ARCHIVUM MATHEMATICUM (BRNO)
Tomus 00 (XXXX), 1–11

TWO-POINTS BOUNDARY VALUE PROBLEMS FOR CARATHÉODORY SECOND ORDER EQUATIONS

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Abstract. Using a suitable version of Mawhin's continuation principle, we obtain an existence result for the Floquet boundary value problem for second order Carathéodory differential equations by means of strictly localized C^2 bounding functions.

1. Introduction

This paper deals with the second order problem

$$\text{(P)} \qquad \begin{cases} x'' = f(t,x,x'), & t \in [0,1] \\ x(1) = Ax(0) \\ x'(1) = Bx'(0) \end{cases}$$

where A and B are $m \times m$ real matrices, with A non-singular, and $f\colon [0,1] \times \mathbb{R}^{2m} \to \mathbb{R}^m$ satisfies the Carathéodory conditions, i.e.

1) $f(t,\cdot,\cdot)$ is continuous for a.e. $t \in [0,1]$;
2) $f(\cdot,x,y)$ is measurable for every $(x,y) \in \mathbb{R}^{2m}$;
3) for every $r > 0$ there exists $g_r \in L^1\big([0,1],\mathbb{R}^{2m}\big)$ such that $\big|f(t,x,y)\big| \le g_r(t)$ for every $|x| \le r$, $|y| \le r$ and a.e. $t \in [0,1]$.

By solution of (P) we mean a classical one, i.e. a function $x\colon [0,1] \to \mathbb{R}^m$ twice differentiable almost everywhere in $[0,1]$ with $x'' \in L^1\big([0,1],\mathbb{R}^m\big)$ and satisfying (P) almost everywhere.

In [14], an existence result for problem (P) is given when the right hand side is continuous (see Theorem 1). It makes use of a suitable version of Mawhin's continuation principle (see [10]) and requires the fulfillment of a transversality condition on the boundary of a suitable open and bounded subset K of $\mathbb{R}^m$. This delicate point is overcome by assuming that K is a bound set defined as the intersection of sub-level sets of certain scalar functions.

2000 *Mathematics Subject Classification:* Primary: 34B15; Secondary: 47H10.

Key words and phrases: continuation principle, coincidence degree, second order differential systems, bound sets, Floquet type boundary conditions.

Received July 2007, revised October 2007. Editor O. Došlý.

The theory of the bound set was introduced by Gaines and Mawhin in [5] for first order as well as for second order equations and by Mawhin in [8] for periodic second order problems.

For a periodic boundary value problem associated to second order differential equations, e.g. when $A = B = I$ in (P), a great deal of existence results was obtained with similar techniques. We remind to [14] for a detailed list of references on this subject. Erbe-Palamides [3] and Erbe-Schmitt [4] applied analogous approaches to the investigation of problem (P) when both A and B are non-singular and satisfy a further assumption.

For natural reasons, when the vector field is of Carathéodory type, in the literature the transversality condition is usually required to be satisfied in a whole neighbourhood of the boundary. In this paper we shall prove that also in this case it is possible to localize the transversality condition on the boundary, extending the results in [14] to the case when the vector field is of Carathéodory type instead that continuous. We will do this following the approach used in Mawhin-Thompson [11] for periodic solutions of first order equations, which makes use of a suitable modification of a Luzin approximation result (see also Scorza Dragoni [13]) given by Thompson [15].

We also assume usual Nagumo growth conditions on the vector field to guarantee the existence of an a priori bound on the first derivative of the possible solutions of problem (P).

As usual $\langle\cdot,\cdot\rangle$ and $|\cdot|$ respectively denote the inner product and the norm of $\mathbb{R}^m$, while $|\cdot|_0$ and $|\cdot|_1$ denotes the norm respectively of $C\big([0,1],\mathbb{R}^m\big)$ and $L^1\big([0,1],\mathbb{R}^m\big)$. Given $\delta > 0$ and $x \in \mathbb{R}^m$, let $B_x^\delta = \big\{y \in \mathbb{R}^m \colon |y - x| \le \delta\big\}$. For $A \subset \mathbb{R}^m$, let $\operatorname{diam} A = \sup_{x\in A} |x|$, χ_A be the characteristic function of A and $\lambda(A)$ the Lebesgue measure of A. Given $V \colon \mathbb{R}^m \to \mathbb{R}^m$ continuous and $A \subset \mathbb{R}^m$, let $V^{-1}(A) = \big\{x \in \mathbb{R}^m \colon V(x) \in A\big\}$.

