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GEOMETRIC APPROACH TO THE PROOF OF FERMAT'S LAST THEOREM

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Abstract

A geometric approach to the proof of Fermat's last theorem is proposed. Instead of integers *a*, *b*, *c*, Fermat's last theorem considers a triangle with side lengths *a*, *b*, *c*. It is proved that in the case of right-angled and obtuse-angled triangles Fermat's equation has no solutions. When considering the case when *a*, *b*, *c* are sides of an acute triangle, it is proved that Fermat's equation has no entire solutions for $p>2$. The numbers $a=k$, $b=k+m$, $c=k+n$, where *k*, *m*, *n* are natural numbers satisfying the inequalities $n>m$, $n<k+m$, exhaust all possible variants of natural numbers *a*, *b*, *c* which are the sides of the triangle.

The proof in this case is carried out by introducing a new auxiliary function $f(k,p)=k^p+(k+m)^p-(k+n)^p$ of two variables, which is a polynomial of degree *p* in the variable *k*. The study of this function necessary for the proof of the theorem is carried out. A special case of Fermat's last theorem is proved, when the variables *a*, *b*, *c* take consecutive integer values. The proof of Fermat's last theorem was carried out in two stages. Namely, all possible values of natural numbers *k*, *m*, *n*, *p* were considered, satisfying the following conditions: firstly, the number (n^p-m^p) is odd, and secondly, this number is even, where the number (n^p-m^p) is a free member of the function $f(k, p)$.

Another proof of Fermat's last theorem is proposed, in which all possible relationships between the supposed integer solution $\tilde{k}$ of the equation $f(k, p)=0$ and the number $(\tilde{n}-\tilde{m})$ corresponding to this supposed solution $\tilde{k}$ are considered.

The proof is carried out using the mathematical apparatus of the theory of integers, elements of higher algebra and the foundations of mathematical analysis. These studies are a continuation of the author's works, in which some special cases of Fermat's last theorem are proved.

Keywords: Fermat's last theorem, geometric approach, number theory, Newton's binomial, Descartes' theorem.

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1. Introduction

In 1637, the French mathematician Pierre Fermat made the following conjecture: for any natural number $p>2$, the equation:

$$a^p+b^p=c^p$$

has no solutions in nonzero integers *a*, *b*, *c*. The results of attempts to prove Fermat's last theorem for more than 300 years are known, a review of which is given, for example, in [1–3]. Pierre Fermat himself proved the absence of integer solutions for the case $p=4$ [4, 5]. As early as 1770, the $p=3$ case was proved by Euler, and enough time had to pass for the theory used by Euler to be convincingly substantiated by Gauss. In 1825, the proof of Fermat's last theorem for the case $p=5$ was presented almost simultaneously by Lejeune Dirichlet and Legendre [1]. For the case $p=7$, the theorem was proved by Lame in 1839. Later, in 1847, Lame announced that he had succeeded in finding a proof of Fermat's last theorem for all prime exponents $p\geq 3$. But Liouville [1] almost immediately found an error in Lame's proof, which Lame later admitted.

Using the ideas of Kummer, with the help of already modern computing tools, the validity of Fermat's last theorem was proved for all prime exponents $p<100000$.

In 1993, Andrew Wiles published the first 130-page proof of Fermat's last theorem. Soon a serious gap was discovered in it, which, with the help of Richard Lawrence Taylor, was quickly eliminated. In 1995, the final version of the proof of Fermat's last theorem [6] was published.

It is known that Fermat's last theorem was proved by Andrew Wiles, professor of mathematics at Princeton University [7]. At the same time, the author used modern mathematical tools. His proof took up an entire issue of the Annals of Mathematics.

Mathematicians appreciated the proof [8]. Thus, the American mathematician Serge Leng included in the third edition of his classic manual on algebra the main constructions of the proof of Fermat's last theorem by Wiles.

Attempts to contribute to the proof of Fermat's last theorem continue. Thus, in [9], the author proves that «there are no nonzero integer solutions of the Fermat equation for exponents greater than the one for which the absence of integer solutions has been proved».

The idea of finding a simpler proof of Fermat's last theorem also remains attractive. The author proposes a geometric approach to the proof of Fermat's last theorem, assuming that the variables a, b, c in Fermat's equation are the lengths of the sides of a certain triangle.

