351. Dynamic orientation of systems for excitation of transverse and longitudinal waves

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Abstract. A theoretical and numerical analysis of dynamic systems for excitation of longitudinal and transverse waves with a self-direction is carried out. Formulas for definition of excitation frequency and a difference of phases depending on system parameters are obtained. Conditions for existence and stability of the steady-state motion regimes are defined.

Keywords: dynamic orientation, elastic system, transverse and longitudinal waves

1. Introduction

The synchronization phenomenon has many applications in mechanics and in physics, in the vibration technologies and in other fields [1-3].

Synchronization is one of mechanisms of self-organization of nonlinear oscillatory systems [1]. The general definition of synchronization properties was offered in [4-5]. In certain cases synchronization arises owing to properties of the system, for example, the frequency synchronization of vibrating or rotating bodies. In such cases the term self-synchronization is used [6]. Machines with independently rotating vibration exciters found wide application. Their rotors can rotate independently, and for normal work of the machine the necessary synchronization of rotations of vibration exciters is reached owing to the phenomenon of self-synchronization of systems [1]. Lately interest is increased to chaotic synchronization where each of synchronized subsystems continues to make complex chaotic vibrations and after the establishment of the synchronous regime [7].

The aim of the present work is the research of the phenomenon of the dynamic orientation of an elastic system for excitation of longitudinal and transverse travelling waves, i.e. studying of the capacity of dynamic system to make vibrations in the directions chosen by the system.

The additional degrees of freedom are arised at fastening vibration exciter to the object by means of connections allowing the motion of the body of the vibration exciter concerning object. Excess degrees of freedom give the possibility to the vibration exciter to choose trajectories of motion or position of its body. Such dynamically defined stable and unstable positions are called the phenomenon of the dynamic orientation. The phenomenon of a dynamic orientation is close to the synchronization phenomenon [8].

For the theoretical research of the phenomenon of orientation the small parameter method is used. The conditions for the existence and stability of the given phenomena for elastic system are received by approximate-analytical methods.

2. Problem statement

The system consisting of a semi-infinite bar with the elastically connected end and n vibration exciters connected to it is analysed.

The bar can perform vibration in a transverse and longitudinal direction as the body of the vibration exciter turns round the joint which connects the body of the vibration exciter to a bar with the center of the axis of rotation in point A_j, and finds a steady direction of vibrations.

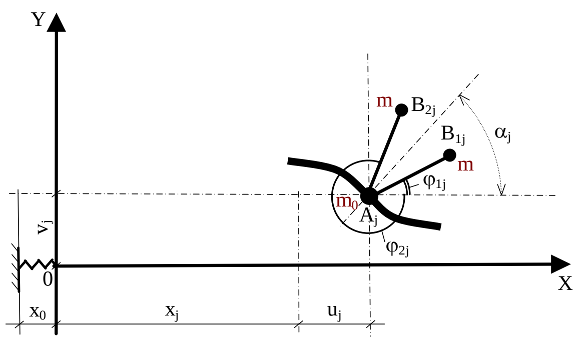

Fig. 1. Model of the system

The exciting masses of vibration exciter are located at points B_{1j} and B_{2j}, and they rotate synchronously in opposite directions so that the excitation along the bar is created [8].

The points A_j, B_{1j}, B_{2j} have the coordinates

$$A_j(x_j+u_j, v_j),\ B_{1j}(x_j+u_j+r\cos\varphi_{1j}, v_j+r\sin\varphi_{1j}),$$
$$B_{2j}(x_j+u_j+r\cos\varphi_{2j}, v_j-r\sin\varphi_{2j}),\quad x_j=const,$$
$$r=\overline{A_jB_{1j}}=\overline{A_jB_{2j}},\quad \alpha_j=\pi+\frac{\varphi_{1j}-\varphi_{2j}}{2},\ j=1,\dots,n.$$

Kinetic energy of the *j*-th vibroexciter is equal to

$$\begin{aligned}T_j=&\frac{m_0+2m}{2}\left((\dot u_j+\dot x_0)^2+\dot v_j^2\right)+\frac{I+mr^2}{2}(\dot\varphi_{1j}^2+\dot\varphi_{2j}^2)-\\&-mr[(\dot u_j+\dot x_0)(\dot\varphi_{1j}\sin\varphi_{1j}+\dot\varphi_{2j}\sin\varphi_{2j})+\\&+\dot v_j(-\dot\varphi_{1j}\cos\varphi_{1j}+\dot\varphi_{2j}\cos\varphi_{2j})],\end{aligned}\qquad(1)$$

where m_0 is the mass concentrated at the point A_j, and m are the masses concentrated at points B_{1j} and B_{2j}, I is the moment of inertia of the rotor of the j-th vibration exciter with respect to the center of the axis of rotation A_j, [9].

The differential equations of motion of the given system is obtained in the form

$$(I+mr^2)\ddot\varphi_{1j}-mr[(\ddot u_j+\ddot x_0)\sin\varphi_{1j}-\ddot v_j\cos\varphi_{1j}]+ +H_{\varphi_{1j}}\dot\varphi_{1j}=M_{\varphi_{1j}},\qquad(2)$$

$$(I+mr^2)\ddot\varphi_{2j}-mr[(\ddot u_j+\ddot x_0)\sin\varphi_{2j}+\ddot v_j\cos\varphi_{2j}]+ +H_{\varphi_{2j}}\dot\varphi_{2j}=M_{\varphi_{2j}},\qquad(3)$$

$$(m_0+2m)(\ddot u_j+\ddot x_0)-mr(\ddot\varphi_{1j}\sin\varphi_{1j}+\dot\varphi_{1j}^2\cos\varphi_{1j}+ +\ddot\varphi_{2j}\sin\varphi_{2j}+\dot\varphi_{2j}^2\cos\varphi_{2j})=F_{in}(u_j),\qquad(4)$$

$$(m_0+2m)\ddot v_j-mr(-\ddot\varphi_{1j}\cos\varphi_{1j}+\dot\varphi_{1j}^2\sin\varphi_{1j}+ +\ddot\varphi_{2j}\cos\varphi_{2j}-\dot\varphi_{2j}^2\sin\varphi_{2j})=F_{in}(v_j),\qquad(5)$$

where $H_{\varphi_{1j}}\dot\varphi_{1j}$ and $H_{\varphi_{2j}}\dot\varphi_{2j}$ are dissipative forces; $M_{\varphi_{1j}}$ and $M_{\varphi_{2j}}$ are the moments of external forces; $F_{in}(u_j)$ and $F_{in}(v_j)$ are inertial forces.