2. Main results

In [14] (see Theorem 1) the authors proved the following continuation theorem for problem (P), which is a suitable version of Mawhin's continuation principle (cf. [10]).

Theorem 1. *Let $f \colon [0,1] \times \mathbb{R}^{2m} \to \mathbb{R}^m$ be a Carathéodory function and A and B a couple of $m \times m$ real matrices. Suppose that $G \subset \mathbb{R}^m$ is an open, bounded and non-empty set such that*

(BS) *there is no solution $x(\cdot)$ for some $\lambda \in (0,1)$ to*

$$(\mathrm{P}_\lambda) \qquad \begin{cases} x'' = \lambda f(t,x,x')\,, & t \in [0,1] \\ x(1) = Ax(0) \\ x'(1) = Bx'(0) \end{cases}$$

such that $x(t) \in \overline{G}$, for all $t \in [0,1]$ and $x(\tilde t) \in \partial G$ for some $\tilde t \in [0,1]$;

(NC) *there is $K > 0$ such that*

$$|x'|_\infty < K$$

for each solution $x(\cdot)$ to (P_λ) *for some $\lambda \in (0,1)$ such that $x(t) \in \overline{G}$, for all $t \in [0,1]$.*
Assume further

$$\ker(I-B) \cap \operatorname{Im}(I-A) = \{0\}$$

and

$$d\big[(I-P_B)\bar{f}, G \cap \ker(I-A), 0\big] \neq 0\,,$$

where d is the Brouwer degree, P_B is the continuous projections of $\mathbb{R}^m$ onto $\operatorname{Im}(I-B)$ *and*

$$\bar{f}(a) := \int_0^1 f(s,a,0)\,ds\,.$$

Then,

$$\text{(P)} \qquad \begin{cases} x'' = f(t,x,x')\,, & t \in [0,1] \\ x(1) = Ax(0) \\ x'(1) = Bx'(0) \end{cases}$$

has at least one solution x with $x(t) \in \bar{G}$, for all $t \in [0,1]$.

Remark 1. Like it is known, when the set $\ker(I-A)$ is invariant for the map $\bar{f}$ and

$$\ker(I-A) \cap \operatorname{Im}(I-B) = \{0\}\,,$$

then

$$\big|d\big[(I-P_B)\bar{f}, G \cap \ker(I-A), 0\big]\big| = \big|d\big[\bar{f}, G \cap \ker(I-A), 0\big]\big|\,.$$

We now reformulate the transversality condition (BS) and the boundedness condition (NC) in order to translate them in more convenient ways, i.e. more easily verifiable in the applications.

Remark 2. When f is independent of the first derivative, the boundedness condition (NC) is trivially satisfied. In fact, let x be a solution of $x'' = f(t,x)$ such that $x(t) \in \overline{G}$ for all $t \in [0,1]$ and denote $R = \operatorname{diam} G$. By Taylor's formula with rest in integral form it holds, for every $i = 1, \ldots, m$ and $t_0, t \in [0,1]$,

$$x_i(t) = x_i(t_0) + x_i'(t_0)(t-t_0) + \int_{t_0}^t (t-s)x_i''(s)\,ds\,,$$

i.e.

$$|t-t_0|\,\big|x_i'(t_0)\big| \leq \big|x_i(t)\big| + \big|x_i(t_0)\big| + \int_0^1 |t-s|\big|f\big(s,x(s)\big)\big|\,ds \leq 2R + |g_R|_1\,.$$

Since for all $t_0 \in [0,1]$ there exists $t \in [0,1]$ with $|t-t_0| \geq \frac{1}{2}$, we get that

$$\big|x_i'(t_0)\big| \leq 2\big(2R + |g_R|_1\big)\,,$$

i.e. that

$$|x'|_0 \leq 2\sqrt{m}\big(2R + |g_R|_1\big)\,.$$

In the general case, a classical hypothesis, known as Nagumo-Hartman growth condition, is known in literature to guarantee (NC). We recall it in the lemma below, because in the following we need a precise estimation of the constant K in (NC), and we remind to Lemma 5.2 in [6] for the proof. Even if it is given for continuous right hand side, indeed it holds also for Carathéodory ones (see [9], p. 728).