2. Materials and methods

The geometric approach, as well as the introduction of an auxiliary function $u=f(k, p)$ of two variables, made it possible to simultaneously apply elements of number theory, higher algebra and mathematical analysis in the proof [10, 11].

3. Results and discussion

The work is a continuation of studies [12, 13].

Fermat's last theorem. For any natural number $p>2$, the equation

$$a^p + b^p = c^p \tag{1}$$

has no solutions in non-zero integers a, b, c.

Let's prove the theorem in an equivalent formulation, stating that Eq. (1) has no natural solutions.

Obviously, $a<c$, $b<c$, $c<a+b$. Let's apply a geometric approach, namely: instead of a triple of numbers a, b, c, consider a triangle with side lengths a, b, c.

Three options are possible: a triangle is right-angled, obtuse-angled or acute-angled.

In the first case:

$$a^2 + b^2 = c^2. \tag{2}$$

In the second case, it follows from the cosine theorem that:

$$a^2 + b^2 < c^2. \tag{3}$$

Combining (2) and (3), let's obtain:

$$a^2 + b^2 \le c^2. \tag{4}$$

Multiplying inequality (4) by c^{p-2}, let's obtain:

$$a^2 \cdot c^{p-2} + b^2 \cdot c^{p-2} \le c^p.$$

Where

$$a^p + b^p < c^p,$$

since $a^2c^{p-2} > a^p$, $b^2c^{p-2} > b^p$. That is, in the first two cases, equation (1) has no solutions for $p>2$.

Let's consider the third case, namely: the triangle is acute. Without loss of generality, let's assume that $a<b$.

For $a=b$, equation (1) takes the form:

$$a^p + a^p = c^p.$$

Where

$$c = a\sqrt[p]{2}.$$

It is known from number theory [10] that c is an irrational number for a and p integers.

Numbers:

$$a = k,\ b = k+m,\ c = k+n,$$

where k, m, n – natural numbers satisfying the inequalities:

$$n > m,\ n < k+m,$$

exhaust all possible variants of natural numbers a, b, c that are sides of the triangle.

Let's formulate the theorem in an equivalent form and with the indicated notation.

Fermat's last theorem. For any natural number $p > 2$, the equation:

$$k^p + (k+m)^p = (k+n)^p$$

has no solutions in natural numbers k, m, n.

In an acute triangle, the following condition is additionally satisfied:

$$k > n - m + \sqrt{2n(n-m)}. \tag{5}$$

Let's prove inequality (5). As is known from the cosine theorem,

$$a^2 + b^2 > c^2,\ k^2 + (k+m)^2 > (k+n)^2,$$

$$k^2 - 2k(n-m) + m^2 - n^2 > 0,$$

where

$$k > n - m + \sqrt{2n(n-m)}.$$

From inequality (5) it follows that $k > 3$.

Let's consider a function of two variables:

$$f(k,p) = k^p + (k+m)^p - (k+n)^p, \tag{6}$$

where $p \geq 2$, $k > 3$.

Assuming the number p to be an integer, let's transform equality (6) according to the Newton binomial formula:

$$f(k,p) = k^p - C_p^1(n-m)k^{p-1} - C_p^2(n^2-m^2)k^{p-2} - \ldots - C_p^1(n^{p-1}-m^{p-1})k - (n^p - m^p). \tag{7}$$

Thus, $f(k, p)$ is a polynomial of degree p of the argument k.

According to Descartes' theorem [11] from the course of higher algebra, the equation:

$$f(k,p) = 0 \tag{8}$$

has a unique positive root for any $p \geq 2$, i.e. the number of positive roots of equation (8) does not depend on p.

Let's prove some assertions.

Proposition 1. For any integer $p \geq 3$ and

$$3 < k \leq \frac{p(n-m)+\sqrt{p^2(n-m)^2+2p(p-1)(n^2-m^2)}}{2}$$

function $f(k,p) < 0$.

Proof. Let's transform equality (7):

$$f(k,p) = k^p - C_p^1(n-m)k^{p-1} - C_p^2(n^2-m^2)k^{p-2} -$$
$$-\left[C_p^3(n^3-m^3)k^{p-3} + \ldots + C_p^1(n^{p-1}-m^{p-1})k + (n^p-m^p)\right].$$

Obviously, at:

$$k^p - C_p^1(n-m)k^{p-1} - C_p^2(n^2-m^2)k^{p-2} \leq 0 \quad (9)$$

function $f(k,p) < 0$.