The longitudinal vibrations of a bar are described by the equation [10]

$$EF\frac{\partial^2u}{\partial x^2}+\xi_1\frac{\partial u}{\partial t}-\rho F\left(\frac{\partial^2u}{\partial t^2}+\ddot x_0\right)= =\sum_j\delta(x-x_j)F_{in}(u_j)\cos\alpha_j,\qquad(6)$$

where $u(x,t)$ is the displacement of the cross section with the abscissa x, ρ is the mass of volume unit, E is the modulus of elasticity of the material, F is the cross-sectional area, ξ_1 is the coefficient of external damping, δ is the Dirac's delta-function.

The boundary condition for the elastically fixed end of the bar has the form

$$c_0x_0=EF\frac{\partial u}{\partial x},\qquad(7)$$

where c_0 is the coefficient of stiffness of the spring.

The transverse vibrations of a bar are described by the equation [10]

$$EJ\frac{\partial^4v}{\partial x^4}+\rho F\frac{\partial^2v}{\partial t^2}+\xi_2\frac{\partial v}{\partial t}=\sum_j\delta(x-x_j)F_{in}(v_j)\sin\alpha_j,\qquad(8)$$

where $v(x,t)$ is the transverse displacement of the points of a bar, EJ is the stiffness of a bar, the term $\xi_2\frac{\partial v}{\partial t}$ characterizes the external damping proportional to speed of displacement of the points of a bar. With $x=0$:

$$EJ\frac{\partial^2v}{\partial x^2}=0,\quad EJ\frac{\partial^3v}{\partial x^3}=0.\qquad(9)$$

Introducing the dimensionless coordinate $\eta=\frac{x}{r}$, the equations (6), (8) and the conditions (7), (9) take the form

$$\frac{EF}{r^2}\frac{\partial^2u}{\partial\eta^2}+\xi_1\frac{\partial u}{\partial t}-\rho F\left(\frac{\partial^2u}{\partial t^2}+\ddot x_0\right)= =\sum_j\delta(\eta-\eta_j)F_{in}(u_j)\cos\alpha_j,\qquad(10)$$

$$c_0x_0=\frac{EF}{r}\frac{\partial u}{\partial\eta}\quad\text{with}\quad\eta=0,\qquad(11)$$

$$\frac{EJ}{r^4}\frac{\partial^4v}{\partial\eta^4}+\rho F\frac{\partial^2v}{\partial t^2}+\xi_2\frac{\partial v}{\partial t}=\sum_j\delta(\eta-\eta_j)F_{in}(v_j)\sin\alpha_j,\qquad(12)$$

$$\frac{EJ}{r^2}\frac{\partial^2v}{\partial\eta^2}=0,\ \frac{EJ}{r^3}\frac{\partial^3v}{\partial\eta^3}=0\quad\text{with}\quad\eta=0.\qquad(13)$$

From equations of motion of the given system it is visible that the vibration exciter on a bar at $\alpha_j=0$ excites longitudinal vibrations, and at $\alpha_j=\frac{\pi}{2}$ - transverse vibrations.

3. Application of the small parameter method

The small parameter method is used for investigation of the steady-state regimes of motion, determined by equations (2) - (5), (10), (12).

The equations (2), (3) take the form

$$(I+mr^2)\ddot{\varphi}_{1j} = \varepsilon\Phi_{1j}, \tag{14}$$

$$(I+mr^2)\ddot{\varphi}_{2j} = \varepsilon\Phi_{2j}, \tag{15}$$

where

$$\Phi_{1j} = mr[(\ddot{u}_j + \ddot{x}_0)\sin\varphi_{1j} - \ddot{v}_j\cos\varphi_{1j}] - H_{\varphi_{1j}}\dot{\varphi}_{1j} + M_{\varphi_{1j}},$$

$$\Phi_{2j} = mr[(\ddot{u}_j + \ddot{x}_0)\sin\varphi_{2j} + \ddot{v}_j\cos\varphi_{2j}] - H_{\varphi_{2j}}\dot{\varphi}_{2j} + M_{\varphi_{2j}},$$

ε is the small parameter.

The steady-state regimes of motion are represented in the form

$$u_j = \sum_{k=0}^{+\infty}\varepsilon^k u_{jk},\ \varphi_{1j} = \sum_{k=0}^{+\infty}\varepsilon^k\varphi_{1j,k},\ F_{in}(u_j) = \sum_{k=0}^{+\infty}\varepsilon^k F_{in_k}(u_{jk}),$$

$$v_j = \sum_{k=0}^{+\infty}\varepsilon^k v_{jk},\ \varphi_{2j} = \sum_{k=0}^{+\infty}\varepsilon^k\varphi_{2j,k},\ F_{in}(v_j) = \sum_{k=0}^{+\infty}\varepsilon^k F_{in_k}(v_{jk}), \tag{16}$$

where $u_j, v_j, \varphi_{1j}, \varphi_{2j}, F_{in}(u_j), F_{in}(v_j)$ are periodic functions of t.

