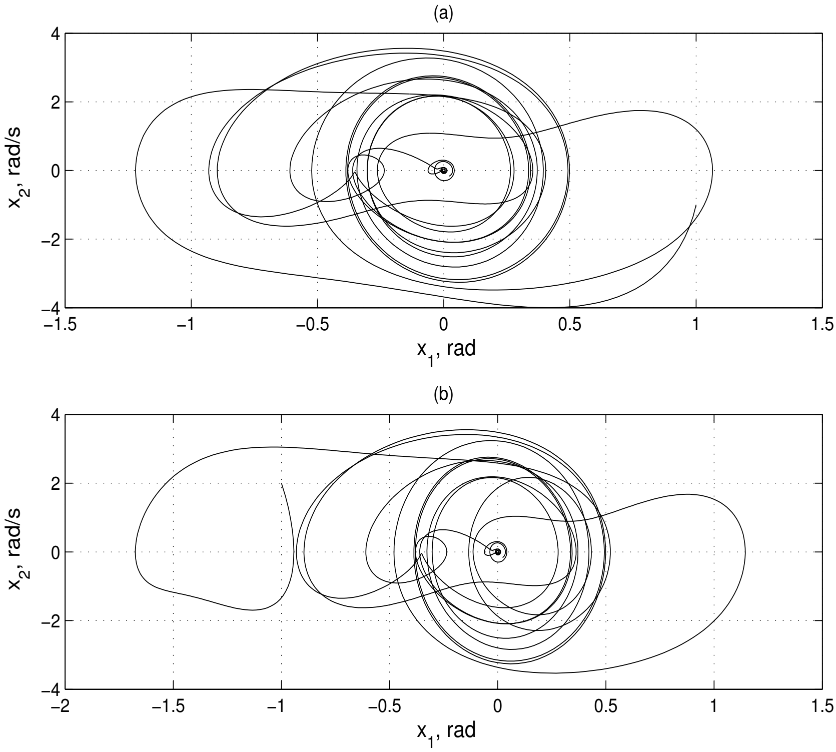

Fig. 1. (a) Main system with initial conditions: $(x_1^{(0)} = 1,\ x_2^{(0)} = -1)$;
(b) Driven system with initial conditions: $(y_1^{(0)} = -1,\ y_2^{(0)} = 2)$

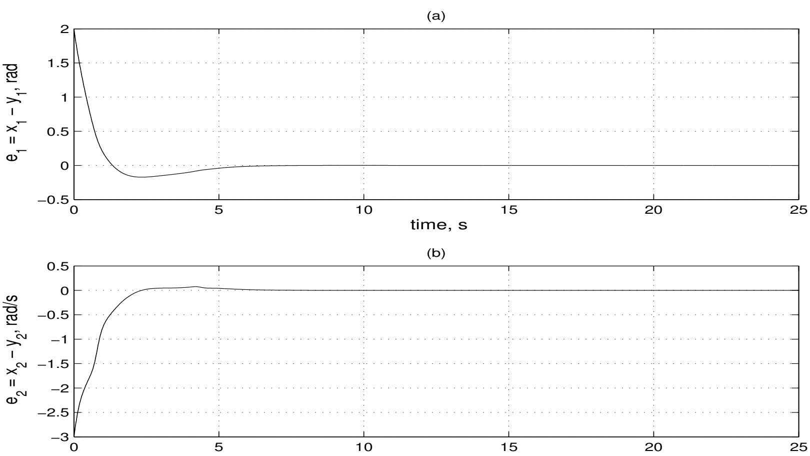

Fig. 2. (a) Angle synchronization error ($e_1 = x_1 - y_1$);
(b) Angular rate synchronization error ($e_2 = x_2 - y_2$)

