Automorphic chromatic index of generalized Petersen graphs

G. Mazzuoccolo, B. Ruini *

Abstract

The automorphic A-chromatic index of a graph Γ is the minimum integer m for which Γ has a proper edge-coloring with m colors which is preserved by a given subgroup A of the full automorphism group of Γ. We compute the automorphic A-chromatic index of each generalized Petersen graph when A is the full automorphism group.

Keywords: graph, edge-coloring, automorphism.
MSC(2000): 05C15, 05C25.

1 Introduction

The generalized Petersen graph $GP(n,k)$, with n and k being integers and $2 \leq 2k < n$, is the simple cubic graph with vertex-set $V(GP(n,k)) = \{u_0, u_1, \ldots, u_{n-1}, v_0, v_1, \ldots, v_{n-1}\}$ and edge-set $E(GP(n,k)) = \{[u_i, u_{i+1}], [u_i, v_i], [v_i, v_{i+k}] : i \in \mathbb{Z}\}$. All subscripts are taken modulo n. The edges of $GP(n,k)$ of types $[u_i, u_{i+1}]$, $[u_i, v_i]$, and $[v_i, v_{i+k}]$ are called outer edges, spokes, and inner edges and they are denoted by O_i,S_i and I_i, respectively; they form three n-sets that we denote by O, S and I, respectively. The n-circuit generated by O is called the outer rim. If d denotes the greatest common divisor of n and k, then I generates a subgraph which is the union of d pairwise-disjoint $(\frac{n}{d})$-circuits, called inner rims. The graph $GP(5,2)$ is the well known Petersen graph. The graph $GP(n,k)$ with n and k relatively prime was introduced by Coxeter in [5], while the generalized Petersen graph $GP(n,k)$ was first considered in [6]. The generalized Petersen graphs have been studied by several authors: Castagna and Prins [3] completed the proof begun by Watkins in [8] that each generalized Petersen graph is 3-edge-colorable, except for the original Petersen graph; Alspach [1] has determined all the Hamiltonian generalized Petersen graphs; the complete classification of their full automorphism groups has been worked out in [6]. There exist chromatic parameters of a graph Γ that depend on the full automorphism group of Γ: the distinguishing number defined in [2], the distinguishing chromatic number defined in [4] and the automorphic chromatic index recently

*Dipartimento di Matematica, Università di Modena e Reggio Emilia, via Campi 213/B, 41125 Modena (Italy)

defined in [7]. In [9], Weigand and Jacobson have studied and completely determined the distinguishing number and the distinguishing chromatic number of each generalized Petersen graph. The aim of this paper is to do the same for the automorphic chromatic index of $GP(n,k)$. We are interested in proper edge-colorings of the generalized Petersen graphs $GP(n,k)$ which are preserved by the full automorphism group $A(n,k)$. In what follows, a coloring always means a proper edge-coloring and a coloring preserved by an automorphism group A is said to be an A-automorphic coloring. The automorphic A-chromatic index of a graph Γ, denoted by χ'_A, is the minimum integer m for which Γ has an A-automorphic coloring with m colors (see [7]). We completely determine the automorphic $A(n,k)$-chromatic index, $\chi'_{A(n,k)}$, of $GP(n,k)$. Our main theorem is the following:

Theorem.

$$\chi'_{A(n,k)} = \begin{cases} 3 & \textit{if } n \textit{ is even, } k \textit{ is odd and } k^2 \not\equiv -1 \textit{ mod } n \\ 5 & \textit{if } n \textit{ is even, } k \textit{ is odd and } k^2 \equiv -1 \textit{ mod } n \\ min\{d_{n,k}, e_{n,k}+2\} & \textit{otherwise,} \end{cases}$$

where $d_{n,k}$ $(e_{n,k})$ *is the smallest odd (even) integer dividing* n *and not dividing* k*, if such an integer exists at all, and* $d_{n,k}$ $(e_{n,k}) = +\infty$ *otherwise.*

The previous theorem is obtained as a consequence of Theorem 2 in Section 4 and of Proposition 5 in Section 5.