Lemma 1. *If there exist a continuous function* $\varphi\colon [0,+\infty) \to (0,+\infty)$*, with*

$$\int^{+\infty} \frac{u}{\varphi(u)}\, du = \infty\,,$$

and $\alpha, \beta \geq 0$ *such that for each* $(t,x,y) \in [0,1] \times \overline{G} \times \mathbb{R}^m$

$$\big|f(t,x,y)\big| \leq \varphi\big(|y|\big)$$

and

$$\big|f(t,x,y)\big| \leq 2\alpha\big[\langle x, f(t,x,y)\rangle + |y|^2\big] + \beta\,,$$

then for each $\lambda \in (0,1)$ *and each solution of* (P_λ) *such that* $x(t) \in \overline{G}$*, for all* $t \in [0,1]$*,*

$$|x'|_0 < \phi^{-1}\Big[\phi\Big(4R + 4\alpha R^2 + \frac{\beta}{4}\Big) + 2R + 4\alpha R^2 + \frac{\beta}{8}\Big]\,,$$

where $\phi(u) = \int_0^u \frac{s}{\varphi(s)}\, ds$ *and* $R = \operatorname{diam} G$*.*

Remark 3. According to Lemma 5.1 of [6], the second inequality of Lemma 1 is not necessary in the scalar case, i.e. when $m = 1$. In this case

$$|x'|_0 < \phi^{-1}\big[\phi(2R) + 2R\big]\,.$$

In next theorem we give an existence result for (P), reformulating the transversality condition (BS) in terms of the so called *bound set* for a boundary value problem, which is an open, bounded and non-empty subset of $\mathbb{R}^m$ having the property that no solution of the problem completely laying in its closure can touch its boundary. Like usually in the literature, we consider bound sets defined as the intersection of sublevel sets of scalar functions said *bounding functions.* Assumptions (H1)–(H4) of Theorem 2 read as the ones corresponding to (BS).

Before going on, we recall the definition of subset having the boundary invariant with respect to the subgroup generated by a non-singular matrix.

Definition 1. An open and bounded subset $G \subset \mathbb{R}^m$ is said to have the boundary invariant with respect to the subgroup of $GL^N(\mathbb{R})$ generated by a non-singular $m \times m$ real matrix A if

$$Au \in \partial G \Leftrightarrow u \in G\,. \tag{IC}$$

Theorem 2. *Let* $f\colon [0,1] \times \mathbb{R}^{2m} \to \mathbb{R}^m$ *be a Carathéodory function and* A *and* B *a couple of* $m \times m$ *real matrices, with* A *non-singular. Let* $G \subset \mathbb{R}^m$ *be an open, bounded and non-empty set whose boundary is invariant with respect to the subgroup generated by* A*.*

Suppose that there exist a continuous function $\varphi\colon [0,+\infty) \to (0,+\infty)$*, with* $\int^{+\infty} \frac{u}{\varphi(u)}\, du = \infty$*, and* $\alpha, \beta \geq 0$ *such that* $\big|f(t,x,y)\big| \leq \varphi\big(|y|\big)$ *and* $\big|f(t,x,y)\big| \leq 2\alpha\big[\langle x, f(t,x,y)\rangle + |y|^2\big] + \beta$ *in* $[0,1] \times \overline{G} \times \mathbb{R}^m$*.*
Denoted now $K = \phi^{-1}\big[\phi\big(4R+4\alpha R^2+\frac{\beta}{4}\big)+2R+4\alpha R^2+\frac{\beta}{8}\big]$*, where* $\phi(u) = \int_0^u \frac{s}{\varphi(s)}\, ds$ *and* $R = \operatorname{diam} G$*, assume further that for each* $u \in \partial G$ *there exist* $V_u\colon \mathbb{R}^m \to \mathbb{R}$ *of class* C^2*,* $a_u \in \big[0, \frac{\pi^2}{4}\big)$ *and* $k_u > 0$ *such that*

(H1) $V_u/_{\overline{G}} \leq 0$*;*
(H2) $V_u(u) = 0$*;*
(H3) $\forall \lambda \in (0,1)$*,* $\forall t \in [0,1]$*,* $\forall x \in \overline{G}\colon V_u(x) > -k_u$*,* $\forall v \in B_0^K$
$$\langle HV_u(x)v, v\rangle + \lambda\langle \nabla V_u(x), f(t,x,v)\rangle \geq -a_u\big[V_u(x) + k_u\big]\,;$$
(H4) $\forall v \in B_0^K : \langle \nabla V_u(u), v\rangle \leq 0 \leq \langle \nabla V_{Au}(Au), Bv\rangle$
$$\langle \nabla V_u(u), v\rangle = 0 \quad \textit{and} \quad \langle \nabla V_{Au}(Au), Bv\rangle = 0\,.$$