We substitute representations of functions $u_j, v_j, \varphi_{1j}, \varphi_{2j}$ in the form (16) into equations (14), (15) and, equalizing coefficients at ε^0 and ε^1, we receive the problem of the first approximation:

$$(I+mr^2)\ddot{\varphi}_{1j,0} = 0,\ (I+mr^2)\ddot{\varphi}_{1j,1} = \Phi_{1j,0},$$

$$(I+mr^2)\ddot{\varphi}_{2j,0} = 0,\ (I+mr^2)\ddot{\varphi}_{2j,1} = \Phi_{2j,0}, \tag{17}$$

where

$$\Phi_{1j,0} = \Phi_{1j}\Big|_{\substack{u_j=u_{j0}\\ v_j=v_{j0}\\ \varphi_{1j}=\varphi_{1j,0}}} = mr[(\ddot{u}_{j0} + \ddot{x}_0)\sin\varphi_{1j,0} - \ddot{v}_{j0}\cos\varphi_{1j,0}] - H_{\varphi_{1j}}\dot{\varphi}_{1j,0} + M_{\varphi_{1j}},$$

$$\Phi_{2j,0} = \Phi_{2j}\Big|_{\substack{u_j=u_{j0}\\ v_j=v_{j0}\\ \varphi_{2j}=\varphi_{2j,0}}} = mr[(\ddot{u}_{j0} + \ddot{x}_0)\sin\varphi_{2j,0} + \ddot{v}_{j0}\cos\varphi_{2j,0}] - H_{\varphi_{2j}}\dot{\varphi}_{2j,0} + M_{\varphi_{2j}}.$$

It follows that $\varphi_{1j,0}$ and $\varphi_{2j,0}$ can be represented in the form

$$\varphi_{1j,0} = \omega t + \overline{\varphi}_{1j,0},\ \varphi_{2j,0} = \omega t + \overline{\varphi}_{2j,0}, \tag{18}$$

where $\overline{\varphi}_{1j,0}$, $\overline{\varphi}_{2j,0}$ are constants.

Periodicity conditions of functions $\varphi_{1j,1}$ and $\varphi_{2j,1}$ according to the equations (17) have the form

$$\overline{\Phi}_{1j,0} = \overline{mr[(\ddot{u}_{j0} + \ddot{x}_0)\sin(\omega t + \overline{\varphi}_{1j,0}) - \ddot{v}_{j0}\cos(\omega t + \overline{\varphi}_{1j,0})] - H_{\varphi_{1j}}\omega + M_{\varphi_{1j}}} = 0,$$

$$\overline{\Phi}_{2j,0} = \overline{mr[(\ddot{u}_{j0} + \ddot{x}_0)\sin(\omega t + \overline{\varphi}_{2j,0}) + \ddot{v}_{j0}\cos(\omega t + \overline{\varphi}_{2j,0})] - H_{\varphi_{2j}}\omega + M_{\varphi_{2j}}} = 0, \tag{19}$$

the upper dash indicates averaging with respect to t.

Substituting representations of functions $F_{in}(u_j)$ and $F_{in}(v_j)$ in the form (16) into (10), (4) and (12), (5) with the account that $\varphi_{1j,0} = \omega t + \overline{\varphi}_{1j,0}$ and $\varphi_{2j,0} = \omega t + \overline{\varphi}_{2j,0}$ and equalizing the coefficients at ε^0, it is obtained

$$\frac{EF}{r^2}\frac{\partial^2 u}{\partial\eta^2} + \xi_1\frac{\partial u}{\partial t} - \rho F\left(\frac{\partial^2 u}{\partial t^2} + \frac{EF}{c_0 r}\frac{\partial^3 u}{\partial\eta\partial t^2}\right) = \sum_j \delta(\eta - \eta_j)F_{in_0}(u_{j0})\cos\alpha_{j0}, \tag{20}$$

$$F_{in_0}(u_{j0}) = (m_0 + 2m)\left(\ddot{u}_{j0} + \frac{EF}{c_0 r}\frac{\partial^3 u(\eta_j, t)}{\partial\eta\partial t^2}\right) - mr\omega^2(\cos(\omega t + \overline{\varphi}_{1j,0}) + \cos(\omega t + \overline{\varphi}_{2j,0})), \tag{21}$$

$$\frac{EJ}{r^4}\frac{\partial^4 v}{\partial\eta^4} + \rho F\frac{\partial^2 v}{\partial t^2} + \xi_2\frac{\partial v}{\partial t} = \sum_j \delta(\eta - \eta_j)F_{in_0}(v_{j0})\sin\alpha_{j0}, \tag{22}$$

$$F_{in_0}(v_{j0}) = (m_0 + 2m)\ddot{v}_{j0} - mr\omega^2(\sin(\omega t + \overline{\varphi}_{1j,0}) - \sin(\omega t + \overline{\varphi}_{2j,0})), \tag{23}$$

where $\alpha_{j0} = \pi + \dfrac{\varphi_{1j,0} - \varphi_{2j,0}}{2} = \pi + \dfrac{\overline{\varphi}_{1j,0} - \overline{\varphi}_{2j,0}}{2}$.

Functions u_{j0} and v_{j0}, $j = 1,\ldots,n$, will be found from expressions (20) - (23), then the constants ω and $\overline{\varphi}_{2j,0} - \overline{\varphi}_{1j,0}$, $j = 1,\ldots,n$, will be defined from periodicity conditions (19) taking into account that if $\alpha_{j0} = 0$, $j = 1,\ldots,n$, at action of the perturbing force developed by the vibration exciters, the bar makes vibrations in a longitudinal direction, and if $\alpha_{j0} = \dfrac{\pi}{2}$ – in a transverse direction.