2 Preliminaries

Since our parameter depends on the full automorphism group of the graph considered, we need to recall the classification of the full automorphism group, $A(n,k)$, of $GP(n,k)$ ([6]). Define the permutations ρ, δ, α on $V(GP(n,k))$ as follows:

$$\begin{array}{ll} \rho(u_i) = u_{i+1}, & \rho(v_i) = v_{i+1}, \\ \delta(u_i) = u_{-i}, & \delta(v_i) = v_{-i}, \\ \alpha(u_i) = v_{ki}, & \alpha(v_i) = u_{ki}. \end{array}$$

with all subscripts taken modulo n. Let $H = <\rho, \delta>$, then $H \leq A(n,k)$ (see [8]), in particular H fixes S set-wise. The following theorem holds (see [6]):

Theorem 1. *If* $(n,k) \notin \{(4,1),(5,2),(8,3),(10,2),(10,3),(12,5),(24,5)\}$ *then*

$$A(n,k) = \begin{cases} H & \textit{if } k^2 \not\equiv \pm 1 \textit{ mod } n \\ <H, \alpha> & \textit{if } k^2 \equiv \pm 1 \textit{ mod } n. \end{cases}$$

The full automorphism groups of the cases not considered in the previous proposition are described in details in the last section.

In this section we outline some properties of an H-automorphic coloring $\mathcal{E}$ of $GP(n,k)$ with color-set $\mathcal{O} \cup \mathcal{I} \cup \mathcal{S}$. Note that there always exists a coloring preserved by H: the one in which each edge has a different color. We indicate with

$\mathcal{O} = \{o_0, o_1, \dots, o_{l-1}\}$, $\mathcal{S} = \{s_0, s_1, \dots, s_{t-1}\}$, $\mathcal{I} = \{i_0, i_1, \dots, i_{m-1}\}$ the set of colors used to color the outer edges, the spokes and the inner edges, respectively. Note that the three sets are not necessarily distinct. The cardinalities of $\mathcal{O}, \mathcal{S}$, and $\mathcal{I}$ are l, t and m.

Lemma 1. *There exists an H-automorphic edge-coloring $\mathcal{E}$ of $GP(n,k)$ if and only if*

(1) $m \mid n$ and $m \nmid k$;

(2) $l \mid n$ and $l > 1$.

Proof. Let $\mathcal{E}$ be an H-automorphic coloring of $GP(n,k)$. The action of $< \rho >$ is transitive on the edge sets O and I, so that (as the coloring is H-automorphic) the colors are used cyclically on both these sets. Thus, we may assume $\mathcal{E}(O_j) = o_j$, $\mathcal{E}(I_j) = i_j$ (where the index j is in each case taken modulo the size of the set being indexed). Thus, l,m each divide $| < \rho > | = n$. Moreover the adjacent edges I_0, I_k and O_i, O_{i+1} have different colors, then m does not divide k and $l > 1$.
Vice versa consider the color-sets $\mathcal{I}$ and $\mathcal{O}$ with m, l such that $m \mid n, m \nmid k$ and $l \mid n, l > 1$. We set $\mathcal{E}(O_j) = o_j$, $\mathcal{E}(I_j) = i_j$, index j taken modulo m and l, respectively. Furthermore, we color all spokes with a unique color $s_0 \notin \mathcal{O} \cup \mathcal{I}$. One can easily check that this coloring is preserved by H. □

In what follows we highlight some basic properties of the action of δ on the coloring $\mathcal{E}$ that will be useful in the next sections.

Lemma 2. *Let $\mathcal{E}$ be an H-automorphic edge-coloring of $GP(n,k)$. If l is even, then no color of $\mathcal{O}$ is fixed by the automorphism δ. If l is odd, then just one color of $\mathcal{O}$ is fixed by the automorphism δ.*

Proof. The action of δ on $\mathcal{O}$ is given by $\delta(o_j) = o_{l-j-1}$. We have $\delta(o_j) = o_j$ if and only if $j \equiv_l l - j - 1$, that is $2j + 1 \equiv_l 0$. If l is even then no colors of $\mathcal{O}$ are fixed by the automorphism δ. If l is odd, since $j \leq l - 1$ and then $2j + 1 \leq 2l - 1$, the unique color fixed by δ is $o_{\frac{l-1}{2}}$. □

Lemma 3. *Let $\mathcal{E}$ be an H-automorphic edge-coloring of $GP(n,k)$. If m is odd, then just one color of $\mathcal{I}$ is fixed by the automorphism δ.*
If m and k are both even, then exactly two colors of $\mathcal{I}$ are fixed by the automorphism δ.