Inequality (9) is satisfied under the condition:

$$\frac{p(n-m)-\sqrt{p^2(n-m)^2+2p(p-1)(n^2-m^2)}}{2} \leq k \leq$$
$$\leq \frac{p(n-m)+\sqrt{p^2(n-m)^2+2p(p-1)(n^2-m^2)}}{2}.$$

Considering that $k > 3$, let's obtain the proof of Proposition 1.

Consequence. Equation (8) has a single positive root k_0 satisfying the inequality:

$$k_0 > \frac{p(n-m)+\sqrt{p^2(n-m)^2+2p(p-1)(n^2-m^2)}}{2}. \quad (10)$$

From inequality (10) it follows that:

$$k_0 > p(n-m).$$

When proving Propositions 2 and 3, the main theorems of mathematical analysis are used.

Proposition 2. Let $f(k,p) < 0 \ \forall p > p_0$ ($k > 3$).

Then $f(k,p)$ decreases monotonically on the interval (p_0,∞) with respect to p.

Proof.

$$f'_p(k,p) = k^p \ln k + (k+m)^p \ln(k+m) - (k+n)^p \ln(k+n) < k^p \ln(k+n) +$$
$$+(k+m)^p \ln(k+n) - (k+n)^p \ln(k+n) = f(k,p)\ln(k+n) < 0.$$

Therefore, $f(k,p)$ is a monotonically decreasing function on the interval (p_0,∞).

Proposition 3. Let $f(k,p) > 0 \ \forall k > k_0 \ (p \geq 2)$. Then the function $f(k,p)$ is monotonically increasing in the variable k on the interval (k_0,∞).

Proof. $f'_k(k,p) = pf(k,p-1)$.

By assumption, $f(k,p-1) > 0$. Therefore, $f'_k(k,p) > 0$. That is, $f(k,p)$ is monotonically increasing on the interval (k_0,∞).

Consequence. The graph of the function $u = f(k,p)$ is concave for any integer $p \geq 3$ on the interval (k_0,∞).

Really,

$$f''_{kk}(k,p) = p(p-1)f(k,p-2) > 0.$$

The graph of the function $u=f(k,p)$ is shown in **Fig. 1**, where k_0 – the minimum point of the function $f(k,p+1)$.

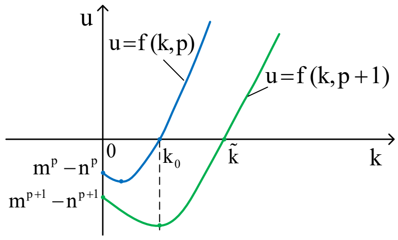

Fig. 1. Mutual arrangement of graphs of functions $u=f(k,p)$ and $u=f(k,p+1)$

Proposition 4. The following recursive formula is valid:

$$f(k,p+1)=kf(k,p)-[n(k+n)^p-m(k+m)^p], \tag{11}$$

for any integer $p\geq 2$ and integer $k>3$.

Proof.

$$f(k,p+1)=k^{p+1}+(k+m)^{p+1}-(k+n)^{p+1}=k\cdot k^p+(k+m)(k+m)^p-$$
$$-(k+n)(k+n)^p=kf(k,p)-\left[n(k+n)^p-m(k+m)^p\right].$$

Obviously, $f(k,p+1)<0$ for $f(k,p)\leq 0$.

Corollary 1. If the triangle is right-angled or obtuse-angled, equation (1) has no solutions in natural numbers a, b, c and positive integer $p>2$.

Indeed, in the first case $f(k,2)=0$, and in the second case $f(k,2)<0$. In both cases $f(k,3)<0$.

Corollary 2. If $f(k,p)>0$, then $f(k,p-1)>0$.

The statement directly follows from equality (11).