4. The forced transverse and longitudinal vibrations of the bar

For case $n=1$ we shall find functions $u(\eta,t)$ and $v(\eta,t)$ - the solutions of the equation (20), (22) with the boundary conditions (11), (13), respectively. And we believe that at the point of the bar with the abscissa $x_1=0$ (or in the dimensionless coordinates $\eta_1=0$) the forces $F_{in0}(u_{10})$ and $F_{in0}(v_{10})$ are applied.

Finding of function $u(\eta,t)$ is reduced to the integration of the differential equation of the longitudinal vibrations of the bar [11]

$$\frac{EF}{r^2}\frac{\partial^2 u}{\partial\eta^2}+\xi_1\frac{\partial u}{\partial t}-\rho F\left(\frac{\partial^2 u}{\partial t^2}+\frac{EF}{c_0 r}\frac{\partial^3 u}{\partial\eta\partial t^2}\right)=0,\ \eta\neq\eta_1, \quad (24)$$

and at the point $\eta=\eta_1=0$ of a bar the force

$$F_{in0}(u_{10})=(m_0+2m)\left(\ddot{u}_{10}+\frac{EF}{c_0 r}\frac{\partial^3 u(0,t)}{\partial\eta\partial t^2}\right)-$$
$$-mr\omega^2(\cos(\omega t+\overline{\varphi}_{10})+\cos(\omega t+\overline{\varphi}_{20})) \quad (25)$$

is applied, where $\varphi_{10}\equiv\varphi_{11,0}$, $\varphi_{20}\equiv\varphi_{21,0}$.

Equation (24) must be integrated with the following conditions:

$$u\to 0 \ \text{ when } \ \eta\to+\infty,$$

$$c_0 x_0=\frac{EF}{r}\frac{\partial u}{\partial\eta}+F_{in0}(u_{10})\cos\alpha_0 \ \text{ when } \ \eta=0, \quad (26)$$

where $\alpha_0\equiv\alpha_{10}$.

The solution of equation (24) is sought in the form

$$u(\eta,t)=\theta(\eta)\cos\omega t+\psi(\eta)\sin\omega t. \quad (27)$$

Substituting the expression (27) into the equation (24), the functions $\theta(\eta)$ and $\psi(\eta)$ are found

$$\begin{cases}\theta(\eta)=e^{-\frac{k}{2}\eta}\left(\begin{aligned}&A_1e^{r_1\eta}+A_2e^{-r_1\eta}+\\&+A_3e^{r_2\eta}+A_4e^{-r_2\eta}\end{aligned}\right),\\ \psi(\eta)=ie^{-\frac{k}{2}\eta}\left(\begin{aligned}&-A_1e^{r_1\eta}-A_2e^{-r_1\eta}+\\&+A_3e^{r_2\eta}+A_4e^{-r_2\eta}\end{aligned}\right),\end{cases} \quad (28)$$

where $r_1=\alpha+i\alpha_1$, $r_2=\alpha-i\alpha_1$, α,α_1 are the positive real numbers

$$\alpha=\frac{\sqrt{\sqrt{(k^2-4\beta^2)^2+16\mu^2}+(k^2-4\beta^2)}}{2\sqrt{2}},$$

$$\alpha_1=\frac{\sqrt{\sqrt{(k^2-4\beta^2)^2+16\mu^2}-(k^2-4\beta^2)}}{2\sqrt{2}}, \quad (29)$$

$$\mu=\frac{\xi_1 r^2\omega}{EF},\ \beta^2=\frac{\rho r^2\omega^2}{E},\ k=\frac{EF\beta^2}{c_0 r}=\frac{\rho Fr\omega^2}{c_0}.$$

The functions $\theta(\eta)$ and $\psi(\eta)$ must satisfy the conditions

$$\theta(\eta)\to 0,\ \psi(\eta)\to 0 \ \text{ when } \ \eta\to+\infty. \quad (30)$$

$$\left(1-\frac{ak}{\beta^2}\cos\alpha_0\right)\frac{\partial\theta}{\partial\eta}(0)-\frac{\beta^2}{k}\left(1+\frac{ak}{\beta^2}\cos\alpha_0\right)\theta(0)=$$
$$=\frac{b}{2}\cos\alpha_0(\cos\overline{\varphi}_{10}+\cos\overline{\varphi}_{20}),$$

$$\left(1-\frac{ak}{\beta^2}\cos\alpha_0\right)\frac{\partial\psi}{\partial\eta}(0)-\frac{\beta^2}{k}\left(1+\frac{ak}{\beta^2}\cos\alpha_0\right)\psi(0)=$$
$$=-\frac{b}{2}\cos\alpha_0(\sin\overline{\varphi}_{10}+\sin\overline{\varphi}_{20}), \quad (31)$$

where $a=\frac{m_0+2m}{EF}r\omega^2=a_1\beta^2$, $a_1=\frac{m_0+2m}{\rho F\cdot r}$,

$$b=\frac{2m}{EF}r^2\omega^2=b_1\beta^2,\ b_1=\frac{2m}{\rho F},$$

are obtained from (27).

Using the conditions (30) and (31), the integration constants A_1-A_4 in (28)

$$A_1=A_3=0,$$

$$A_2=-\frac{bk}{4}\frac{\cos\alpha_0(e^{-i\overline{\varphi}_{10}}+e^{-i\overline{\varphi}_{20}})}{\left(\frac{k^2}{2}+\beta^2+kr_1\right)-\frac{ak}{\beta^2}\cos\alpha_0\left(\frac{k^2}{2}-\beta^2+kr_1\right)},$$

$$A_4=-\frac{bk}{4}\frac{\cos\alpha_0(e^{i\overline{\varphi}_{10}}+e^{i\overline{\varphi}_{20}})}{\left(\frac{k^2}{2}+\beta^2+kr_2\right)-\frac{ak}{\beta^2}\cos\alpha_0\left(\frac{k^2}{2}-\beta^2+kr_2\right)}$$

are determined.

