Chapter

Dissipative Quantum System and Energy Balance

Jishad Kumar

Abstract

We discuss how various parts of a quantum many-body system exchange energies at thermal equilibrium. To show this, we assume a quantum system is coupled to a many-body environment (at thermal equilibrium with a bigger environment) consisting of a large number of independent and non-interacting quantum harmonic oscillators above a stable ground state. Once the coupling to a large environment is switched on, the system dissipates its energy continuously to the environment until it reaches equilibrium with the latter. We use the Quantum Langevin equation to show such energy exchange at equilibrium. We conclude that different parts of a physical system can exchange energies even at absolute zero temperature.

Keywords: open systems, quantum dissipation, fluctuations, instantaneous power, the charged oscillator in a magnetic field

1. Introduction

Isolating a quantum system from its environment is not possible since the coupling energy plays a pivotal role in the low-temperature properties of the system. Moreover, a complete understanding or control of the huge environment is also not feasible. How one will study the properties of the quantum system? A working method is to partition the whole system into two different parts, viz., the *system* and the *environment or heat bath*, and eliminate the bath degrees of freedom after carefully considering the effects of the heat bath on the system parameters. This makes the system essentially an *open* one and the study of open systems has been very crucial in many applications of quantum mechanics [1]. There are consequences due to the establishment of a coupling between the system and the heat bath. Firstly, there may be an irreversible transfer of energy from the system to the environment (*dissipation*), next there is *Brownian motion* - fluctuations in the system's degrees of freedom due to the noisy force exerted on the system by the environment. There is *decoherence*, a purely quantum mechanical phenomenon where the system-bath coupling destroys the coherent superposition of states. The first two processes have classical counterparts which have been extensively studied by many authors in the literature. A more detailed understanding shows that the fluctuating force from the environment induces decoherence and damping in the system properties.

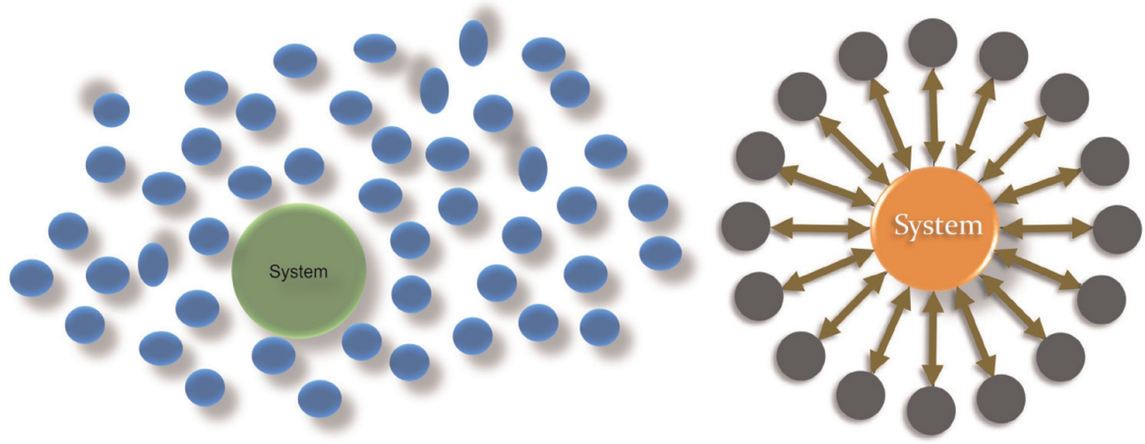

Figure 1.
(a) Schematic representation of Brownian motion. The jittering motion of a large mass immersed in the medium containing a large number of particles/molecules is the phenomenon of Brownian motion. The random kicks the mass received from its environmental particles make the motion stochastic. (b) The famous system-plus-bath arrangement. The test system with one or few degrees of freedom is assumed to be in contact with its environment containing a large number of independent and identical harmonic oscillators. The individual masses of the oscillators in the environment $m_1, m_2, m_3 \ldots m_N$ are assumed to be smaller than the mass M of the test system.

The well-known model of the system-plus-bath approach to open quantum systems stems from classical physics. The underlying phenomenon is the classical Brownian motion where a test system undergoes random motion due to the "kicks" received by the former from the surrounding particles when immersed in a suitable medium (**Figure 1**). In the classical setting, dissipation is introduced in the system "by hand" by inserting a time-dependent damping term into the equation of motion. However, this naive approach never works in the quantum domain where everything is governed by certain principles like Heisenberg's uncertainty relation. Moreover, the physical quantities are operators in quantum mechanics and these operators must satisfy certain commutation relations. The damping term inserted in the equation of motion violates the uncertainty principle. The role of fluctuating/random forces is crucial in order to preserve the canonical structure. The knowledge of the details of the processes including dissipation in the system may not be explicitly known so sometimes the dissipation mechanism is globally described by friction, resistance, viscosity, etc. These parameters are introduced in order to compensate for the information loss due to dissipation. In this microscopic system-bath model, *friction* comes about by the irreversible transfer of energy from the system to its environment. The environment is modeled such that no energy, which is transferred, may come back to the system within any physically relevant time periods. That means the so-called Poincaré recurrence time is infinity. A really necessary condition for the full Hamiltonian is that, under certain conditions, the known classical results must be recovered. Quantum dynamics at arbitrarily low temperatures and (or) with strong damping can be studied within the system-plus-bath approach, regardless of whether the bath is ohmic, sub-ohmic, or super-ohmic. The key thermodynamical quantity of a quantum dissipative system is the reduced density operator $\rho(t) = tr_B \rho_T(t)$, i.e., the partial trace of the total system plus bath density operator ρ_T over the bath degrees of freedom. Here t denotes time. Quantum dissipation theory describes not only the evolution of $\rho(t)$ but also the equilibrium behavior of the reduced system as $\rho(t \to \infty) = \rho_{eq}(T) \propto tr_B e^{-\mathcal{H}_T/k_B T}$, where T is the temperature, k_B is the Boltzmann constant and $\mathcal{H}_T$ is the total Hamiltonian. The latter property is also referred as the detailed balance

relation of the quantum dissipation theory [2]. The description of the dissipative system is recovered by the reduced density matrix obtained by eliminating the bath degrees of freedom which imparts damping and fluctuations.

The system plus reservoir (bath) approach to open quantum systems, was originally introduced by many authors [3–9] and popularized later in the literature by many others [10–12]. The idea here is to couple a system with a finite degree of freedom (system under study) with a reservoir consisting of an infinite number of independent and non-interacting harmonic oscillators. This model has been discussed in the literature both for harmonic systems [7, 8, 13, 14] and anharmonic systems [15]. Once the coupling is established the reservoir imparts fluctuations in the system coordinates which thereby causes the system under observation to lose energy rapidly (and is irrevocable) to the bath. Because of this fluctuating or random force, the system undergoes Brownian motion. The reservoir is commonly known as a heat bath because the system dissipates its energy continuously and the former distributes this dissipated energy it its various energy-infusing modes. The relevant variables of the heat bath are averaged out later from the larger Hilbert space of the full system-plus-bath setup, to obtain an effective description of the test system alone. The projected dynamics of the test quantum system belonging to the truncated Hilbert space appear dissipative due to the bath-induced decoherence effects. Usually, either a formal path integral approach in the Schrödinger picture [16] or the quantum Langevin equation in the Heisenberg picture [12] is used to eliminate the heat bath degrees of freedom.