Proposition 5. Equation (8) has no positive integer roots for any integer $p>2$ for the case:

$$a=k-1,\ b=k,\ c=k+1.$$

Proof. Equation (8) will take the form:

$$k^p-\left[(k+1)^p-(k-1)^p\right]=0. \tag{12}$$

For odd p:

$$k^p-2\left[C_p^1k^{p-1}+C_p^3k^{p-3}+...+1\right]=0. \tag{13}$$

For even p:

$$k^{p-1}-2\left[C_p^1k^{p-2}+C_p^3k^{p-4}+...+C_p^1\right]=0. \tag{14}$$

In both cases, for $k\leq 2p$, the function $f(k,p)<0$. Therefore, equation (12) has a root $k_0\geq 2p+1$.

In equation (13) the intercept is (–2), and in equation (14) it is ($-2p$). As is known, equations (13) and (14) can have integer roots that are divisors of free terms. But $k_0\geq 2p+1$, therefore, equation (12) has no integer positive roots for integer p.

Proposition 6. Equation (8) has no positive integer roots for $n-m=2s+1$, where s is a positive integer, and for any integer $p>2$.

Proof. Thus, the numbers a, b, c take the values:

$$a=k,\ b=k+m,\ c=k+m+2s+1.$$

It is easy to show that the free term of equation (8), equal to:

$$m^p-(m+2s+1)^p,$$

is an odd number.

Indeed, for any positive integer m, one of the numbers m or $(m+2s+1)$ is odd, therefore, one of the numbers m^p or $(m+2s+1)^p$ is an odd number.

Whence it follows that:

$$m^p-(m+2s+1)^p,$$

is an odd number.

Thus, equation (8) in the considered case does not have even roots.

Let's show that equation (8) has no odd positive roots for $n-m=2s+1$ and any integer $p>2$.

Assume the opposite, that is, equation (8) has an odd positive root $\tilde{k}$ (with corresponding values $\tilde{m}$ and $\tilde{n}$) for some $\tilde{p}>2$.

Obviously,

$$2\delta<\tilde{k}<2\delta+2,$$

where δ – positive integer.

Moreover, it is necessary to consider only the case $f(2\delta,\tilde{p})<0$, $f(2\delta+2,\tilde{p})>0$.

Let's do **Fig. 2**, on which draw a curve $u=f(k,\tilde{p})$ on a segment $[2\delta, 2\delta+2]$.

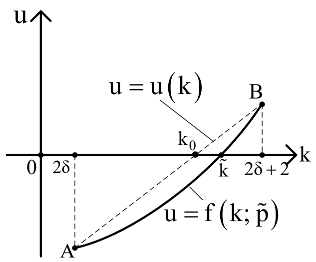

Fig. 2. Mutual arrangement of numbers k_0 and $\tilde{k}$

Let's make the equation of a straight line:

$$A[2\delta; f(2\delta,\tilde{p})],\ B[2\delta+2; f(2\delta+2,\tilde{p})];$$

$$u(k)-f(2\delta,\tilde{p})=\frac{f(2\delta+2,\tilde{p})-f(2\delta,\tilde{p})}{2}(k-2\delta). \tag{15}$$

Where

$$k_0=2\delta-\frac{2f(2\delta,\tilde{p})}{f(2\delta+2,\tilde{p})-f(2\delta,\tilde{p})}. \tag{16}$$

From equality (16) it follows:

1) If $|f(2\delta,\tilde{p})|>f(2\delta+2,\tilde{p})$, then $k_0>2\delta+1$.

2) If $|f(2\delta,\tilde{p})|=f(2\delta+2,\tilde{p})$, then $k_0=2\delta+1$.

3) If $|f(2\delta,\tilde{p})|<f(2\delta+2,\tilde{p})$, then $k_0<2\delta+1$.

In the first and second cases, the number $\tilde{k}$ is not an integer, since:

$$2\delta+1<\tilde{k}<2\delta+2.$$

Let's consider the third case, namely:

$$-\frac{f(2\delta,\tilde{p})}{f(2\delta+2,\tilde{p})}<1.$$

It is easy to verify that the numerator and denominator are odd numbers (if there is a common divisor, then it must be reduced by it).

In this way,

$$-\frac{f(2\delta,\tilde{p})}{f(2\delta+2,\tilde{p})}=\frac{r}{q}<1, \qquad (17)$$

where r/q is an irreducible fraction, and $r<q$. Let's recall that, by assumption:

$$f(2\delta+1,\tilde{p})=0.$$

To illustrate case 3, let's perform **Fig. 3**, where $E(k_0;\,0)$, $M(2\delta+1;0)$, $C(2\delta+2;0)$, $D(2\delta;0)$.