Function $u(\eta,t)$ is represented in the form

$$u(\eta,t) = A(\eta)\cos(\omega t + \gamma_0(\eta)) , \tag{32}$$

where $A(\eta) = \sqrt{\theta^2(\eta) + \psi^2(\eta)}$ is the amplitude of vibrations of the points of a bar, and $\gamma_0(\eta) = -arctg\dfrac{\psi(\eta)}{\theta(\eta)}$ is the initial phase.

Presenting $u(\eta,t)$ in the form (32), it is obtained

$$u(\eta,t) = \frac{bk}{2\Delta} e^{-\left(\frac{k}{2}+\alpha\right)\eta} \cos\alpha_0 [\cos(\omega t + \overline{\varphi}_{10} + \alpha_1\eta + \Lambda) +$$

$$+\cos(\omega t + \overline{\varphi}_{20} + \alpha_1\eta + \Lambda)] , \tag{33}$$

where $\Lambda = arctg\dfrac{k\alpha_1}{k\alpha + \dfrac{k^2}{2} + \beta^2 \cdot D}$, $D = \dfrac{1 + \dfrac{ak}{\beta^2}\cos\alpha_0}{1 - \dfrac{ak}{\beta^2}\cos\alpha_0}$,

$$\Delta = \sqrt{\mu_1^2 + (k\alpha_1)^2}\,\Delta_1 ,$$

$$\Delta_1 = \sqrt{\left(\frac{ak}{\beta^2}\cos\alpha_0 - \frac{\mu_1\mu_2 + (k\alpha_1)^2}{\mu_1^2 + (k\alpha_1)^2}\right)^2 + \left(\frac{2k\alpha_1\beta^2}{\mu_1^2 + (k\alpha_1)^2}\right)^2} ,$$

$$\mu_1 = k\alpha + k^2/2 - \beta^2 , \; \mu_2 = k\alpha + k^2/2 + \beta^2 .$$

Differentiating the function $u(\eta,t)$ with respect to η at $\eta = \eta_1 = 0$, the value of the function $x_0(t) = \dfrac{k}{\beta^2}\dfrac{\partial u}{\partial \eta}(0,t)$ is found, then adding $x_0(t)$ and $u_{10}(t) = u(0,t)$, it is obtained

$$u_{10}(t) + x_0(t) = \frac{bk}{2\beta^2\Delta_1}\cos\alpha_0[\cos(\omega t + \overline{\varphi}_{10} + \Lambda_1) +$$

$$+\cos(\omega t + \overline{\varphi}_{20} + \Lambda_1)] , \tag{34}$$

where $\Lambda_1 = \Lambda - arctg\dfrac{k\alpha_1}{\mu_1}$.

Finding the function $v(\eta,t)$ is reduced to the integration of the differential equation of the transverse vibrations of the bar

$$\frac{EJ}{r^4}\frac{\partial^4 v}{\partial \eta^4} + \rho F\frac{\partial^2 v}{\partial t^2} + \xi_2\frac{\partial v}{\partial t} = 0 , \; \eta \neq \eta_1 , \tag{35}$$

valid everywhere, except at the point $\eta = \eta_1 = 0$, where the force

$$F_{in_0}(v_{10}) = (m_0 + 2m)\ddot{v}_{10} - $$
$$- mr\omega^2(\sin(\omega t + \overline{\varphi}_{10}) - \sin(\omega t + \overline{\varphi}_{20})) \tag{36}$$

is applied.

Equation (35) must be integrated with the conditions

$v \to 0$, when $\eta \to +\infty$,

$$\frac{\partial^2 v}{\partial \eta^2}(0,t) = 0 , \; \frac{\partial^3 v}{\partial \eta^3}(0,t) = -\frac{F_{in_0}(v_{10})r^3\sin\alpha_0}{EJ} . \tag{37}$$

The solution of equation (35) is sought in the form

$$v(\eta,t) = X(\eta)\cos\omega t + Y(\eta)\sin\omega t . \tag{38}$$

Designating $\beta_1^2 = \dfrac{\rho F r^4\omega^2}{EJ}$, $\mu_1 = \dfrac{\xi_2 r^4\omega}{EJ}$ and substituting the expression (38) into the equation (35), the system of the ordinary differential equations

$$\begin{cases} X^{(4)}(\eta) - \beta_1^2 X(\eta) + \mu_1 Y(\eta) = 0, \\ Y^{(4)}(\eta) - \beta_1^2 Y(\eta) - \mu_1 X(\eta) = 0 \end{cases} \tag{39}$$

is obtained.

Solving the system of ordinary differential equations (39), the functions $X(\eta)$ and $Y(\eta)$ are found:

$$\begin{cases} X(\eta) = D_1K_1 + D_2K_2 + iD_3K_3 + iD_4K_4 + \\ \quad + D_5K_5 + D_6K_6 + iD_7K_7 + iD_8K_8, \\ Y(\eta) = D_1K_7 + D_2K_4 - iD_3K_5 - iD_4K_2 + \\ \quad + D_5K_3 + D_6K_8 - iD_7K_1 - iD_8K_6, \end{cases} \tag{40}$$

where

$$K_1(\eta) = \frac{1}{2}(ch\gamma\eta\cos\gamma_1\eta + ch\gamma_1\eta\cos\gamma\eta) ,$$

$$K_2(\eta) = \frac{1}{2}(sh\gamma\eta\cos\gamma_1\eta + ch\gamma_1\eta\sin\gamma\eta) ,$$

$$K_3(\eta) = \frac{1}{2}(sh\gamma\eta\sin\gamma_1\eta + sh\gamma_1\eta\sin\gamma\eta) ,$$

$$K_4(\eta) = \frac{1}{2}(ch\gamma\eta\sin\gamma_1\eta + sh\gamma_1\eta\cos\gamma\eta) ,$$

$$K_5(\eta) = \frac{1}{2}(ch\gamma\eta\cos\gamma_1\eta - ch\gamma_1\eta\cos\gamma\eta) ,$$

$$K_6(\eta) = \frac{1}{2}(sh\gamma\eta\cos\gamma_1\eta - ch\gamma_1\eta\sin\gamma\eta) ,$$

$$K_7(\eta) = \frac{1}{2}(sh\gamma\eta\sin\gamma_1\eta - sh\gamma_1\eta\sin\gamma\eta) ,$$

$$K_8(\eta) = \frac{1}{2}(ch\gamma\eta\sin\gamma_1\eta - sh\gamma_1\eta\cos\gamma\eta) , \tag{41}$$