2. Theoretical framework

The system-plus-bath model for dissipative quantum systems is described as follows. A quantum system of a finite degree of freedom is coupled to a heat bath consisting of independent and non-interacting harmonic excitations above a stable ground state. The interaction between the quantum system and an individual oscillator of the heat bath is inversely proportional to the total volume $\mathcal{V}$ of the bath, thereby ensuring that the individual coupling is a linear function of bath coordinates. This nice property further allows one to eliminate the bath degrees of freedom easily. Because the number of oscillators in the bath is very large, the weak perturbation of any individual bath oscillator on the quantum system does not necessarily mean that the coupling of the system and the bath is weak. This model, even if the individual oscillators of the bath couple weakly to the system, allows the inclusion of strong damping also [11].

We write the total Hamiltonian for the "full" system as

$$H = H_S + H_B + H_{SB}, \tag{1}$$

where the system Hamiltonian is given by

$$H_S = \frac{p^2}{2M} + V(q), \tag{2}$$

where M is the mass of the quantum system which is moving in a potential $V(q)$, with q being the coordinate of the system. The heat bath Hamiltonian is written as the sum of N non-interacting oscillators

$$H_B = \sum_{j=1}^{N} \left(\frac{p_j^2}{2m_j} + \frac{m_j}{2} \omega_j^2 x_j^2 \right). \tag{3}$$

The possibility of revival of the initial state after a course of time, since one can pass on to the normal coordinates with the heat bath consisting of harmonic oscillators and $V(q)$ as a harmonic potential, can be overcome with heat bath having sufficiently many oscillators so that the Poincaré recurrence time becomes infinity [17]. The third contribution, the interaction Hamiltonian can be written as

$$H_{SB} = -q \sum_{j=1}^{N} C_j x_j + q^2 \sum_{j=1}^{N} \frac{C_j^2}{2m_j \omega_j^2}, \tag{4}$$

which is bilinear in the system and bath coordinates. The last term (which depends on the coupling constants C_j and only contains an operator in the system Hilbert space) in the interaction Hamiltonian is included as a counter term to ensure that the dissipation is homogeneous in all spaces. Since the coupling is via the position variables, if this term is not included, then the coupling becomes different wherever the quantum particle is located. Or in other words, the model is not translationally invariant and that will result in an unphysical renormalization of the potential. Therefore, one must understand that the counter term in the interaction Hamiltonian is included to make sure of the fact that dissipation has been introduced solely by the coupling to the reservoir not by a renormalization of $V(q)$. If it was not included, then the minimum of the potential surface of the global system for a given q is at $x_j = C_j q / m_j \omega_j^2$ for all j . This result in an 'effective' potential renormalized by the coupling which is given by

$$V_{\mathit{eff}}(q) = V(q) - \sum_{j=1}^{N} \frac{C_j^2 q^2}{2m_j \omega_j^2}. \tag{5}$$

This becomes clear if we consider the minimum of the Hamiltonian with respect to the system and environment coordinates. From the requirement

$$\frac{\partial H}{\partial x_j} = m_j \omega_j^2 x_j - C_j q = 0, \tag{6}$$

we obtain

$$x_j = \frac{C_j}{m_j \omega_j^2} q. \tag{7}$$

Using this result, we determine the minimum of the Hamiltonian with respect to the system coordinate and is given by

$$\frac{\partial H}{\partial q} = \frac{\partial V}{\partial q} - \sum_{j=1}^{N} C_j x_j + q \sum_{j=1}^{N} \frac{C_j^2}{m_j \omega_j^2} = \frac{\partial V}{\partial q}. \tag{8}$$

The second term in Eq. (4) thus ensures that this minimum is determined by the potential $V(q)$ only. Before we start writing equations of motion, for simplicity, we write the full Hamiltonian as

$$\mathcal{H} = \frac{p^2}{2M} + V(q) + \sum_{j=1}^{N}\left\{\frac{p_j^2}{2m_j} + \frac{1}{2}m_j\omega_j^2\left(x_j - \frac{C_j q}{m_j\omega_j^2}\right)^2\right\}. \tag{9}$$

2.1 Quantum mechanical derivation

We need to look at the reduced equation of motion for the system coordinate. In this section, we derive the quantum Langevin equation for our test system coordinate under the influence of the heat bath induced fluctuation effects. In the quantum domain, all the parameters are quantum variables and are operators. In the literature, Magalinskiĭ [14] showed that the elimination of the environmental degrees of freedom leads to a damped equation of motion for the system coordinate. The time evolution of an operator A, in the Heisenberg picture, is given by

$$\frac{dA}{dt} = \frac{i}{\hbar}[H, A]. \tag{10}$$

From Eq. (9) we obtain the equations of motion for the bath degrees of freedom

$$\dot{p}_j = -m_j\omega_j^2 x_j + C_j q \tag{11}$$

$$\dot{x}_j = \frac{p_j}{m_j}, \tag{12}$$

and, similarly, the equations of motion for the system degree of freedom are given by

$$\dot{p} = -\frac{\partial V}{\partial q} + \sum_{j=1}^{N} C_j x_j - q\sum_{j=1}^{N}\frac{C_j^2}{m_j\omega_j^2}, \tag{13}$$

$$\dot{q} = \frac{p}{M}. \tag{14}$$

We treat the system coordinate $q(t)$ as if it were a given function of time, we then solve the environmental equations of motion and it turns out to be an ordinary second order linear inhomogeneous differential equation with the solution of the form

$$x_j(t) = x_j(0)\cos\left(\omega_j t\right) + \frac{p_j(0)}{m_j\omega_j}\sin\left(\omega_j t\right) + \frac{C_j}{m_j\omega_j}\int_0^t ds\,\sin\left(\omega_j(t-s)\right)q(s). \tag{15}$$

Inserting Eq. (15) into Eq. (13), we obtain an effective equation of motion for the system coordinate

$$M\ddot{q} - \int_0^t ds \sum_{j=1}^{N} \frac{C_j^2}{m_j\omega_j} \sin\big(\omega_j(t-s)\big)q(s) + \frac{\partial V}{\partial q} + q\sum_{j=1}^{N} \frac{C_j^2}{m_j\omega_j^2} \tag{16}$$
$$= \sum_{j=1}^{N} C_j \left[x_j(0)\cos\big(\omega_j t\big) + \frac{p_j(0)}{m_j\omega_j} \sin\big(\omega_j t\big) \right].$$

This is further simplified by partially integrating the second term of the LHS and that yields

$$M\ddot{q} + M\int_0^t ds\gamma(t-s)\dot{q}(s) + \frac{\partial V}{\partial q} = f(t), \tag{17}$$

where the damping kernel is given by

$$\gamma(t) = \frac{1}{M}\sum_{j=1}^{N} \frac{C_j^2}{m_j\omega_j^2} \cos\big(\omega_j t\big). \tag{18}$$

It can also be expressed alternatively as

$$\gamma(t-t') = \Theta(t-t')\frac{1}{M}\sum_{j=1}^{N} \frac{C_j^2}{m_j\omega_j^2} \cos\big[\omega_j(t-t')\big]. \tag{19}$$

The operator-valued random force in Eq. (17) takes the form

$$f(t) = \sum_{j=1}^{N} C_j \left[\left(x_j(0) - \frac{C_j}{m_j\omega_j^2} q(0) \right) \cos\big(\omega_j t\big) + \frac{p_j(0)}{m_j\omega_j} \sin\big(\omega_j t\big) \right]. \tag{20}$$