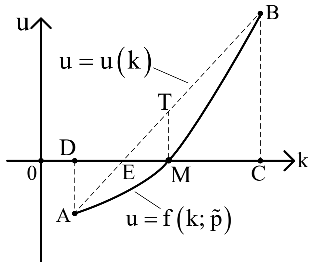

Fig. 3. Graphical interpretation of case 3

From equality (15) it follows:

$$u(2\delta+1)=\frac{f(2\delta,\tilde{p})+f(2\delta+2,\tilde{p})}{2}.$$

Triangles ETM and EBC are similar, so:

$$\frac{BC}{MT}=\frac{2\delta+2-k_0}{2\delta+1-k_0}.$$

Where

$$MT=f(2\delta+2,\tilde{p})\left(1-\frac{1}{2\delta+2-k_0}\right),$$

so

$$1-\frac{1}{2\delta+2-k_0}=\frac{f(2\delta,\tilde{p})+f(2\delta+2,\tilde{p})}{2f(2\delta+2,\tilde{p})}.$$

After performing the transformations, let's obtain:

$$k_0=2\delta+\frac{2r}{r+q}.$$

Wherein

$$2\delta+1-k_0=\frac{q-r}{q+r}>0. \qquad (18)$$

Therefore, for any r, q satisfying condition (17), $f(k_0,\tilde{p})<0$.

That's why:

$$\lim_{r\to q} f\left(k_0,\,\tilde{p}\right)=f\left(2\delta+1,\,\tilde{p}\right)\leq 0.$$

On the other hand, let's arrive at case 2, in which $f\left(2\delta+1,\,\tilde{p}\right)\neq 0$.

Therefore, equation (8) has no odd positive roots for $n-m=2s+1$ and any integer $p>2$.

Proposition 7. Equation (8) has no positive integer roots for $n-m=2s$, where s is a positive integer, and any integer $p>2$.

Proof.

Equation (8) in expanded form has the form:

$$k^p=C_p^1\left(n-m\right)k^{p-1}+C_p^2\left(n^2-m^2\right)k^{p-2}+\ldots+C_p^1\left(n^{p-1}-m^{p-1}\right)k+n^p-m^p.$$

Equation (8) has no odd roots, since for any odd k, the left side of the equality is an odd number, and the right side is an even number.

By analogy with proposition (6), it can be shown that equation (8) has no even positive roots.

Combining propositions (6), (7) let's arrive at the following statement:

Fermat's last theorem. For any natural number $p>2$, the equation $f(k,p)=0$ has no solutions in the natural numbers k, m, n.

Let's offer one more proof of Fermat's last theorem.

Proposition 8. Equation (8) has no positive integer roots for any integer $p>2$.

Proof.

Let's assume the opposite, that is, there are integers $\tilde{p},\tilde{k}$ (with corresponding values $\tilde{m}$ and $\tilde{n}$) such that $f\left(\tilde{k},\,\tilde{p}\right)=0$.

The following options are possible:

1. Numbers $\tilde{k}$ and $\left(\tilde{n}-\tilde{m}\right)$ do not have common divisors provided $\left(\tilde{n}-\tilde{m}\right)\neq 1$.
2. Numbers $\tilde{k}$ and $\left(\tilde{n}-\tilde{m}\right)$ have GCD equal to μ, and $\mu<\tilde{n}-\tilde{m}$.
3. GCD of $\tilde{k}$ and $\left(\tilde{n}-\tilde{m}\right)$ numbers and equal to $\left(\tilde{n}-\tilde{m}\right)$.