γ, γ_1, $\tilde{\alpha}$, $\tilde{\alpha}_1$ are the positive real constants:

$$\gamma = \sqrt{\frac{\sqrt{\tilde{\alpha}^2 + \tilde{\alpha}_1^2} + \tilde{\alpha}}{2}} , \; \gamma_1 = \sqrt{\frac{\sqrt{\tilde{\alpha}^2 + \tilde{\alpha}_1^2} - \tilde{\alpha}}{2}} ,$$

$$\tilde{\alpha} = \sqrt{\frac{\sqrt{\beta_1^4 + \mu_1^2} + \beta_1^2}{2}}, \quad \tilde{\alpha}_1 = \sqrt{\frac{\sqrt{\beta_1^4 + \mu_1^2} - \beta_1^2}{2}}. \qquad (42)$$

$K_1(0) = 1$, $K_2'(0) = \gamma$, $K_4'(0) = \gamma_1$, $K_3''(0) = \tilde{\alpha}_1$, $K_5''(0) = \tilde{\alpha}$, $K_6'''(0) = \delta_1$, $K_8'''(0) = \delta_2$, where $\delta_1 \equiv \tilde{\alpha}\gamma - \tilde{\alpha}_1\gamma_1$, $\delta_2 \equiv \tilde{\alpha}\gamma_1 + \tilde{\alpha}_1\gamma$, values of all other functions and their derivatives to the third order inclusive at $\eta = 0$ are equal to zero.

Applying the connection conditions

$$\begin{cases} X(\eta) \to 0 \\ Y(\eta) \to 0 \end{cases} \text{when } \eta \to +\infty, \qquad \begin{cases} X''(0) = 0, \\ Y''(0) = 0, \end{cases}$$

$$\begin{cases} X'''(0) = \left(a_2 X(0) + \frac{b_2}{2}(\sin\overline{\varphi}_{10} - \sin\overline{\varphi}_{20}) \right) \sin\alpha_0, \\ Y''(0) = \left(a_2 Y(0) + \frac{b_2}{2}(\cos\overline{\varphi}_{10} - \cos\overline{\varphi}_{20}) \right) \sin\alpha_0, \end{cases} \qquad (43)$$

where $a_2 = \frac{m_0 + 2m}{EJ} r^3\omega^2 = a_1\beta_1^2$, $b_2 = \frac{2m}{EJ} r^4\omega^2 = b_1\beta_1^2$, are obtained from conditions (37); the integration constants $D_1 - D_8$ of functions $X(\eta)$ and $Y(\eta)$ (40) are determined from algebraic system concerning these constants:

$$D_1 = -D_2 - D_6, \; D_3 = D_5 = 0, \; D_4 = -iD_6,$$

$$D_7 = -iD_2 + iD_6, \; D_8 = iD_2,$$

$$D_2 = \frac{b_2 \sin\alpha_0}{2\Delta_2}((a_2 \sin\alpha_0 - \delta_2)(\sin\overline{\varphi}_{10} - \sin\overline{\varphi}_{20}) + $$
$$+(a_2 \sin\alpha_0 + \delta_1)(\cos\overline{\varphi}_{10} - \cos\overline{\varphi}_{20})),$$

$$D_6 = \frac{b_2 \sin\alpha_0}{2\Delta_2}((a_2 \sin\alpha_0 + \delta_1)(\sin\overline{\varphi}_{10} - \sin\overline{\varphi}_{20}) - $$
$$-(a_2 \sin\alpha_0 - \delta_2)(\cos\overline{\varphi}_{10} - \cos\overline{\varphi}_{20})),$$

$$\Delta_2 = (a_2 \sin\alpha_0 + \delta_1)^2 + (a_2 \sin\alpha_0 - \delta_2)^2.$$

Functions $X(\eta)$ and $Y(\eta)$ take the form

$$\begin{cases} X(\eta) = D_2(K_2 - K_1 + K_7 - K_8) + D_6(K_4 - K_1 + K_6 - K_7), \\ Y(\eta) = D_2(K_4 - K_1 + K_6 - K_7) - D_6(K_2 - K_1 + K_7 - K_8). \end{cases}$$

Therefore, the function $v(\eta, t)$ is equal to

$$v(\eta,t) = (D_2 \cos\omega t - D_6 \sin\omega t)(K_2 - K_1 + K_7 - K_8) +$$
$$+(D_6 \cos\omega t + D_2 \sin\omega t)(K_4 - K_1 + K_6 - K_7). \qquad (44)$$

Functions $v(\eta,t)$ are represented in the form $v(\eta,t) = A(\eta)\cos(\omega t + \gamma_0(\eta))$. It is obtained

$$v(\eta,t) = \frac{b_2 \sin\alpha_0}{\sqrt{\Delta_2}} \sin\frac{\overline{\varphi}_{10} - \overline{\varphi}_{20}}{2} \times$$
$$\times\left((K_4 - K_1 + K_6 - K_7)\cos\left(\omega t + \frac{\overline{\varphi}_{10} + \overline{\varphi}_{20}}{2} - \Lambda_2 \right) - \right.$$
$$\left. -(K_2 - K_1 + K_7 - K_8)\sin\left(\omega t + \frac{\overline{\varphi}_{10} + \overline{\varphi}_{20}}{2} - \Lambda_2 \right) \right), \qquad (45)$$

where $\Lambda_2 = arctg \frac{a_2 \sin\alpha_0 - \delta_2}{a_2 \sin\alpha_0 + \delta_1}$.