Suppose finally that $\ker(I-B) \cap \operatorname{Im}(I-A) = \{0\}$ *and* $d\big[(I-P_B)\bar{f}, G \cap \ker(I - A), 0\big] \neq 0$*, where* P_B *is the continuous projections of* $\mathbb{R}^m$ *onto* $\operatorname{Im}(I-B)$ *and* $\bar{f}(a) := \int_0^1 f(s,a,0)\, ds$*. Then,* (P) *has at least one solution* x *with* $x(t) \in \bar{G}$*, for all* $t \in [0,1]$*.*

For the proof of Theorem 2 we refer to Theorem 6 in [14], which contains the same result as the previous one for continuous right hand sides. In the quoted theorem assumptions (H3) and (H4) are required to be satisfied for all v in $\mathbb{R}^m$ instead that only in a ball. They are used to avoid that a solution of (P_λ) completely laying in the closure of G reaches its boundary. Like pointed out in [14], at this aim, the vector v is not an arbitrary point of $\mathbb{R}^m$, but, in fact, plays the role of the first derivative of the solution. We point out that, according to Lemma 1, for every $\lambda \in (0,1), K$ is a bound for the first derivative of each solution of (P_λ), completely contained in the closure of G. Hence (H3) and (H4) are sufficient to guarantee that such solution does not touch the boundary of G and it is easy to verify that the same proof given in [14], Theorem 6 holds true also for Carathéodory right hand sides.

We now restrict ourselves to consider candidate bound sets defined as the sublevel sets of one only scalar function, called *guiding function*, which is equivalent to consider all the bounding functions equal among them.

Corollary 1. *Let* $f\colon [0,1] \times \mathbb{R}^{2m} \to \mathbb{R}^m$ *be a Carathéodory function and* A *and* B *a couple of* $m \times m$ *real matrices, with* A *non-singular.*
Suppose that there exist a function $V\colon \mathbb{R}^m \to \mathbb{R}$ *of class* C^2 *and* $k > 0$ *such that*

i) $V^{-1}(-\infty, 0)$ *is non-empty and bounded;*
ii) $V^{-1}(0)$ *is invariant with respect to the subgroup generated by* A*;*
iii) $\forall x \in V^{-1}(0) \nabla V(x) \neq 0$*;*
iv) $\forall x \in V^{-1}(-k, 0]\, HV(x)$ *is positive semidefinite.*

Assume also that there exist a continuous function $\varphi\colon [0,+\infty) \to (0,+\infty)$*, with* $\int^{+\infty} \frac{u}{\varphi(u)}\, du = \infty$*, and* $\alpha, \beta \geq 0$ *such that* $|f(t,x,y)| \leq \varphi\big(|y|\big)$ *and* $|f(t,x,y)| \leq 2\alpha\big[\langle x, f(t,x,y)\rangle + |y|^2\big] + \beta$ *in* $(t,x,y) \in [0,1] \times V^{-1}(-\infty, 0] \times \mathbb{R}^m$*.*

Denoted now $K = \phi^{-1}\big[\phi\big(4R+4\alpha R^2+\frac{\beta}{4}\big)+2R+4\alpha R^2+\frac{\beta}{8}\big]$, *where* $\phi(u) = \int_0^u \frac{s}{\varphi(s)}\,ds$ *and* $R = \operatorname{diam} V^{-1}(-\infty,0)$, *suppose further that*

v) $\forall (t,x,v) \in [0,1] \times V^{-1}(-k,0] \times B_0^K$

$$\langle \nabla V(x), f(t,x,v)\rangle \geq 0\,,$$

vi) $\forall x \in V^{-1}(0)$, $\forall v \in B_0^K : \langle \nabla V(x), v\rangle \leq 0 \leq \langle \nabla V(Ax), Bv\rangle$

$$\langle \nabla V(x), v\rangle = 0 \quad \textit{and} \quad \langle \nabla V(Ax), Bv\rangle = 0\,.$$

Assume finally

(K1) $\ker(I-A)$ *invariant for* $\bar f$,

(K2) $\ker(I-B)\cap \operatorname{Im}(I-A) = \ker(I-A)\cap\operatorname{Im}(I-B) = \{0\}$,

(K3) $d\big[\nabla V, V^{-1}(-\infty,0)\cap \ker(I-A), 0\big] \neq 0$.

Then, (P) has at least one solution x *with* $V\big(x(t)\big) \leq 0$, *for all* $t \in [0,1]$.