The statistical average of this fluctuating force vanishes when the average is taken over the total density matrix of the bath degrees of freedom and the coupling. That is

$$\langle f(t)\rangle_{\rho_{(B+SB)}} = \frac{Tr_B[f(t)\exp\left(-\beta(\mathcal{H}_B + \mathcal{H}_{SB})\right)]}{Tr_B[\exp\left(-\beta(\mathcal{H}_B + \mathcal{H}_{SB})\right)]} = 0, \tag{21}$$

where $\rho_{(B+SB)}$ is the shifted canonical equilibrium distribution of the heat bath which is given by

$$\rho_B = Z_B^{-1}\exp\left[-\beta\sum_j \left(\frac{p_j(0)^2}{2m_j} + \frac{m_j\omega_j^2}{2}\left(x_j(0) - \frac{C_j}{m_j\omega_j^2} q(0) \right)^2 \right) \right], \tag{22}$$

where Z_B is the partition function of the bath oscillators. Also

$$\langle f(t)f(t')\rangle_{\rho_{(B+SB)}} = Mk_BT\gamma(t-t'). \tag{23}$$

This is the fluctuation-dissipation relation. For weak coupling, we seperate the transient term (or the initial slip) $M\gamma(t)q(0)$ which is of second order in the coupling constant C_j and write the random force as [18].

$$f(t) = \zeta(t) - m\gamma(t)q(0). \tag{24}$$

Therefore the generalized Langevin equation takes the form

$$M\ddot{q}(t) + M\int_0^t dt'\gamma(t-t')\dot{q}(t') + V'(q) = \zeta(t) - M\gamma(t)q(0), \tag{25}$$

where

$$\zeta(t) = \sum_j C_j\left(x_j(0)\cos\left(\omega_j t\right) + \frac{p_j(0)}{m_j\omega_j}\sin\left(\omega_j t\right)\right). \tag{26}$$

The integration in the forward direction of time in Eq. (25) ensures that it breaks the time reversal invariance explicitly thereby introducing irreversibility in the problem. When taken an average of the Eq. (26) with respect to the canonical classical equilibrium density operator of the unperturbed bath

$$\rho_B(0) = Z_B^{-1}\exp\left[-\beta\sum_j\left(\frac{p_j(0)^2}{2m_j} + \frac{m_j\omega_j^2}{2}x_j(0)^2\right)\right], \tag{27}$$

we obtain

$$\langle\zeta(t)\rangle_B = \frac{Tr_B[\zeta(t)\exp\left(-\beta\mathcal{H}_B\right)]}{Tr_B[\exp\left(-\beta\mathcal{H}_B\right)]} = 0, \tag{28}$$

and the unequal time correlation

$$\langle\zeta(t)\zeta(t')\rangle_{\rho_B(0)} = Mk_BT\gamma(t-t'). \tag{29}$$

From Eq. (20), it is clear that the force operator depends explicitly on the initial conditions of the bath positions and momenta and also on an inhomogeneous slip term $M\gamma(t)q(0)$. Usually, this term is neglected under the Markovian/Ohmic limit, when the friction assumes the ohmic form $\gamma(t) \to 2\gamma\delta(t)$ [18].

Now, we calculate the correlation function of the random force. We may use either $f(t)$ with respect to $\rho_{(B+SB)}$ or equivalently $\zeta(t)$ with respect to ρ_B. Eqs. (20) and (24) gives

$$\langle\zeta(t)\zeta(0)\rangle_B = \sum_{j,l} C_jC_l\left\langle\left(x_j(0)\cos\left(\omega_j t\right) + \frac{p_j(0)}{m_j\omega_i}\sin\left(\omega_j t\right)\right)x_l(0)\right\rangle_B. \tag{30}$$

At thermal equilibrium, the second moments of the position and momentum of the bath are calculated and yields

$$\left\langle x_j(0)x_l(0)\right\rangle_B = \delta_{jl}\frac{\hbar}{2m_j\omega_j}\coth\left(\frac{\hbar\beta\omega_j}{2}\right), \tag{31}$$

$$\left\langle p_j(0)x_l(0)\right\rangle_B = -\frac{i\hbar}{2}\delta_{jl}. \tag{32}$$

Incorporating Eqs. (31) and (32), the noise correlation in Eq. (30) can be expressed as

$$\langle\zeta(t)\zeta(0)\rangle_B = \sum_{j=1}^{N}\frac{\hbar C_j^2}{2m_j\omega_j}\left[\coth\left(\frac{\hbar\beta\omega_j}{2}\right)\cos\left(\omega_j t\right) - i\sin\left(\omega_j t\right)\right]. \tag{33}$$

It is to be noted that the noise correlation contains an imaginary part. This is due to the fact that $\zeta(t)$ and $\zeta(0)$ are now operators in quantum mechanics and, in general, do not commute with each other. We have obtained all the relations corresponding to the classical counterparts, here. Now, we need the reduced description of the quantum system alone and to get such an expression we have to eliminate the bath degrees of freedom from the total picture. The formal way of doing that is to introduce the "spectral density" of the bath oscillators which contains all the information about the heat bath. The spectral density is given by

$$J(\omega) = \pi \sum_{j=1}^{N} \frac{C_j^2}{2m_j\omega_j} \delta\big(\omega - \omega_j\big), \tag{34}$$

so that the damping kernel takes a form

$$\gamma(t) = \frac{1}{M} \sum_{j=1}^{N} \frac{C_j^2}{m_j\omega_j^2} \cos\big(\omega_j t\big) = \frac{2}{M} \int_0^\infty \frac{d\omega}{\pi} \frac{J(\omega)}{\omega} \cos(\omega t). \tag{35}$$

The most widely used form of the spectral density is of the following form

$$J(\omega) = M\gamma\omega, \tag{36}$$

which is called the *ohmic* and it apparently produces a memoryless friction

$$\gamma(t) = 2\gamma\delta(t). \tag{37}$$

But this is an ideal situation. In real physical situations, the spectral density falls off in the $\omega \to \infty$ limit. The above form (cf. Eq. (36)) of the spectral density, in the large ω limit, gives divergences in certain physical quantities like the momentum dispersion. It is then customary to introduce a cut-off to the spectrum with which the specturm vanishes above that cutoff. Such a spectrum is known as a Drude regularized spectral density [1] and is given by

$$J(\omega) = M\gamma\omega \frac{\omega_D^2}{\omega^2 + \omega_D^2}, \tag{38}$$

where ω_D is a cutoff to the spectrum of bath oscillators above which the spectral density vanishes. From Eq. (35), for positive arguments $t > 0$, the damping or memory kernel takes a form

$$\gamma(t) = \gamma\omega_D \exp(-\omega_D t). \tag{39}$$

We need $\tilde{\gamma}(\omega)$, the Fourier transform of Eq. (39), for our forthcoming calculations and is written as

$$\tilde{\gamma}(\omega) = \frac{\gamma\omega_D}{\omega_D - i\omega}, \quad \text{and } \Re[\tilde{\gamma}(\omega)] = \frac{\gamma\omega_D^2}{\omega_D^2 + \omega^2}. \tag{40}$$

The "tilde" sign is used to denote the Fourier transform of a function throughout the chapter. One can still use the terminology "ohmic damping" even if the Eq. (38)

does not hold above a critical frequency, provided all the typical frequencies appearing in the dynamics should be much lower than this critical frequency. In the strict 'Ohmic' limit the generalized Langevin equation becomes memoryless and corresponds to the classical Langevin equation.