Then the function $v_{10}(t) = v(0,t)$ is equal to

$$v_{10}(t) = \frac{b_2 \sin\alpha_0}{\sqrt{2\Delta_2}}[\cos(\omega t + \overline{\varphi}_{20} - \Lambda_3) -$$
$$-\cos(\omega t + \overline{\varphi}_{10} - \Lambda_3)], \qquad (46)$$

where $\Lambda_3 = arctg \frac{2a_2 \sin\alpha_0 + \delta_1 - \delta_2}{\delta_1 + \delta_2}$.

5. Conditions of the existence and stability of the solutions

It is necessary to make use of periodicity conditions (19) at definition of frequency ω for longitudinal and transverse vibrations which at $n = 1$ takes the form

$$\overline{\Phi}_{10} = \overline{[(\ddot{u}_{10} + \ddot{x}_0)\sin(\omega t + \overline{\varphi}_{10}) - \ddot{v}_{10}\cos(\omega t + \overline{\varphi}_{10})] -}$$
$$\overline{-h_1\omega + M_1} = 0,$$

$$\overline{\Phi}_{20} = \overline{[(\ddot{u}_{10} + \ddot{x}_0)\sin(\omega t + \overline{\varphi}_{20}) + \ddot{v}_{10}\cos(\omega t + \overline{\varphi}_{20})] -}$$
$$\overline{-h_2\omega + M_2} = 0, \qquad (47)$$

where $\overline{\Phi}_{k0} \equiv \overline{\Phi}_{kj,0}$, $h_k = \frac{H_{\varphi kj}}{mr}$, $M_k = \frac{M_{\varphi kj}}{mr}$, $j = 1$, $k = 1,2$.

Substituting the functions $u_{10} + x_0$ (34) and v_{10} (46) into the system (47), the given system is transformed to the form

$$\left(\frac{bk}{2\beta^2\Delta_1}\cos\alpha_0 \cos\Lambda_1 + \frac{b_2}{\sqrt{2\Delta_2}}\sin\alpha_0 \sin\Lambda_3 \right)\sin(\overline{\varphi}_{20} - \overline{\varphi}_{10}) =$$
$$= \frac{h_1\omega - M_1 - h_2\omega + M_2}{\omega^2},$$

$$\frac{bk}{2\beta^2\Delta_1}\cos\alpha_0\sin\Lambda_1(1+\cos(\overline{\varphi}_{20}-\overline{\varphi}_{10}))-\frac{b_2}{\sqrt{2\Delta_2}}\sin\alpha_0\cos\Lambda_3\times$$

$$\times(1-\cos(\overline{\varphi}_{20}-\overline{\varphi}_{10}))=\frac{h_1\omega-M_1+h_2\omega-M_2}{\omega^2}. \qquad (48)$$

From the given system (48) it follows that the condition of existence of solutions is the inequality

$$\left|\frac{h_1\omega-M_1-h_2\omega+M_2}{\frac{bk\omega^2}{2\beta^2\Delta_1}\cos\alpha_0\cos\Lambda_1+\frac{b_2\omega^2}{\sqrt{2\Delta_2}}\sin\alpha_0\sin\Lambda_3}\right|<1.$$

a) For $\alpha_0=\frac{\pi}{2}$ $\quad\overline{\varphi}_{20}-\overline{\varphi}_{10}=\pi$,

from system (48) it is obtained that

$$h_1\omega-M_1=h_2\omega-M_2,$$

$$-\frac{b_2\omega^2}{\sqrt{2\Delta_2}}\cos\Lambda_3\Big|_{\alpha 0=\frac{\pi}{2}}=h_1\omega-M_1,$$

and as

$$\sqrt{\Delta_2}\Big|_{\alpha 0=\frac{\pi}{2}}=\sqrt{(a_2+\delta_1)^2+(a_2-\delta_2)^2},$$

$$\cos\Lambda_3\Big|_{\alpha 0=\frac{\pi}{2}}=\frac{\delta_1+\delta_2}{\sqrt{2((a_2+\delta_1)^2+(a_2-\delta_2)^2)}},$$

that for transverse vibrations of the bar frequency ω is defined from the equality

$$\frac{b_2\omega^2(\delta_1+\delta_2)}{(a_2+\delta_1)^2+(a_2-\delta_2)^2}+2(h_1\omega-M_1)=0. \qquad (49)$$

b) For $\alpha_0=0$ $\quad\overline{\varphi}_{20}-\overline{\varphi}_{10}=0$,

from the system (48) we have

$$h_1\omega-M_1=h_2\omega-M_2,$$

$$\frac{bk\omega^2}{2\beta^2\Delta_1}\sin\Lambda_1\Big|_{\alpha 0=0}=h_1\omega-M_1.$$

As

$$\Delta_1\Big|_{\alpha 0=0}=\sqrt{\left(\frac{ak}{\beta^2}-\frac{\mu_1\mu_2+(k\alpha_1)^2}{\mu_1^2+(k\alpha_1)^2}\right)^2+\left(\frac{2k\alpha_1\beta^2}{\mu_1^2+(k\alpha_1)^2}\right)^2},$$

$$\sin\Lambda_1\Big|_{\alpha 0=0}=\frac{2k\alpha_1\beta^2}{\left(\mu_1^2+(k\alpha_1)^2\right)\Delta_1\Big|_{\alpha 0=0}}$$

that for longitudinal vibrations of the bar frequency ω is defined from the expression

$$\frac{bk^2\alpha_1\omega^2}{\left(\mu_1^2+(k\alpha_1)^2\right)\cdot[\Delta_1\big|_{\alpha 0=0}]^2}-(h_1\omega-M_1)=0. \qquad (50)$$