Proof. The assumptions on V imply that $G := V^{-1}(-\infty,0)$ is an open, bounded and non-empty subset whose boundary $\partial G = V^{-1}(0)$ is invariant with respect to the subgroup generated by A. For every $u \in V^{-1}(0)$ set $V_u = V$, $a_u = 0$ and $k_u = k$. iv) and v) imply then that

$$\langle HV(x)v, v\rangle + \lambda\langle \nabla V(x), f(t,x,v)\rangle \geq 0$$

for all $\lambda \in (0,1)$ and $(t,x,v) \in [0,1]\times V^{-1}(-k,0]\times B_0^K$.
By (K1) and (K2), according to Remark 1, we obtain that

$$\big|d\big[(I-P_B)\bar f, V^{-1}(-\infty,0)\cap\ker(I-A),0\big]\big| = \big|d\big[\bar f, V^{-1}(-\infty,0)\cap\ker(I-A),0\big]\big|\,.$$

Moreover for each $x \in V^{-1}(0)$

$$\langle \nabla V(x), \bar f(x)\rangle = \int_0^1 \langle \nabla V(x), f(s,x,0)\rangle ds \geq 0\,,$$

because of v).
Applying now Poincaré-Bohl theorem (see [7], Theorem 2.1.5) we get

$$d\big[\bar f, V^{-1}(-\infty,0)\cap\ker(I-A),0\big] = d\big[\nabla V, V^{-1}(-\infty,0)\cap\ker(I-A),0\big]$$

and the thesis follows by (K3) and Theorem 2. $\square$

Now consider again Theorem 2. When the vector field f is continuous, in [14] (see Theorem 5 and Corollary 1) an existence result for (P) was obtained requiring, instead of (H3), the following assumption:

(H3') $\forall \lambda \in (0,1)$, $\forall t \in (0,1)$, $\forall v \in \mathbb{R}^m : \langle \nabla V_u(u), v\rangle = 0$

$$\langle HV_u(u)v, v\rangle + \lambda\langle \nabla V_u(u), f(t,u,v)\rangle > 0\,.$$

Like pointed out in the quoted paper, (H3), when localized at u, is weaker than (H3'). On the other hand, (H3) must be satisfied in a whole neighbourhood of the point, while (H3') is required to be satisfied only at u. When the vector field is of Carathéodory type, for natural reasons usually in the literature the hypothesis is assumed in a neighbourhood of the boundary of the candidate bound set. In the next corollary we shall prove that Corollary 1 holds also when assuming condition v) satisfied only in the boundary of the bound set, instead of in a neighbourhood of it. We will do this making use of a suitable modification of a Luzin approximation result (see also Scorza Dragoni [13]) given by Thompson [15]. On this subject we remind to Mawhin-Thompson [11], where this technique was employed to generalize the existence results for periodic solutions of a first order differential equation in Mawhin-Ward [12]. We also recall Andres-Malaguti-Taddei [2], where the same technique allowed to generalize the results in [1] for solutions of a Floquet problem associated with an inclusion.

Corollary 2. *Let $f\colon [0,1]\times\mathbb{R}^{2m}\to\mathbb{R}^m$ be a Carathéodory function and A and B a couple of $m\times m$ real matrices, with A non-singular.*
Suppose that there exists a function $V\colon\mathbb{R}^m\to\mathbb{R}$ of class C^2 such that $V^{-1}(-\infty,0)$ is bounded and non-empty, $V^{-1}(0)$ is invariant with respect to the subgroup generated by A, $\langle x,\nabla V(x)\rangle>0$ for every $x\in V^{-1}(0)$, $HV(x)$ is positive semidefinite in $V^{-1}(-h,0]$ for some $h>0$.
Assume also that there exist a continuous function $\varphi\colon[0,+\infty)\to(0,+\infty)$, with $\int^{+\infty}\frac{u}{\varphi(u)}\,du=\infty$, and $\alpha,\beta\geq 0$ such that $\big|f(t,x,y)\big|\leq\varphi\big(|y|\big)$ and $\big|f(t,x,y)\big|\leq 2\alpha\big[\langle x,f(t,x,y)\rangle+|y|^2\big]+\beta$ in $(t,x,y)\in[0,1]\times V^{-1}(-\infty,0]\times\mathbb{R}^m$.
Denoted now $K=\phi^{-1}\big[\phi\big(4R+12\alpha R^2+\frac{3}{4}\beta\big)+2R+12\alpha R^2+\frac{3}{8}\beta\big]$, where $\phi(u)=\frac{1}{3}\int_0^u\frac{s}{\varphi(s)}\,ds$ and $R=\operatorname{diam}V^{-1}(-\infty,0)$, suppose further that

i) $\forall(t,x,v)\in[0,1]\times V^{-1}(0)\times B_0^K$

$$\langle\nabla V(x),f(t,x,v)\rangle>0\,;$$

ii) $\forall x\in V^{-1}(0)$, $\forall v\in B_0^K\colon\langle\nabla V(x),v\rangle\leq 0\leq\langle\nabla V(Ax),Bv\rangle$

$$\langle\nabla V(x),v\rangle=0\quad\textit{and}\quad\langle\nabla V(Ax),Bv\rangle=0\,.$$