Proof. The action of δ on $\mathcal{I}$ is given by $\delta(i_j) = i_{m-k-j}$. We have $\delta(i_j) = i_j$ if and only if $j \equiv_m m - k - j$, that is $2j \equiv_m -k$.
The color i_j is fixed by δ if and only if one of the following holds

$$j \equiv_m \frac{m-k}{2} \tag{1}$$

$$j \equiv_m -\frac{k}{2} \tag{2}$$

If m is odd then exactly one of (1) and (2) has a solution, according to the fact that k is odd or even respectively, in both cases we have a unique color fixed by δ.
If m is even and k is even then both (1) and (2) have a solution, then we have exactly two colors fixed by δ. □

In Section 3 we will use the previous lemmas to determine an H-automorphic coloring of $GP(n,k)$ with the minimum number of colors. In particular it will be useful to establish in which cases one can use the same set of colors for inner and outer edges: nevertheless in some of these cases it will be more convenient to select $\mathcal{O}$ and $\mathcal{I}$ distinct in order to minimize the total number of colors.

3 χ'_H of $GP(n,k)$

In this section we determine χ'_H of $GP(n,k)$. In the rest of the paper we make use of $d_{n,k}$ and $e_{n,k}$ defined in the Main Theorem.

Proposition 1. *If n is odd, then $\chi'_H = d_{n,k}$.*

Proof. Let $\mathcal{E}$ be an H-automorphic coloring of $GP(n,k)$. By Lemma 1 it follows that m is odd, $m \geq d_{n,k}$ and then $\chi'_H \geq d_{n,k}$. Now we furnish a $d_{n,k}$-coloring preserved by H to prove the assertion. By Lemmas 2 and 3 we know that δ fixes exactly one color both in $\mathcal{O}$ and $\mathcal{I}$, then we consider $m = l = s = d_{n,k}$ and $\mathcal{I} \equiv \mathcal{O} \equiv \mathcal{S} = \{o_0, \ldots, o_{l-1}\}$. We set $\mathcal{E}(O_j) = o_j$,$\mathcal{E}(I_j) = i_j$ with $i_j = o_{\frac{n+k+l-1}{2}+j}$ if k is odd and $i_j = o_{\frac{k+l-1}{2}+j}$ if k is even. Finally, we color the spokes S_j with the color $s_j = o_{\frac{l-1}{2}+j}$. It is easy to check that both ρ and δ preserve the coloring and then the assertion follows. □

In Figure 1, we show an H-automorphic 5-coloring of $GP(15,3)$.

Proposition 2. *If n is even and k is odd, then*

$$\chi'_H = 3.$$

Proof. We furnish an H-automorphic 3-coloring of $GP(n,k)$. Color O_j and I_j either with the color o_0 or with the color o_1, according to the parity of j. Observe that this is a proper coloring by the fact that n is even and k is odd. Finally, color all spokes with the color $s_0 \notin \{o_0, o_1\}$.
It is straightforward to check that this coloring is preserved by H. □

Proposition 3. *If n and k are even, then*

$$\chi'_H = min\{d_{n,k}, e_{n,k} + 2\}.$$

Proof. If $e_{n,k} < +\infty$ there is an H-automorphic $(e_{n,k} + 2)$-coloring, say $\mathcal{E}_1$, with colors $\mathcal{O}_1 \cup \mathcal{I}_1 \cup \mathcal{S}_1$, $m_1 = e_{n,k}$ $l_1 = 2$, $\mathcal{S}_1 = \mathcal{I}_1 = \{i_0, i_1, \ldots, i_{m_1-1}\}$, where $s_j = i_{m_1-\frac{k}{2}+j}$ (the color of the spoke S_j), and O_j colored with o_0 or o_1 according to the parity of j. If $d_{n,k} < +\infty$, there is an H-automorphic $d_{n,k}$-coloring, say $\mathcal{E}_2$, with colors $\mathcal{O}_2 \cup \mathcal{I}_2 \cup \mathcal{S}_2$, $m_2 = d_{n,k} = l_2 = s_2$, $\mathcal{O}_2 = \mathcal{I}_2 = \mathcal{S}_2$ where $i_j = o_{\frac{l_2-1+k}{2}+j}$ (the color of the inner edge I_j) and with $s_j = o_{\frac{l_2-1}{2}+j}$ (the color of the spoke S_j).