Let's write equation (8) in expanded form:

$$k^p=C_p^1\left(n-m\right)k^{p-1}+C_p^2\left(n^2-m^2\right)k^{p-2}+\ldots+C_p^1\left(n^{p-1}-m^{p-1}\right)k+n^p-m^p. \tag{19}$$

Let's consider the first option. Substituting the solution $\tilde{k}$ into equation (19), let's obtain the identity:

$$\tilde{k}^{\tilde{p}}=C_{\tilde{p}}^1\left(\tilde{n}-\tilde{m}\right)\tilde{k}^{\tilde{p}-1}+C_{\tilde{p}}^2\left(\tilde{n}^2-\tilde{m}^2\right)\tilde{k}^{\tilde{p}-2}+\ldots+C_{\tilde{p}}^1\left(\tilde{n}^{\tilde{p}-1}-\tilde{m}^{\tilde{p}-1}\right)\tilde{k}+\tilde{n}^{\tilde{p}}-\tilde{m}^{\tilde{p}}. \tag{20}$$

By virtue of the assumption, the number $\tilde{k}^{\tilde{p}}$ is not divisible by $\left(\tilde{n}-\tilde{m}\right)$. The right side of equality (20) is divisible by the number $\left(\tilde{n}-\tilde{m}\right)$. It is a contradiction.

Let's consider the second option. Number μ is the GCD of numbers $\tilde{k}$ and $\left(\tilde{n}-\tilde{m}\right)$ therefore:

$$\tilde{k}=\mu t_0,\;\; \tilde{n}-\tilde{m}=\mu s_0,$$

moreover, the numbers t_0 and s_0 do not have common divisors.

Substituting into equation (19), let's obtain:

$$\begin{gathered}\left(\mu t_0\right)^{\tilde{p}}=C_{\tilde{p}}^1\left(\mu s_0\right)\left(\mu t_0\right)^{\tilde{p}-1}+C_{\tilde{p}}^2\left(\mu s_0\right)\left(\tilde{n}+\tilde{m}\right)\left(\mu t_0\right)^{\tilde{p}-2}+\ldots\\ +C_{\tilde{p}}^1\left(\mu s_0\right)\left(\tilde{n}^{\tilde{p}-2}+m\tilde{n}^{\tilde{p}-3}+\ldots+\tilde{n}\tilde{m}^{\tilde{p}-3}+\tilde{m}^{\tilde{p}-2}\right)\mu t_0+\\ +\left(\mu s_0\right)\left(\tilde{n}^{\tilde{p}-1}+\tilde{m}\tilde{n}^{\tilde{p}-2}+\ldots+\tilde{m}^{\tilde{p}-2}\tilde{n}+\tilde{m}^{\tilde{p}-1}\right).\end{gathered} \tag{21}$$

The number on the right side of equality (21) has a divisor s_0, but the number on the left side of equality (21) has no such divisor. They came to a contradiction.

In the third case $\tilde{k} = \lambda(\tilde{n} - \tilde{m})$, where $\lambda > \tilde{p}$.

Let's substitute into equation (19):

$$\left[\lambda(\tilde{n}-\tilde{m})\right]^{\tilde{p}} = C_{\tilde{p}}^{1}(\tilde{n}-\tilde{m})\left[\lambda(\tilde{n}-\tilde{m})\right]^{\tilde{p}-1} + C_{\tilde{p}}^{2}\left(\tilde{n}^{2}-\tilde{m}^{2}\right)\left[\lambda(\tilde{n}-\tilde{m})\right]^{\tilde{p}-2} + \ldots + C_{\tilde{p}}^{1}\left(\tilde{n}^{\tilde{p}-1}-\tilde{m}^{\tilde{p}-1}\right)\left[\lambda(\tilde{n}-\tilde{m})\right] + \tilde{n}^{\tilde{p}} - \tilde{m}^{\tilde{p}}. \quad (22)$$

The left and right sides of equality (22) are integers. The left side is the product of two factors. The right side has a divisor $(\tilde{n} - \tilde{m})$.

Dividing equality (22) by $(\tilde{n} - \tilde{m})$ let's obtain:

$$\lambda\left[\lambda(\tilde{n}-\tilde{m})\right]^{\tilde{p}-1} = C_{\tilde{p}}^{1}(\tilde{n}-\tilde{m})\lambda\left[\lambda(\tilde{n}-\tilde{m})\right]^{\tilde{p}-2} + C_{\tilde{p}}^{2}\left(\tilde{n}^{2}-\tilde{m}^{2}\right)\lambda\left[\lambda(\tilde{n}-\tilde{m})\right]^{\tilde{p}-3} + \ldots + C_{\tilde{p}}^{1}\left(\tilde{n}^{\tilde{p}-1}-\tilde{m}^{\tilde{p}-1}\right)\lambda + \left(\tilde{n}^{\tilde{p}-1} + \tilde{m}\tilde{n}^{\tilde{p}-2} + \ldots + \tilde{n}\tilde{m}^{\tilde{p}-2} + \tilde{m}^{\tilde{p}-1}\right). \quad (23)$$

The last term in equality (23) does not explicitly contain divisors $(\tilde{n} - \tilde{m})$ and λ. All other terms on the right side and the left side of the equality are divided into numbers $(\tilde{n} - \tilde{m})$ and λ. In this case, too, we have arrived at a contradiction.