Assume finally $\ker(I-A)$ *invariant for $\bar f$ and for ∇V,* $\ker(I-B)\cap\operatorname{Im}(I-A)=\ker(I-A)\cap\operatorname{Im}(I-B)=\{0\}$ *and $d\big[\nabla V,V^{-1}(-\infty,0)\cap\ker(I-A),0\big]\neq 0$.*
Then, (P) has at least one solution x with $V\big(x(t)\big)\leq 0$, for all $t\in[0,1]$.

Proof. Since $V\in C^2(\mathbb{R}^m)$ and $V^{-1}(-\infty,0)$ is bounded, then $V^{-1}(0)$ is compact. By the hypothesis on V we then get that there exist $k\in(0,h]$ such that $\langle x,\nabla V(x)\rangle>0$ for every $x\in V^{-1}(-k,k)$. Let now $\mu\in C\big(\mathbb{R}^m,[0,1]\big)$ be such that $\mu\equiv 1$ in $V^{-1}\big(-\frac{k}{2},\frac{k}{2}\big)$ and $\mu\equiv 0$ in $\mathbb{R}^m\backslash V^{-1}(-k,k)$.
Take $\{\epsilon_n\}_n$ monotone decreasing to zero. Since f is a Carathéodory function, then Theorem 2.3 in [11] implies that there exists a monotone decreasing sequence $\{\theta_n\}_n$ of open subsets of $[0,1]$ such that $\lambda(\theta_n)\leq\epsilon_n$ and $f\in C\big(([0,1]\backslash\theta_n)\times\mathbb{R}^{2m}\big)$ for every $n\in\mathbb{N}$. Obviously $\cap_{n=1}^\infty\theta_n$ has null Lebesgue measure and $\lim_{n\to\infty}\chi_{\theta_n}(t)=0$ for every $t\notin\cap_{n=1}^\infty\theta_n$.
Define now for each $n\in\mathbb{N}$ and $(t,x,y)\in[0,1]\times\mathbb{R}^{2m}$,

$$f_n(t,x,y)=f(t,x,y)+\mu(x)g(t,x,y)\chi_{\theta_n}(t)\frac{\nabla V(x)}{|\nabla V(x)|}$$

where

$$g(t,x,y)=2\min\big\{\varphi(|y|),2\alpha[\langle x,f(t,x,y)\rangle+|y|^2]+\beta\big\}\,.$$

Since

$$\big|f_n(t,x,y)-f(t,x,y)\big|=\mu(x)g(t,x,y)\chi_{\theta_n}(t)=0$$

when $x \in \mathbb{R}^m \backslash V^{-1}(-k,k)$ and $\nabla V(x) \neq 0$ when $x \in V^{-1}(-k,k)$, it follows that f_n is well defined. Since f is a Carathéodory function and μ, φ and ∇V are continuous, f_n is a Carathéodory function.

Let us now prove that each problem

$$(\mathrm{P}_n) \qquad \begin{cases} x'' = f_n(t,x,x'), & t \in [0,1] \\ x(1) = Ax(0) \\ x'(1) = Bx'(0), \end{cases}$$

satisfies the assumptions of Theorem 1.

First notice that g is positive in $[0,1] \times V^{-1}(-\infty,0] \times \mathbb{R}^m$. In fact, by hypothesis,

$$g(t,x,y) \geq 2\big|f(t,x,y)\big| \geq 0\,.$$

If $g(t,x,y) = 0$, then $\big|f(t,x,y)\big| = 0$, which implies that

$$g(t,x,y) = 2\min\big\{\varphi\big(|y|\big), 2\alpha|y|^2 + \beta\big\} > 0\,,$$

because α and β are positive constants, and φ is a positive function.