3. The model and the calculation of instantaneous power

We discussed the theoretical descriptions to understand how a dissipative system behaves when connected to a heat bath at equilibrium. The quantum system was considered to be not in equilibrium prior to the coupling. Once the coupling is established the quantum system continuously transfers its energy to the equilibrium bath and eventually, the system reaches thermal equilibrium (asymptotically) with the heat bath. Thermal equilibrium is said to have reached when a quantum system explores its phase-space fully. From here onwards we discuss the various energy exchanges that happen in a many-body system at equilibrium. The thermal properties of the quantum system can be calculated by assuming that the entire system-plus-bath arrangement is embedded in an infinitely large environment which provides the working temperature. Therefore, from here onwards we denote the quantum system as a subsystem of the bigger bath. Of course, the heat bath with N oscillator modes can also be considered as a subsystem of the bigger bath. Put it differently, the quantum system of our interest and the heat bath with which it is connected became the constituents of a large environment (**Figure 2**). Now we discuss how the energy exchange processes within this 'envelope'. We show here how the random force balances the energy lost by the quantum subsystem to the heat bath. It is enough to calculate the work done by the random force to that it compensates for the energy lost from the subsystem. Moreover, this work done by the random force is necessary to maintain equilibrium.

To proceed further, we need a working model for the subsystem. We choose the charged oscillator in a magnetic field as our quantum subsystem. Hence our system of

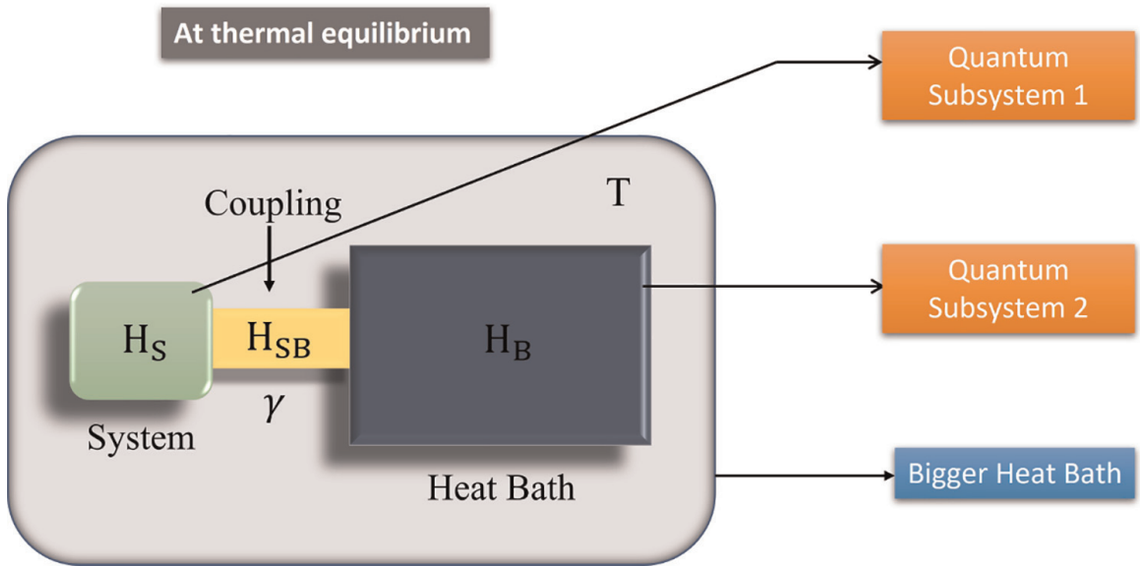

Figure 2.
Pictorial representation of various constituents of a bigger bath. The quantum system as well as the heat bath are now two different parts of the larger environment. The terminology "subsystem" makes better sense in this context.

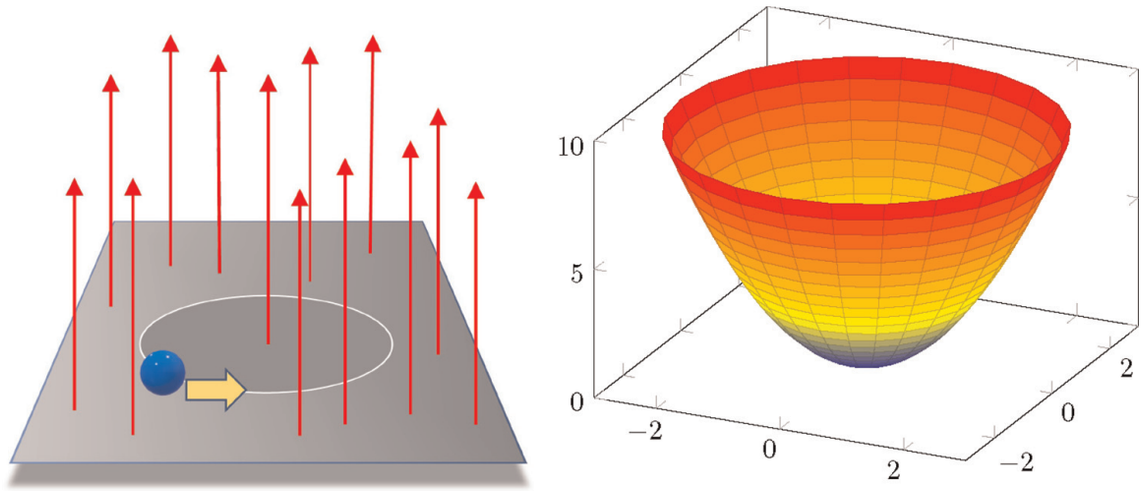

Figure 3.
(a) Electron motion under a perpendicular uniform magnetic field. The trajectories are helical in nature, but the projection of an individual electron's trajectory onto a two-dimensional plane shows a circular motion around the magnetic line of force. (b) The mexican-hat potential. We take this potential to confine the electron under perpendicular magnetic field and the whole arrangement can be simply called a "charged oscillator in a magnetic field". This real physical model is very useful and studied extensively in the condensed matter realm in various contexts. Studies on quantum dots and wires rely on this model heavily.

study is the dissipative charged oscillator in a magnetic field not the uncoupled charged oscillator in a magnetic field. A damped harmonic oscillator was used as a quantum subsystem in the literature [19, 20]. Magnetic field effects in the domain of dissipative quantum physics are of great interest in various phenomena including the quantum Hall effect [21] and superconductivity [22]. Studies on the dissipative charged oscillator in a magnetic field using the system-plus-reservoir approach were originally carried out by Li et al. [23, 24]. This model, later, was used by many others [25–30] in different contexts (**Figure 3**).