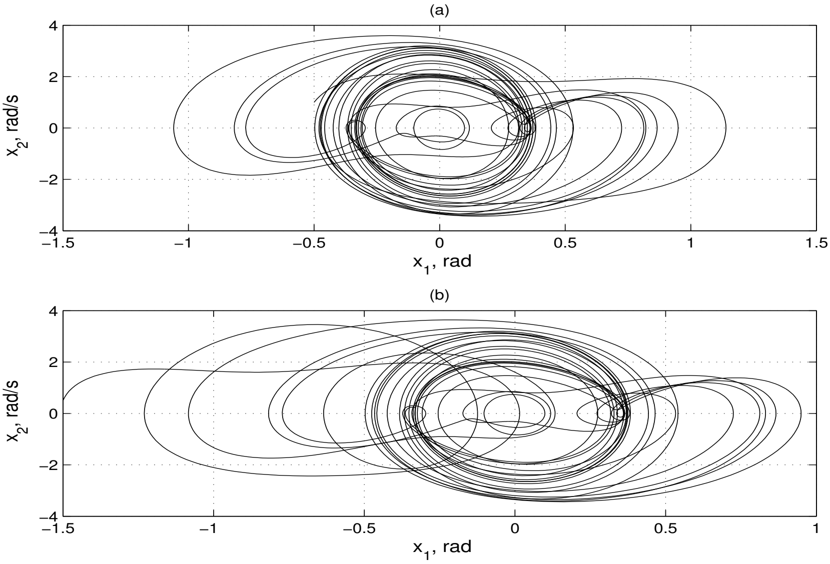

Fig. 3. (a) Main system with initial conditions: $(x_1^{(0)} = -0.5,\ x_2^{(0)} = 1)$;
(b) Driven system with initial conditions: $(y_1^{(0)} = -1.5,\ y_2^{(0)} = 0.5)$

The simulation results of the synchronization errors across-time are displayed in Fig. 4(a), representing the synchronization error while tracking the angle, whereas Fig. 4(b) displays the error related to the synchronization of the angular rate. Here too, the synchronization errors converge to zero. The practical settling time in both cases is equal to 6.96 seconds corresponding to the 1 % static error, which is consistent with the required $T_s = 7$ s settling time. Thus the specifications about the settling time and the static error are again practically fulfilled as in the first example using the proposed control law (equation 10).

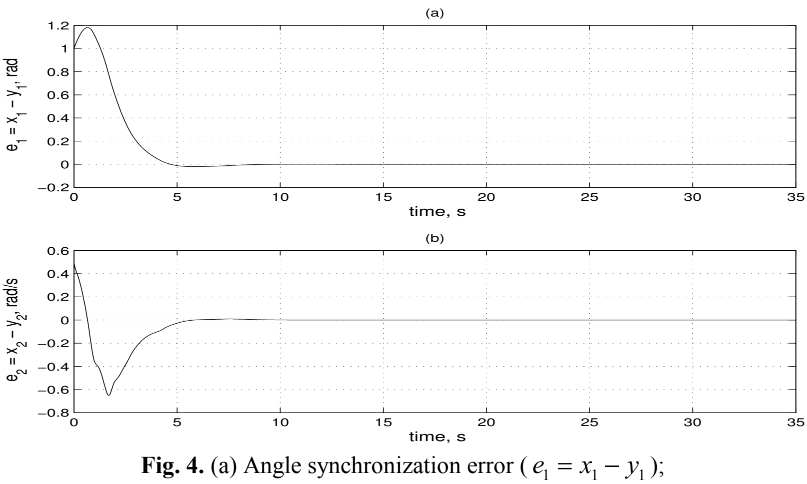

Fig. 4. (a) Angle synchronization error ($e_1 = x_1 - y_1$);
(b) Angular rate synchronization error ($e_2 = x_2 - y_2$)

Conclusion

A method that guarantees the effective synchronization of a chaotic gyroscopic system within a specified time limit has been proposed in the paper. The method is based on compelling the tracking error to have a linear second order dynamics that stabilizes to zero. The advantages of the proposed method over existing ones are threefold: first, it does not resort to Lyapunov function candidates that are usually cumbersome to find for time-varying systems like the considered gyroscopic system; second, it does not require the linearization of the actual nonlinear model; and finally, it guarantees actual synchronization within a specified time limit. Two illustrative examples have shown the effectiveness of the proposed method in practice. Further extensions of the method will deal with higher order gyroscopic systems in the framework of aerospace system design.

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