The Nagumo growth condition on f, imply that, for all $(t,x,y) \in [0,1] \times V^{-1}(-\infty,0] \times \mathbb{R}^m$,

$$\big|f_n(t,x,y)\big| \leq \big|f(t,x,y)\big| + g(t,x,y) \leq 3\varphi\big(|y|\big)$$

and

$$\begin{aligned} &6\alpha\big[\langle x, f_n(t,x,y)\rangle + |y|^2\big] + 3\beta \\ =&6\alpha\Big\{\langle x, f(t,x,y)\rangle + \frac{\mu(x)\chi_{\theta_n(t)}g(t,x,y)}{|\nabla V(x)|}\langle x, \nabla V(x)\rangle + |y|^2\Big\} + 3\beta \\ \geq&6\alpha\big[\langle x, f(t,x,y)\rangle + |y|^2\big] + 3\beta \\ \geq&\big|f(t,x,y)\big| + g(t,x,y) \geq \big|f_n(t,x,y)\big|\,, \end{aligned}$$

because $\mu \equiv 0$ in $\mathbb{R}^m \backslash V^{-1}(-k,k)$ and $\langle x, \nabla V(x)\rangle > 0$ for all $x \in V^{-1}(-k,k)$. Hence the conditions of Lemma 1 are satisfied by the positive continuous function 3φ and the positive constants 3α and 3β. According to the quoted lemma, for all x solution of (P_n) with $x(t) \in \overline{G}$ for all t it holds $\|x'\|_0 \leq K$.

Moreover, since

$$\overline{f_n}(a) = \overline{f}(a) + \frac{\mu(a)\int_{\theta_n} g(s,a,0)\,ds}{|\nabla V(a)|}\,\nabla V(a)\,,$$

it follows that $\ker(I-A)$ is invariant for $\overline{f_n}$, because it is invariant both for $\overline{f}$ and ∇V and it is a linear subspace.

To apply the continuation principle it remains to prove condition (BS) of Theorem 1. Suppose by contradiction that there exist $\lambda \in (0,1), x$ solution of

$$\begin{cases} x'' = \lambda f_n(t,x,x'), & t \in [0,1] \\ x(1) = Ax(0) \\ x'(1) = Bx'(0), \end{cases}$$

and $t_0 \in [0,1]$ such that $x(t) \in V^{-1}(-\infty,0]$ for all t and $x(t_0) \in V^{-1}(0)$. According to the invariance of $V^{-1}(0)$ with respect to the subgroup generated by A, $t_0 \in]0,1[$ or both $x(0)$ and $x(1) \in V^{-1}(0)$.
Let us consider the function $v(t) = V\big(x(t)\big)$. Since $V \in C^2(\mathbb{R}^m)$ and f_n is a Carathéodory function, v is of class C^1 and v' is absolutely continuous in $[0,1]$. Trivially, t_0 is a local maximum point for v. If $t_0 \in]0,1[$, then $v'(t_0) = 0$. If $t_0 \in \{0,1\}$, it holds

$$\big\langle \nabla V\big(x(0)\big), x'(0)\big\rangle = v'(0) \le 0$$

and

$$0 \le v'(1) = \big\langle \nabla V\big(x(1)\big), x'(1)\big\rangle = \big\langle \nabla V\big(Ax(0)\big), Bx'(0)\big\rangle\,.$$

Thus ii) implies that $v'(0) = v'(1) = 0$.
Moreover, for a.a. $t \in [0,1]$,

$$\begin{aligned} v''(t) =& \big\langle HV\big(x(t)\big)x'(t), x'(t)\big\rangle + \lambda\big\langle \nabla V\big(x(t)\big), f_n\big(t, x(t), x'(t)\big)\big\rangle \\ =& \big\langle HV\big(x(t)\big)x'(t), x'(t)\big\rangle + \lambda\big[\big\langle \nabla V\big(x(t)\big), f\big(t, x(t), x'(t)\big)\big\rangle \\ &+ \mu\big(x(t)\big) g\big(t, x(t), x'(t)\big)\chi_{\theta_n(t)}\big|\nabla V\big(x(t)\big)\big|\big]\,. \end{aligned}$$