$\mathcal{H}_S$ represents the Hamiltonian of the charged magneto-oscillator. It is given by

$$\mathcal{H}_S = \frac{1}{2M}\left(\mathbf{p} - \frac{e\mathbf{A}}{c}\right)^2 + \frac{1}{2}M\omega_0^2\mathbf{r}^2, \tag{41}$$

where M is the mass of the quantum subsystem. The two dimensional vectors $\mathbf{p}$ and $\mathbf{r}$ represent the momentum and position coordinates of the subsystem respectively. Here $\mathbf{A}$ is denotes the magnetic vector potential, e is the charge of the electron, and c is the velocity of light. The Hamiltonian for the bath and the coupling, i.e., $\mathcal{H}_B + \mathcal{H}_{SB}$, can be expressed as

$$\mathcal{H}_B + \mathcal{H}_{SB} = \sum_{j=1}^{N}\left\{\frac{\mathbf{p}_j^2}{2m_j} + \frac{1}{2}m_j\omega_j^2\left(\mathbf{x}_j - \frac{C_j\mathbf{r}}{m_j\omega_j^2}\right)^2\right\}. \tag{42}$$

m_js and ω_js are the masses and frequencies of the individual bath oscillators respectively. C_js are the coupling between the system and the heat bath oscilators. The two dimensional vectors $\mathbf{p}_j$s and $\mathbf{x}_j$s represent the momentum and the position coordinates of the bath oscillators respectively. The position and momentum vectors of the subsystem and the heat bath are operators and they satisfy the following commutation relations

$$\left[\mathbf{r}_i, \mathbf{p}_k\right] = i\hbar\delta_{ik}, \quad \left[\mathbf{x}_{ji}, \mathbf{p}_{lk}\right] = i\hbar\delta_{jl}\delta_{ik}. \tag{43}$$

Following the steps given in the previous section, we use the Heisenberg equations of motion from the total Hamiltonian, we obtain the equations of motion for the subsystem and the bath coordinates. Eliminating the bath variables yields the generalized equation for the system coordinate (which is an operator) and that is given by [30]:

$$M\ddot{\mathbf{r}} + \int_0^t dt'\dot{\mathbf{r}}(t')\gamma(t-t') - \frac{e}{c}(\dot{\mathbf{r}} \times \mathbf{B}) + M\omega_0^2\mathbf{r} = -M\gamma(t)\mathbf{r}(0) + \mathbf{F}(t). \tag{44}$$

This is an initial value equation and has an exact solution as well [30, 31]. Like we saw in the previous section, the spurious initial slip term is here as well. We may define an auxiliary random force $\mathcal{K}(t) = -M\gamma(t)\mathbf{r}(0) + \mathbf{F}(t)$ just for esthetics. The Langevin equation is shown to be independent of any particular choice of gauge since it does not explicitly contain the magnetic vector potential $\mathbf{A}$. The appearance of external magnetic field is via the quantum version of the Lorentz force term in the equation. The *memory friction function* $\gamma(t)$ is defined already in the previous section. The stochastic Brownian noise $\mathbf{F}(t)$, depends explicitly on initial coordinates and the momenta of the bath oscillators, is given by

$$\mathbf{F}(t) = \sum_j \left\{ C_j\mathbf{x}_j(0)\cos\left(\omega_j t\right) + \frac{C_j\mathbf{p}_j(0)}{m_j\omega_j}\sin\left(\omega_j t\right) \right\}. \tag{45}$$

It is necessary to note that the damping $\gamma(t)$ and the operator valued Gaussian random force remain unchanged by the external magnetic field. The random force (45) satisfies the following relations [23, 24, 26, 28–30].

$$\begin{aligned}\left\langle\{F_\alpha(t), F_\kappa(t')\}\right\rangle &= \delta_{\alpha\kappa}\frac{2}{\pi}\int_0^\infty d\omega\Re[\tilde{\gamma}(\omega + i0^+)]\hbar\omega \\ &\quad\times\coth\left(\frac{\hbar\omega}{2k_BT}\right)\cos\left[\omega(t-t')\right],\end{aligned} \tag{46}$$

$$\begin{aligned}\left\langle[F_\alpha(t), F_\kappa(t')]\right\rangle &= \delta_{\alpha\kappa}\frac{2}{i\pi}\int_0^\infty d\omega\Re[\tilde{\gamma}(\omega + i0^+)]\hbar\omega \\ &\quad\times\sin\left[\omega(t-t')\right],\end{aligned} \tag{47}$$

where $\alpha, \kappa = x, y, z$ and $\tilde{\gamma}(z) = \int_0^\infty dt\exp(izt)\gamma(t)$, $(z > 0)$. For the Drude regularized Ohmic spectral density, using Eq. (38), the Eqs. (46) and (47) can be written as

$$\begin{aligned}\left\langle\{F_\alpha(t), F_\kappa(t')\}\right\rangle &= \delta_{\alpha\kappa}\frac{2\gamma\omega_D^2}{\pi}\int_0^\infty d\omega\frac{\hbar\omega}{\omega_D^2 + \omega^2} \\ &\quad\times\coth\left(\frac{\hbar\omega}{2k_BT}\right)\cos\left[\omega(t-t')\right],\end{aligned} \tag{48}$$

$$\begin{aligned}\left\langle[F_\alpha(t), F_\kappa(t')]\right\rangle &= \delta_{\alpha\kappa}\frac{2\gamma\omega_D^2}{i\pi}\int_0^\infty d\omega\frac{\hbar\omega}{\omega_D^2 + \omega^2} \\ &\quad\times\sin\left[\omega(t-t')\right].\end{aligned} \tag{49}$$

The anti-symmetric correlation can also be written as

$$\langle\{F_\alpha(t), F_\kappa(t')\}\rangle = 2\delta_{\alpha\kappa}\int_0^\infty d\omega \mathcal{G}(\omega)\cos[\omega(t-t')], \tag{50}$$

where $\mathcal{G}(\omega)$ is called the power spectrum, and is given by

$$\begin{aligned}\mathcal{G}(\omega) &= \frac{\gamma\omega_D^2}{\pi}\frac{\hbar\omega}{\omega_D^2+\omega^2}\coth\left(\frac{\hbar\omega}{2k_BT}\right)\\ &= \frac{2\gamma\omega_D^2}{\pi\left(\omega_D^2+\omega^2\right)}\left(\frac{\hbar\omega}{2}+\frac{\hbar\omega}{e^{\hbar\omega/k_BT}-1}\right).\end{aligned} \tag{51}$$

The fluctuation dissipation theorem ensures that the symmetric combination and the commutator structure of the random force are (i) proportional to the friction constant γ, and (ii) are independent of the external potential $V(\mathbf{r})$. From Eq. (52), we observe that the power spectrum of the random force is the Planck spectrum, with an extra term due to the zero point fluctuations. Needless to say, in the limit of $\hbar \to 0$ the spectrum becomes the flat spectrum of the white noise.

We now calculate the expectation value of the instantaneous power supplied by the random force. Since we work with operators, we use the symmetric form of the power. The power can be written as

$$\mathcal{P}_F = \frac{1}{2}\langle\mathbf{v}(t)\mathbf{F}(t)+\mathbf{F}(t)\mathbf{v}(t)\rangle, \tag{52}$$

with $\mathbf{v}(t) = \dot{\mathbf{r}}(t)$. We define the variables $Z = x + iy$, $F = F_x + iF_y$, and $\overline{\gamma}(t) = \frac{\gamma(t)}{m} + i\omega_c$, to re-write the Langevin equation given in Eq. (44) as [30].

$$\ddot{Z} + \int_0^t dt'\overline{\gamma}(t-t')\dot{Z}(t') + \omega_0^2 Z = -\gamma(t)Z(0) + \frac{F(t)}{M}, \tag{53}$$

where $\omega_c = eB/mc$ is the magnetic cyclotron frequency. The new position coordinate is Z and its derivative with respect to time represents the velocity operator. Using Laplace transform and with the aid of fundamental solutions, we find the solution of the Langevin equation given in Eq. (53) as [30, 31].

$$Z(t) = M\dot{\chi}(t)Z(0) + M\chi(t)\dot{Z}(0) + \int_0^t d\tau\chi(t-\tau)F(\tau), \tag{54}$$

where $\chi(t)$ is called the response function of the system and is defined as [30, 31].

$$\chi(t-\tau) = \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} ds\hat{\chi}(s)\exp(s(t-\tau)). \tag{55}$$