Now let $\mathcal{E}$ be an H-automorphic coloring of $GP(n,k)$. If m is even (hence $e_{n,k} < +\infty$), by Lemma 2 and 3 the two sets $\mathcal{O}$ and $\mathcal{I}$ are disjoint, then $m \geq e_{n,k}$, $l \geq 2$; therefore $\chi'_H \geq e_{n,k} + 2$. If m is odd (hence $d_{n,k} < +\infty$), by Lemma 1 $m \geq d_{n,k}$; hence $\chi'_H \geq d_{n,k}$. Therefore, the statement follows. □

Note that there exist values of n and k, which satisfy the hypothesis of the previous proposition, such that both $d_{n,k}$ and $e_{n,k}$ are finite (for instance if $n = 12$ and $k = 2$ then $e_{n,k} = 4$ and $d_{n,k} = 3$).

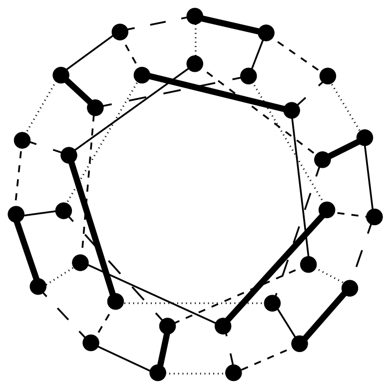

Figure 1: An H-automorphic 5-coloring of $GP(15,3)$

4 General case

In this section we compute $\chi'_{A(n,k)}$ with $(n,k) \notin \{(4,1),(5,2),(8,3),(10,2),(10,3),(12,5),(24,5)\}$. Recall that $H \leq A(n,k)$ for each n and k.

Theorem 2. *If* $(n,k) \notin \{(4,1),(5,2),(8,3),(10,2),(10,3),(12,5),(24,5)\}$ *then*

$$\chi'_{A(n,k)} = \begin{cases} 3 & \text{if } n \text{ is even } k, \text{ is odd and } k^2 \not\equiv -1 \text{ mod } n \\ 5 & \text{if } n \text{ is even } k, \text{ is odd and } k^2 \equiv -1 \text{ mod } n \\ min\{d_{n,k}, e_{n,k}+2\} & \text{otherwise.} \end{cases}$$

Proof. If n is even, k is odd and $k^2 \not\equiv \pm 1$ mod n, we get $A(n,k) = H$ (see Proposition 1); hence from Proposition 2 it follows $\chi'_{A(n,k)} = \chi'_H = 3$.

Let n be even, k odd and $k^2 \equiv 1$ mod n. From Proposition 1 we have $A(n,k) =< H, \alpha >$. Let us consider the 3-coloring $\mathcal{E}$ of $GP(n,k)$ of the proof of

Proposition 2. It is preserved by H. Since α fixes the set of spokes, $\alpha(O_j) = I_{kj}$, $\alpha(I_j) = O_{kj}$ and j and kj have same parity, then α preserves the coloring $\mathcal{E}$. It follows that $\chi'_{A(n,k)} = 3$ in this case.

If n is even, k odd and $k^2 \equiv -1 \bmod n$, from Proposition 1 the automorphic group $A(n,k)$ is $< H, \alpha >$. Suppose there exists an H-automorphic 3-coloring $\mathcal{E}$ of $GP(n,k)$. Two colors o_0, o_1 are used to color the outer rim and a color $s_0 \notin \{o_0, o_1\}$ is used to color the spokes. Since $\alpha(O_0) = I_0$ and $\alpha(I_0) = O_{-1}$, then the color of the edge I_0 must be different from o_0, o_1, s_0. Therefore there not exist a 3-coloring of $GP(n,k)$ preserved by $< H, \alpha >$. Moreover, an edge-coloring preserved by $< H, \alpha >$ must have at least two colors o_0, o_1 for the outer rim and at least two other different colors i_0, i_1 for the inner rim. Hence, a 4-coloring $\mathcal{E}$ of $GP(n,k)$ preserved by $< H, \alpha >$ does not exist: the spokes required at least a fifth different color.

Now we describe a 5-coloring of $GP(n,k)$ preserved by $A(n,k)$ with set-color $\{o_0, o_1, i_0, i_1, s_0\}$. Color O_j either with the color o_0 or o_1 according to the parity of j. Color also I_j either with the color i_0 or i_1 according to the parity of j. Color all the spokes with the same color s_0. This coloring is preserved by $A(n,k)$. In particular α acts on the set of colors as the permutation (o_0, i_0, o_1, i_1). Therefore, in this case $\chi'_{A(n,k)} = 5$.