If $t_0 \in [0,1]\backslash\theta_n$, since f is continuous in $\big([0,1]\backslash\theta_n\big) \times \mathbb{R}^{2m}$, v is twice differentiable in t_0. Thus $v''(t_0) \ge 0$, because t_0 is a local maximum point for v and $v'(t_0) = 0$. On the other hand i) implies that

$$\begin{aligned} v''(t_0) =& \big\langle HV\big(x(t_0)\big)x'(t_0), x'(t_0)\big\rangle \\ &+ \lambda\big\langle \nabla V\big(x(t_0)\big), f\big(t_0, x(t_0), x'(t_0)\big)\big\rangle > 0\,, \end{aligned}$$

because HV is positive semi definite in $V^{-1}(-k,0]$, and we get a contradiction.
If $t_0 \in \theta_n$, according to the invariance condition (IC), we do not lose in generality assuming that $t_0 < 1$. Since θ_n is an open set and x is a continuous function, there exists $t_1 > t_0$ such that $[t_0,t_1] \subset \theta_n, x(t) \in V^{-1}\big(-\frac{k}{2}, 0\big]$ for all $t \in [t_0,t_1]$ and $v(t_1)$ is the minimum of v in $[t_0,t_1]$. Then

$$\begin{aligned} 0 \ge& v'(t_1) = \int_{t_0}^{t_1} v''(s)\,ds \\ =& \int_{t_0}^{t_1} \big[\big\langle HV\big(x(s)\big)x'(s), x'(s)\big\rangle \\ &+ \lambda\big\langle \nabla V\big(x(s)\big), f\big(s, x(s), x'(s)\big)\big\rangle + g\big(s, x(s), x'(s)\big)\big|\nabla V\big(x(s)\big)\big|\big]\,ds \\ \ge& \lambda \int_{t_0}^{t_1} \big[-\big|f\big(s, x(s), x'(s)\big)\big| + g\big(s, x(s), x'(s)\big)\big]\big|\nabla V\big(x(s)\big)\big|\,ds \\ \ge& \frac{\lambda}{2} \int_{t_0}^{t_1} g\big(s, x(s), x'(s)\big)\big|\nabla V\big(x(s)\big)\big|\big]\,ds > 0\,, \end{aligned}$$

because in $V^{-1}\big(-\frac{k}{2}, 0\big]$, $\mu \equiv 1$, HV is positive semi definite, ∇V is different from zero and $g(t,x,y) \ge 2\big|f(t,x,y)\big|$ and positive for all $(t,x,y) \in [0,1] \times V^{-1}\big(-\frac{k}{2}, 0\big] \times \mathbb{R}^m$. Therefore we get again a contradiction and also (BS) is proved.
Applying Theorem 1 we obtain that for every $n \in \mathbb{N}$ there exists x_n solution of (P_n) such that $x_n(t) \in V^{-1}(-\infty,0]$ and $\big|x_n'(t)\big| \le K$ for each $t \in [0,1]$. Since

f_n is of Carathéodory type and $V^{-1}(-\infty,0)]$ is bounded, Ascoli-Arzelá theorem implies that $\{x_n\}_n \to x$ uniformly in $C^1\big([0,1]\big)$. Moreover $\{f_n\}_n \to f$ a.e., because $\lim_{n\to\infty}\chi_{\theta_n}(t) = 0$ for every $t \notin \cap_{n=1}^{\infty}\theta_n$ and $\lambda(\cap_{n=1}^{\infty}\theta_n) = 0$. Finally, for every $n \in \mathbb{N}$ and a.e. $t \in [0,1]$, $\big|f_n\big(t, x_n(t), x_n'(t)\big)\big| \le 3\sup_{B_0^K}\varphi$, and we can conclude by Lebesgue's dominated convergence theorem that x is a solution of (P). □

Remark 4. With respect to Corollary 1, in Corollary 2 we assume the stronger conditions that $\langle x, \nabla V(x)\rangle > 0$ for every $x \in V^{-1}(0)$ and that $\ker(I-A)$ is invariant for ∇V. However those assumption are not restrictive. In fact, in literature, V is often defined as $V(x) = |x|^2 - R^2$ for some $R > 0$. In this case $\nabla V = 2I$ and $V^{-1}(0) = \big\{x \in \mathbb{R}^m \colon |x| = R\big\}$. Thus both the conditions are satisfied.

Remark 5. According respectively to Remark 3 and 2, when $m = 1$ or the vector field f is independent from the first derivative, Corollary 2 can be proved without assuming the condition on the positive constants α and β and, in the second case, neither the condition on the positive function φ. It is then sufficient to define $g(t,x,y)$ respectively equal to $2\varphi\big(|y|\big)$ and $2g_R$, K respectively equal to $\phi^{-1}\big[\phi(2R) + 2R\big]$ and $2\sqrt{m}\big(2R + 3|g_R|_1\big)$ and reason as above.

Acknowledgement. I wish to thank Prof. F. Zanolin for fruitful discussions and helpful suggestions.

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