The Laplace transform of the response function is written as

$$\hat{\chi}(s) = \frac{1}{M}\frac{1}{s^2+\omega_0^2+s\overline{\gamma}(s)}, \tag{56}$$

with the Fourier transform

$$\chi(t-\tau) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} d\omega \tilde{\chi}(\omega)\exp\left(-i\omega(t-\tau)\right), \tag{57}$$

where the dynamical susceptibility is given by

$$\tilde{\chi}(\omega) = \frac{1}{M}\frac{1}{-\omega^2+\omega_0^2-i\omega\overline{\gamma}(\omega)}. \tag{58}$$

We have, here, $\overline{\gamma}(\omega) = i\omega_c + \tilde{\gamma}(\omega)$. The memory kernel $\gamma(t)$ typically falls to zero in the bath relaxation time. As a result, the initial slip term in Eq. (44) (and subsequently in Eq. (53)) vanishes for very long times (can be experimental time scales) compared to the system's characteristic decay time. Therefore, for longer times compared to the system's relaxation time, the quantum Langevin equation (cf. Eq. (53)) becomes a stationary one with the lower limit of integration $-\infty$, i.e., [30]

$$\ddot{Z} + \int_{-\infty}^{t} dt' \overline{\gamma}(t-t')\dot{Z}(t') + \omega_0^2 Z = \frac{F(t)}{M}, \tag{59}$$

with the solution, which is a stationary process, is given by

$$Z(t) = \int_{-\infty}^{t} d\tau \chi(t-\tau)F(\tau). \tag{60}$$

We note from the expressions (Eqs. (53) and (57)) that, so long as ω_0 remains non-zero, the response function $\chi(t-\tau)$ vanishes exponentially for longer times. This, in fact, is in close proximity with the Tauberian theorem which states the asymptotic behavior of a function depends on the low frequency behavior of its Fourier transform. Hence for very longer times, the dependence of $Z(t)$ on the initial values (cf. Eq. (54)) completely disappears and that yield [30, 31].

$$Z(t) = \int_{0}^{t} d\tau \chi(t-\tau)F(\tau). \tag{61}$$

Upon comparing Eq. (61) with Eq. (60), we conclude that $Z(t)$ in Eq. (61) is the solution of the stationary Langevin equation in Eq. (59). With these remarks in mind, we write the Langevin equation in Eq. (44) as

$$M\ddot{\mathbf{r}} + M\int_{-\infty}^{t} dt' \dot{\mathbf{r}}(t')\gamma(t-t') - \frac{e}{c}(\dot{\mathbf{r}}\times\mathbf{B}) + M\omega_0^2\mathbf{r} = \mathbf{F}(t). \tag{62}$$

Using Eqs. (52) and (62), we write the power as

$$\mathcal{P}_F = \frac{d}{dt}\left\langle \frac{1}{2M}\left(\mathbf{P} - \frac{e\mathbf{A}}{c}\right)^2 + \frac{1}{2}M\omega_0^2\mathbf{r}^2 \right\rangle + \int_{-\infty}^{t} dt' \gamma(t-t')\frac{1}{2}\langle \dot{\mathbf{r}}(t)\dot{\mathbf{r}}(t') + \dot{\mathbf{r}}(t')\dot{\mathbf{r}}(t)\rangle. \tag{63}$$

Since the Langevin equation describes a stationary process the first term is zero because for a stationary process the expectation values of time dependent quantities must be time translational invariant or, in other words, they are constant. This becomes clear when we evaluate the expectation value of an arbitrary time dependent operator $W(t)$. We write [19].

$$\langle W(t)\rangle = \frac{\mathrm{Tr}\left\{e^{-\beta H}e^{iHt/\hbar}W(0)e^{-iHt/\hbar}\right\}}{\mathrm{Tr}e^{-\beta H}} = \frac{\mathrm{Tr}\{W(0)\}}{\mathrm{Tr}e^{-\beta H}}, \tag{64}$$

where $\beta = 1/k_BT$. The cyclic property of the Trace operation is used to get the desired result. Thus it becomes clear that $\langle W(t)\rangle$ is time independent for a canonical ensemble and the derivative of it must be zero. Our task is just to find the second term in Eq. (64). We use Eq. (61) to compute the velocity autocorrelation function. It is to be noted that the real part of the quantity $\frac{1}{2}\left\langle\left\{\dot{Z}(t), \dot{Z}^{\dagger}(t')\right\}\right\rangle$ is equivalent to $\frac{1}{2}\langle\dot{\mathbf{r}}(t)\dot{\mathbf{r}}(t') + \dot{\mathbf{r}}(t')\dot{\mathbf{r}}(t)\rangle$.

We write [30].

$$\frac{1}{2}\left\langle\left\{Z(t), Z^{\dagger}(t')\right\}\right\rangle = \frac{1}{2m^2}\int_0^t d\tau\int_0^{t'} d\tau'\chi(t-\tau)\chi^*(t'-\tau')\left\langle\left\{F(\tau), F^{\dagger}(\tau')\right\}\right\rangle, \tag{65}$$

where, using (47), we write [30].

$$\left\langle\left\{F(\tau), F^{\dagger}(\tau')\right\}\right\rangle = \langle\{F_x(\tau), F_x(\tau')\}\rangle + \left\langle\left\{F_y(\tau), F_y(\tau')\right\}\right\rangle,$$
$$= \frac{4}{\pi}\int_0^{\infty} d\omega\Re[\tilde{\gamma}(\omega + i0^+)]\hbar\omega \times \coth\left(\frac{\hbar\omega}{2k_BT}\right)\cos[\omega(\tau-\tau')].$$

After some algebra, we obtain [30].

$$\frac{1}{2}\left\langle\left\{Z(t), Z^{\dagger}(t')\right\}\right\rangle = \frac{\hbar}{M\pi}\int_{-\infty}^{+\infty} d\omega\,\omega\Re\left[\frac{\tilde{\gamma}(\omega)}{m}\right]\tilde{\chi}(\omega)\tilde{\chi}^*(\omega) \times\coth\left(\frac{\hbar\omega}{2k_BT}\right)e^{-i\omega(t-t')}, \tag{67}$$

where

$$\tilde{\chi}(\omega) = -\frac{1}{M}\,\frac{(\omega + i\omega_D)}{(\omega + i\lambda_1)(\omega + i\lambda_2)(\omega + i\lambda_3)}, \tag{68}$$

Here $\lambda_j s$ are the roots of the cubic equation

$$\omega^3 + i\omega^2(\omega_D + i\omega_c) - \omega\left(\omega_0^2 + \gamma\omega_D + i\omega_c\omega_D\right) - i\omega_0^2\omega_D = 0. \tag{69}$$

Similarly, the complex conjugate $\tilde{\chi}^*(\omega)$ of Eq. (68) is given by

$$\tilde{\chi}^*(\omega) = -\frac{1}{M}\,\frac{(\omega - i\omega_D)}{(\omega - i\lambda_{1'})(\omega - i\lambda_{2'})(\omega - i\lambda_{3'})}, \tag{70}$$

with the λ_j's are the complex conjugates of λ_js. Since Eqs. (68) and (70) satisfy the following relation

$$M\omega\Re[\tilde{\gamma}(\omega)]\tilde{\chi}(\omega)\tilde{\chi}^*(\omega) = \frac{1}{2i}[\tilde{\chi}(\omega) - \tilde{\chi}^*(\omega)] = \tilde{\chi}''(\omega), \tag{71}$$

we obtain [30].