Let n odd and $k^2 \not\equiv \pm 1 \bmod n$ or n even and k even. From Propositions 1 and 3 and Theorem 1 in Section 1 the statement follows (Note that if n is odd then $e_{n,k} = +\infty$).

If n is odd and $k^2 \equiv \pm 1 \bmod n$, Proposition 1 states that $\chi'_H = d_{n,k}$. The full automorphism group is $< H, \alpha >$. The minimal $d_{n,k}$-coloring described in the proof of Proposition 1 is preserved not only by H but also by α. In particular α acts on the set of colors as following:

$$\alpha(o_j) = \begin{cases} o_{\frac{k+l-1}{2}+kj} & \text{if } k \text{ is even} \\ o_{\frac{n+k+l-1}{2}+kj} & \text{if } k \text{ is odd}, \end{cases}$$

and the color $o_{\frac{l-1}{2}}$ is fixed by α. Therefore, in this case $\chi'_{A(n,k)} = d_{n,k}$. Since $e_{n,k} = +\infty$ the statement follows. $\square$

5 Exceptional cases

In this section we compute $\chi'_{A(n,k)}$ with $(n,k) \in \{(4,1), (5,2), (8,3), (10,2), (10,3), (12,5), (24,5)\}$.

Define the permutation λ on $V(GP(10,2))$ to have the following cycle structure $\lambda = (u_0, v_2, v_8)(u_1, v_4, u_8)(u_2, v_6, u_9)(u_3, u_6, v_9)(u_4, u_7, v_1)(u_5, v_7, v_3)$. Moreover, for $(n,k) \in \{(4,1), (8,3), (12,5), (24,5)\}$ let σ be the permutation defined on $V(GP(n,k))$ as follows: $\sigma(u_{4i}) = u_{4i}, \sigma(v_{4i}) = u_{4i+1}, \sigma(u_{4i+1}) = u_{4i-1}, \sigma(u_{4i-1}) = v_{4i}, \sigma(u_{4i+2}) = v_{4i-1}, \sigma(v_{4i-1}) = v_{4i+5}, \sigma(v_{4i+1}) = u_{4i-2}, \sigma(v_{4i+2}) = v_{4i-6}$; finally, define the permutations μ and μ' on $V(GP(10,3))$ and $V(GP(5,2))$ to have the following cycle structures $\mu = (u_2, v_1)(u_3, v_4)(u_7, v_6)$

$(u_8, v_9)(v_2, v_8)$ (v_3, v_7) and $\mu' = (u_2, v_1)(u_3, v_4)(v_2, v_3)$, respectively. Then, the following result holds (see [6]):

Proposition 4. *$A(10, 2) =< \rho, \lambda >$, $A(10, 3) =< \rho, \mu >$, $A(5, 2) =< \rho, \mu' >$ and $A(n, k) =< \rho, \delta, \sigma >$ for $(n, k) \in \{(4, 1), (8, 3), (12, 5), (24, 5)\}$.*

Note that $H \leq A(n, k)$ holds also for all exceptional cases. The following proposition complete the proof of the Main Theorem.

Proposition 5.

$$\chi'_{A(n,k)} = \begin{cases} 3 \text{ if } (n, k) \in \{(4, 1), (8, 3), (12, 5), (24, 5)\} \\ 5 \text{ if } (n, k) \in \{(5, 2), (10, 2), (10, 3)\}. \end{cases}$$

Proof. If $(n, k) \in \{(4, 1), (8, 3), (12, 5), (24, 5)\}$ then $\chi'_{A(n,k)} \geq \chi'(GP(n, k)) = 3$. In Figure 2 an $A(n, k)$-automorphic 3-coloring of $GP(n, k)$ is shown.