Whence follows the assertion of Fermat's last theorem.

Indeed, for natural numbers p, m, n, the number k is not a natural number.

Let's give one more (incomplete) proof of the third option, when $\tilde{k} = \lambda(\tilde{n} - \tilde{m})$.

Let $(\tilde{n} - \tilde{m}) = 2s$, and $\tilde{n}, \tilde{m}$ – odd numbers. Substituting into equation (19), let's obtain:

$$[2s\lambda]^{\tilde{p}} = \tilde{p}(2s)[2s\lambda]^{\tilde{p}-1} + \frac{\tilde{p}(\tilde{p}-1)}{2}(2s)(2\tilde{m}+2s)[2s\lambda]^{\tilde{p}-2} + \ldots + \tilde{p}(2s) \times \left[(\tilde{p}-1)\tilde{m}^{\tilde{p}-2}(2s) + \frac{(\tilde{p}-1)(\tilde{p}-2)}{2}(2s)^{2}\tilde{m}^{\tilde{p}-3} + \ldots + (\tilde{p}-1)\tilde{m}(2s)^{\tilde{p}-2} + (2s)^{\tilde{p}-1}\right] \times [2s\lambda] + \left[\tilde{p}\tilde{m}^{\tilde{p}-1}(2s) + \frac{\tilde{p}(\tilde{p}-1)}{2}\tilde{m}^{\tilde{p}-2}(2s)^{2} + \ldots + \tilde{p}\tilde{m}(2s)^{\tilde{p}-1} + (2s)^{\tilde{p}}\right]. \quad (24)$$

Let's divide equality (24) by $2s$. In this case, the number on the left side of the resulting equality and all terms on the right side (except for the term $\tilde{p}\tilde{m}^{\tilde{p}-1}$) are even. Moreover, the term $\tilde{p}\tilde{m}^{\tilde{p}-1}$ is odd if the number $\tilde{p}$ is odd and even if the number $\tilde{p}$ is even. In the first case, let's came to a contradiction, since there is an even number on the left side of the resulting equality, and an odd number on the right.

Let $\tilde{p}$ be an even number. The number $\tilde{p}$ can be represented as:

$$\tilde{p} = 2^{i} \cdot j,$$

where i, j – integers $(i < \tilde{p} - 1,\ j \geq 1)$, j – odd number.

Dividing the resulting equality by 2^{i} let's obtain an odd number on the right side, and an even number on the left side. They came to a contradiction.

Let $\tilde{n}$ and $\tilde{m}$ even numbers having GCD γ i.e.

$$\tilde{n} = n_{1}\gamma,\ \tilde{m} = m_{1}\gamma.$$

In this case, one of the numbers m_1, n_1 is odd, or both are odd.

In the first case $(n_1 - m_1)$ is an odd number, in the second case $(n_1 - m_1)$ is even.

It can be shown that equation (8) for any integer k has a unique root p_0, which is not a positive integer.

By virtue of the previous theorem $f(k, p) \neq 0$, for any integers $p > 2$ and $k > 3$. As is known, in an acute triangle $f(k,2) > 0$ for any k satisfying inequality (5). The number $f(k,3)$ can be negative or positive. Let $f(k,3) < 0$. Due to the continuity of the function $f(k,p)$ with respect to the variable p, there exists a unique number p_0 $(2 < p_0 < 3)$ such that $f(k,p_0) = 0$. If $f(k,3) > 0$, then let's continue the indicated process, considering $f(k,4)$ and so on.

4. Conclusions

Intersecting statements are deliberately given. Perhaps they will help to find a more concise proof of Fermat's last theorem.

This is one of many attempts to prove Fermat's Last Theorem.

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