$$\frac{1}{2}\langle\{Z(t), Z^\dagger(t')\}\rangle = \frac{\hbar}{M\pi}\int_{-\infty}^{+\infty} d\omega\, \tilde{\chi}''(\omega)\coth\left(\frac{\hbar\omega}{2k_BT}\right)e^{-i\omega(t-t')}, \tag{72}$$

where $\tilde{\chi}''(\omega)$ is given by

$$\tilde{\chi}''(\omega) = \frac{1}{M}\,\frac{\omega\tilde{\gamma}(\omega)}{\left(\omega_0^2 - \omega^2 + \omega\omega_c\right)^2 + \omega^2\tilde{\gamma}^2(\omega)}. \tag{73}$$

Taking the derivatives of Eq. (72) with respect to t and t' yields the velocity autocorrelation

$$\frac{1}{2}\left\langle\left\{\dot{Z}(t), \dot{Z}^\dagger(t')\right\}\right\rangle = \frac{\hbar}{M\pi}\int_{-\infty}^{\infty} d\omega\, \omega^2\, \tilde{\chi}''(\omega)\coth\left(\frac{\hbar\omega}{2k_BT}\right)e^{-i\omega(t-t')}. \tag{74}$$

Substituting the real part of the velocity autocorrelation into the equation for the power yields

$$\begin{aligned}\mathcal{P}_F = &\int_{-\infty}^{t} dt'\gamma(t-t')\frac{\hbar}{M\pi}\int_{-\infty}^{\infty} d\omega\, \omega^2\, \tilde{\chi}''(\omega)\\ &\times\coth\left(\frac{\hbar\omega}{2k_BT}\right)\cos\left[\omega(t-t')\right].\end{aligned} \tag{75}$$

Since the damping $\gamma(t)$ is 0 for negative t and $\tilde{\gamma}(\omega) = \int_0^\infty dt\gamma(t)e^{i\omega t}$, $\omega > 0$, the upper limit of the integral in Eq. (44) can be replaced with $+\infty$. Therefore

$$\begin{aligned}\mathcal{P}_F = &\frac{\hbar}{M\pi}\int_{-\infty}^{+\infty} d\omega\int_{-\infty}^{\infty} dt'\, \gamma(t-t')\cos\left[\omega(t-t')\right]\\ &\times\omega^2\, \tilde{\chi}''(\omega)\coth\left(\frac{\hbar\omega}{2k_BT}\right).\end{aligned} \tag{76}$$

It is possible to write

$$\int_{-\infty}^{+\infty} dt'\gamma(t-t')\cos\left[\omega(t-t')\right] = \Re\left\{\int_{-\infty}^{+\infty} dt'\gamma(t-t')e^{i\omega(t-t')}\right\}. \tag{77}$$

Since the bracketed term is just the Fourier transform $\tilde{\gamma}(\omega)$ of $\gamma(t-t')$, we write Eq. (77) as

$$\mathcal{P}_F = \frac{\hbar}{M\pi}\int_{-\infty}^{\infty} d\omega\, \omega^2\, \tilde{\chi}''(\omega)\Re[\tilde{\gamma}(\omega)]\coth\left(\frac{\hbar\omega}{2k_BT}\right). \tag{78}$$

It is immediately observed that only the even part of the above integral contributes. We know that $\tilde{\gamma}(-\omega)^* = \tilde{\gamma}(\omega)$ and $\tilde{\chi}(-\omega) = \tilde{\chi}(\omega)$, so that $\tilde{\chi}''(\omega)$ is an odd function of ω and $\Re[\tilde{\gamma}(\omega)]$ is an even function of the frequency ω. Upon using the real part of $\tilde{\gamma}(\omega)$, we write

$$\mathcal{P}_F = \frac{\hbar}{M\pi}\int_{-\infty}^{\infty} d\omega\, \omega^2\, \tilde{\chi}''(\omega)\, \frac{\gamma\omega_D^2}{\omega_D^2+\omega^2} \coth\left(\frac{\hbar\omega}{2k_BT}\right). \tag{79}$$

This is the instantaneous power supplied by the random force for our particular model and the above expression is positive quantity always.

To check the result, we evaluate the power supplied by the random force at high temperatures (or in the classical case). After some tedious mathematical manipulations we obtain [32].

$$\mathcal{P}_F = \frac{4k_BT\gamma}{M}\, \frac{\left(2+\omega_0^2/\omega_D^2+\gamma/\omega_D\right)}{\left[\left(2+\omega_0^2/\omega_D^2+\gamma/\omega_D\right)^2+\left(2\omega_c/\omega_D\right)^2\right]}, \tag{80}$$

which in the large cutoff (strict ohmic) limit ($\omega_D \to \infty$) yields the form

$$\mathcal{P}_F = \frac{4k_BT\gamma}{M}. \tag{81}$$

This clearly indicates to us that the rate of work done by the random force on the quantum subsystem is indeed proportional to the damping/dissipation strength γ. This in turn tells us that, at equilibrium, the energy lost by the quantum subsystem due to dissipation is compensated by an amount of energy received from the random force. As a consequence of the fluctuation-dissipation theorem, the Eq. (81) does not contain any term from the external potential (and magnetic field), in the strict ohmic limit.

4. Discussion of the result

The full quantum many-body system has an infinite number of degrees of freedom each with its corresponding zero-point oscillations. The full system must be in the ground state at the absolute zero of temperature ($T = 0$). This further implies that no work is done on or by the system. However, for any finite coupling (irrespective of the strength of the coupling), $\mathcal{H}_S$ does not commute with $\mathcal{H}$. That means the ground state of $\mathcal{H}$ is not the ground state of $\mathcal{H}_S$. Therefore, even at absolute zero temperature, the energy of the charged oscillator in a magnetic field must fluctuate. Mathematically, this statement can be expressed as

$$\Delta\mathcal{H}_S^2 = \langle\mathcal{H}_S^2\rangle - \langle\mathcal{H}_S\rangle^2 \neq 0, \tag{82}$$

the mean square fluctuations of $\mathcal{H}_S$ is not equal to zero. No matter how weak the coupling between the subsystem and the bath, the fluctuation of the subsystem Hamiltonian does not vanish at $T = 0$. This fluctuation is obviously driven by the random force. Therefore, the work done per unit time by this random force is, indeed, balanced by the dissipative loss of the subsystem. Hence there is no net work done on the subsystem. To evaluate the RHS of Eq. (82) we resort to Eq. (41). We write

$$\langle \mathcal{H}_S \rangle = \frac{1}{2M}\left\langle \left(\mathbf{p} - \frac{e\mathbf{A}}{c}\right)^2 \right\rangle + \frac{1}{2}M\omega_0^2\langle \mathbf{r}^2 \rangle, \tag{83}$$

In the new variable $Z = x + iy$, the subsystem Hamiltonian can be written as [30].

$$\mathcal{H}_S = \frac{1}{2}M\dot{Z}\dot{Z}^\dagger - \frac{1}{2}\hbar\omega_c + \frac{1}{2}M\omega_0^2 ZZ^\dagger. \tag{84}$$

Therefore

$$\langle \mathcal{H}_S \rangle = \frac{1}{2}M\left\langle \dot{Z}\dot{Z}^\dagger \right\rangle - \frac{1}{2}\hbar\omega_c + \frac{1}{2}M\omega_0^2\langle ZZ^\dagger \rangle. \tag{85}$$

We need to compute the unequal time correlation functions $\langle Z(t)Z^\dagger(t')\rangle$ and $\left\langle \dot{Z}(t)\dot{Z}^\dagger(t')\right\rangle$. To get this, we only evaluate the symmetric combination of $Z(t)$ and $Z^\dagger(t')$. The anti-symmetric combination vanishes in the analytic continuation $t' = t$ to obtain the equilibrium values [30].