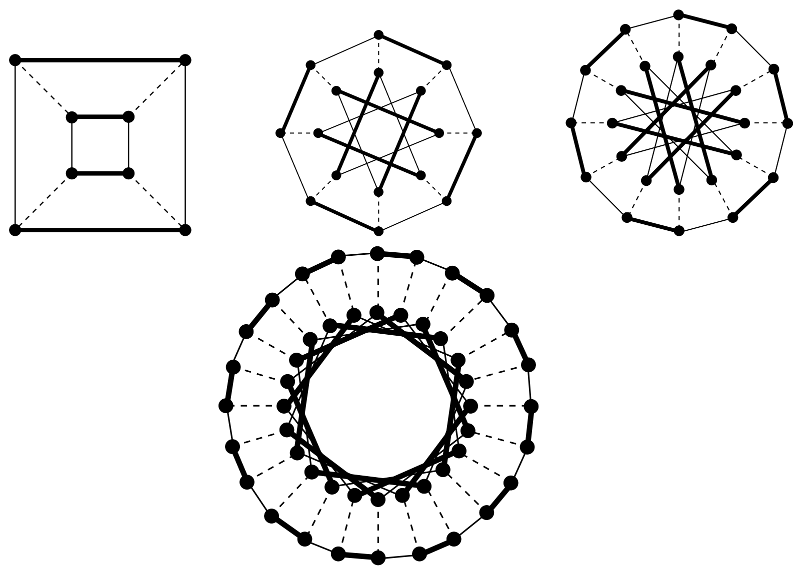

Figure 2: Automorphic 3-colorings of $GP(4, 1)$, $GP(8, 3)$, $GP(12, 5)$ and $GP(24, 5)$.

If $(n, k) \in \{(5, 2), (10, 2)\}$, then $H \leq A(n, k)$ and $\chi'_{A(5,2)} \geq \chi'_H = d_{5,2} = 5$ (see Proposition 1) and $\chi'_{A(10,2)} \geq \chi'_H = min\{d_{10,2}, e_{10,2} + 2\} = 5$ (see Proposition 3). In Figure 3 an $A(n, k)$-automorphic 5-coloring of $GP(n, k)$ is shown.

If $(n, k) = (10, 3)$, then $\chi'_{A(10,3)} \geq 3$, but the unique H-automorphic 3-coloring of $GP(10, 3)$ (see Proposition 2) is not preserved by the automorphism μ of $A(10, 3)$. Moreover, an H-automorphic 4-coloring of $GP(10, 3)$ does not

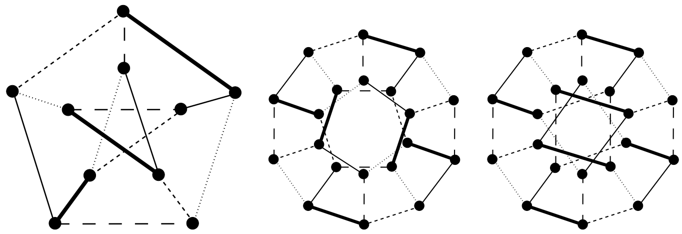

Figure 3: Automorphic 5-colorings of $GP(5,2), GP(10,2)$ and $GP(10,3)$

exist. An $A(10,3)$-automorphic coloring of $GP(10,3)$ uses m colors to colors the inner rim with $m \mid 10$ and $m \nmid 3$ (see Lemma 1). Since $d_{10,3} = 5$, then $\chi'_{A(10,3)} \geq 5$. In Figure 3 an $A(10,3)$-automorphic 5-coloring is shown. $\square$

References

[1] B. Alspach, The classification of Hamiltonian generalized Petersen graphs, *J. Combin. Theory Ser. B*, **34** (1983), 293-312.

[2] M.O. Albertson, K.L. Collins, Symmetry Breaking in Graphs, *Electron. J. of Combin.*, 3 (1996), #R18.

[3] F. Castagna, G. Prins, Every generalized Petersen graph has a Tait coloring, *Pacific. J. Math.*, **40** (1972), 53-58.

[4] K.L.Collins, A.N.Trenk, The distinguishing chromatic number, *Electron. J. of Combin.*, **13** (2006), #R16.

[5] H.S.M. Coxeter, Self-dual configurations and regular graphs, *Bull. Amer. Math. Soc.*, **56** (1950), 413–455.

[6] R. Frucht, J.E. Graver, M.E. Watkins, The groups of the generalized Petersen graphs, *Proc. Cambridge Philos. Soc.*, **70** (1971), 211–218.

[7] C. Fiori, G. Mazzuoccolo, B. Ruini, On the automorphic chromatic index of a graph, submitted.

[8] M.E. Watkins, A Theorem on Tait colorings with an application to the generalized Petersen graphs, *J. Combin. Theory*, **6** (1969), 152–164.

[9] J.Weigand and M.S.Jacobson, Distinguishing and Distinguishing Chromatic Numbers of Generalized Petersen Graphs, *AKCE J.Graphs Combin.*, **5** (2008), 199-211.