The solution was stable when it should satisfy the condition

$$\frac{\partial(\overline{\Phi}_{20}-\overline{\Phi}_{10})}{\partial(\overline{\varphi}_{20}-\overline{\varphi}_{10})}<0. \qquad (51)$$

On the basis of this inequality it is received that in the case if $\alpha_0=\frac{\pi}{2}$ the solution is stable when the inequality

$$\frac{b_2\omega^2(2a_2+\delta_1-\delta_2)}{2((a_2+\delta_1)^2+(a_2-\delta_2)^2)}<0 \text{ or } 2a_2+\delta_1-\delta_2<0 \qquad (52)$$

is fulfilled; in the case when $\alpha_0=0$ the solution should satisfy the inequality

$$\frac{\omega^2bk}{2\beta^2[\Delta_1\big|_{\alpha 0=0}]^2}\left(\frac{ak}{\beta^2}-\frac{\mu_1\mu_2+(k\alpha_1)^2}{\mu_1^2+(k\alpha_1)^2}\right)>0$$

$$\text{or}\quad\frac{ak}{\beta^2}-\frac{\mu_1\mu_2+(k\alpha_1)^2}{\mu_1^2+(k\alpha_1)^2}>0. \qquad (53)$$

6. Experimental analysis results

As initial data it is accepted:

$a_1=0.1$, $b_1=0.2$, $\rho_1=2$, $n=0.5$, $\xi_0=0$, $h_1=h_2=2$, $M_1=M_2=1$, $\alpha_0=0$.

Entered into consideration the dimensionless coordinates are connected with corresponding dimensional coordinates by the formulas

$$a_1=\frac{m_0+2m}{\rho F\cdot r},\ b_1=\frac{2m}{\rho F},\ \rho_1=\sqrt{\frac{\rho}{E}}\cdot r,\ n=\frac{EF}{c_0\cdot r},$$

$$h_j=\frac{H_{\varphi j}}{mr},\ M_j=\frac{M_{\varphi j}}{mr},\ \xi_0=\frac{\xi_1\cdot r^2}{EF}.$$

In the given case $\overline{\varphi}_{20}-\overline{\varphi}_{10}=2\pi$. The equation for determination of frequency ω has the solutions:

$\omega_1 \approx 0.5040$, $\omega_2 \approx 2.6571$. The frequency ω_2 is the frequency of resonance.

The three-dimensional graph of the function $u(\eta, t)$ with the given parameters is represented in Fig. 2.

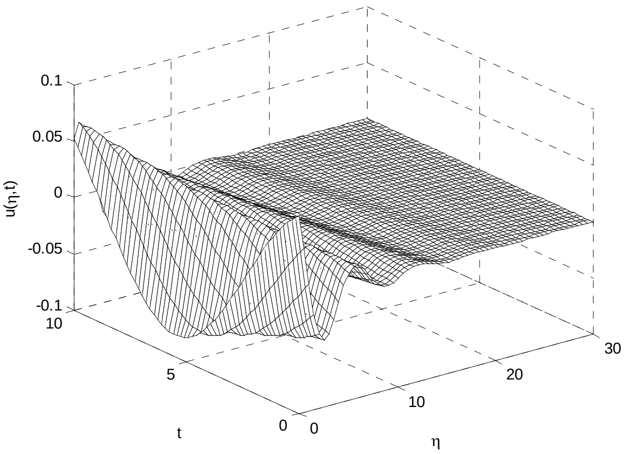

Fig. 2. The graph of the function $u(\eta, t)$

Following dimensionless coordinates

$$c_1 = 0.3,\ c_2 = 1,\ h_1 = h_2 = 0.1,\ M_1 = M_2 = 1,\ \alpha_0 = \frac{\pi}{2},$$

are connected with corresponding dimensional coordinates by the formulas

$$c_1 = \sqrt{\frac{\rho F}{EJ}} \cdot r^2,\ c_2 = \frac{\xi_2 \cdot r^4}{EJ}.$$

In this case $\overline{\varphi}_{20} - \overline{\varphi}_{10} = \pi$. The equation for determination of frequency ω has the solution: $\omega \approx 3.4029$.

The three-dimensional graph of the function $v(\eta, t)$ with the given parameters is represented in Fig.3.

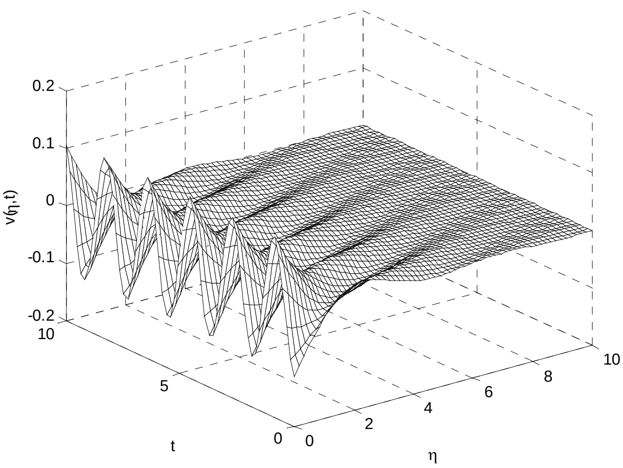

Fig. 3. The graph of the function $v(\eta, t)$

7. Conclusions

The dynamic model of the system is constructed, the exact solution of a problem about determination of longitudinal and transverse vibrations of the bar with the elastically connected end is found in the work. Using the given exact solutions, the expressions for the definition of the excitation frequency are received, leading to the results of calculations with any entered parameters of the system.

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