Using Eqs. (72) and (74), at $T = 0$, we compute Eq. (85). We get [30].

$$\begin{aligned}\langle \mathcal{H}_S \rangle_{T=0} = {} & \frac{\hbar\gamma}{\pi}\ln\left(\frac{\omega_D}{\omega_0}\right) + \frac{\hbar\omega_0^2}{\pi}\left\{\frac{1}{\Lambda}\ln\left(\frac{\lambda_1}{\lambda_2}\right) + \frac{1}{\Lambda'}\ln\left(\frac{\lambda_1'}{\lambda_2'}\right)\right\} \\ & - \frac{\hbar}{4\pi}\left\{\frac{\overline{\gamma}^2}{\Lambda}\ln\left(\frac{\lambda_1}{\lambda_2}\right) + \frac{\overline{\gamma}^{*2}}{\Lambda'}\ln\left(\frac{\lambda_1'}{\lambda_2'}\right)\right\},\end{aligned} \tag{86}$$

where $\Lambda = \sqrt{\overline{\gamma}^2 - 4\omega_0^2}$ and $\Lambda' = \sqrt{\overline{\gamma}^{*2} - 4\omega_0^2}$. Also $\lambda_{1,2} = \overline{\gamma}/2 \pm i\sqrt{\omega_0^2 - \overline{\gamma}^2/4}$ and $\lambda_{1,2}' = \overline{\gamma}^*/2 \mp i\sqrt{\omega_0^2 - \overline{\gamma}^{*2}/4}$. For weak dissipation $\overline{\gamma}$, $\overline{\gamma}^* \ll 2\omega_0$,

$$\begin{aligned}\langle \mathcal{H}_S \rangle_{T=0} = {} & \frac{\hbar\gamma}{\pi}\ln\left(\frac{\omega_D}{\omega_0}\right) - \frac{\hbar\gamma\omega_c}{2\pi\omega_0}\ln\left\{\frac{(\overline{\gamma}/2 + i\omega_0)(\overline{\gamma}^*/2 - i\omega_0)}{(\overline{\gamma}/2 - i\omega_0)(\overline{\gamma}^*/2 + i\omega_0)}\right\} \\ & + \frac{i\hbar(\gamma^2 + \omega_c^2 - 4\omega_0^2)}{4\pi\omega_0}\ln\left\{\frac{(\gamma^2 + \omega_c^2 - 4\omega_0^2) + 4i\gamma\omega_0}{(\gamma^2 + \omega_c^2 - 4\omega_0^2) - 4i\gamma\omega_0}\right\}.\end{aligned} \tag{87}$$

In the absence of the magnetic field ($\omega_c = 0$) we get

$$\langle \mathcal{H}_S \rangle_{T=0} = \hbar\omega_0 + \frac{\hbar\gamma}{\pi}\ln\left(\frac{\omega_D}{\omega_0}\right), \tag{88}$$

which is nothing but the result for a two dimensional isotropic oscillator. Since $\mathbf{r}$ and $\dot{\mathbf{r}}$ (or Z and $\dot{Z}$) are Gaussian operators with zero mean values, $\langle \mathcal{H}_S^2 \rangle$ (cf, Eq. (41)) can be readily expressed in terms of products of $\langle \mathbf{r}^2 \rangle$ and $\langle \dot{\mathbf{r}}^2 \rangle$ (or using Z^2 and $\dot{Z}^2$ with Eq. (84)). We evaluate the $\langle \mathcal{H}_S^2 \rangle$ in the absence of magnetic field to just show the concerned point. Magnetic field has no significant role in the present evaluation to show that the mean square fluctuation of the subsystem Hamiltonian does not vanish. With the aid of Eq. (88), we write

$$\Delta\mathcal{H}_S^2 = \langle \mathcal{H}_S \rangle \frac{\hbar\gamma}{\pi}\ln\left(\frac{\omega_D}{\omega_0}\right). \tag{89}$$

Even for a very weak damping, the mean of the subsystem energy is above its ground state energy and that the fluctuations in this energy do not vanish. This tells us that one part of a physical system in its ground state can and does exchange energy with another part. The formalism we have chosen to show the energy balance is an exact one. We use the Langevin equation to calculate the expectation values after taking the time derivative of the subsystem Hamiltonian.

$$\frac{d}{dt}\langle \mathcal{H}_S \rangle + \gamma \langle \dot{\mathbf{r}}^2(t) \rangle = \frac{1}{2}\langle \dot{\mathbf{r}}(t)\mathbf{F}(t) + \mathbf{F}(t)\dot{\mathbf{r}}(t) \rangle. \tag{90}$$

The exact equation given above is obtained under the strict ohmic limit of the Langevin Equation in Eq. (62). The expectation values appearing above are equal time expectation values and are independent of time as the total physical system is invariant under the time translation. The first term in the above relation is zero. The rest of the equation then describe that the power (RHS) is actually proportional to the dissipation constant γ. Power does not vanish even at absolute zero temperature and this indicates that different parts of the physical system can and do exchange energy even at absolute zero temperature.

Author details

Jishad Kumar
Department of Chemical and Biological Physics, Weizmann Institute of Science, Rehovot, Israel

*Address all correspondence to: kumar.jishad@gmail.com

References

[1] Weiss U. *Quantum Dissipative Systems*. 3rd ed. Singapore: World Scientific; 2008

[2] Dittrich T, Hänggi P, Ingold G-L, Kramer B, Schön G, Zwerger W. *Quantum Transport and Dissipation*. New York, NY: Wiley-VCH; 1998

[3] Nakajima S. Progress in Theoretical Physics. 1958;**20**:948

[4] Zwanzig R. The Journal of Chemical Physics. 1960;**33**:1338

[5] Zwanzig R. In: Brittin WE, Downs BW, Down J, editors. *Lectures in Theoretical Physics*. Vol. 3. Boulder, New York: Interscience; 1961; Also see, Zwanzig R. Nonequilibrium Statistical Mechanics. New York: Oxford University Press; 2001

[6] Prigogine J, Resibois P. Physica. 1961; **27**:629

[7] Senitzky IR. Physics Review. 1961; **119**:670; **124** 642

[8] Ford GW, Kac M, Mazur P. Journal of Mathematical Physics. 1965;**6**:504

[9] Mori H. Progress in Theoretical Physics. 1965;**33**:423

[10] Caldeira AO, Leggett AJ. Physical Review Letters. 1981;**46**:211

[11] Caldeira AO, Leggett AJ. Annals of Physics (NY). 1983;**149**:374

[12] Ford GW, Lewis JT, O'Connell RF. Physical Review A. 1988;**37**:4419

[13] Ullersama P. Physica. 1966;**32**:27, 56, 74, 90

[14] Magalinski VB. Zhurnal Eksperimental'noi i Teoreticheskoi Fiziki. 1959;**36**:1942 [Sov. Phys. JETP **9** 1381]

[15] Zwanzig R. Journal of Statistical Physics. 1973;**9**:215

[16] Grabert H, Schramm P, Ingold G-L. Physics Reports. 1988;**168**:115

[17] Hemmer PC, Maximon LC, Wergeland H. Physics Review. 1958;**111**: 689

[18] Hänggi P. Lecture Notes in Physics. 1997;**484**:15-22

[19] Li XL, Ford GW, O'Connell RF. Physical Review E. 1993;**48**:1547

[20] Li XL, Ford GW, O'Connell RF. Physical Review E. 1995;**51**:5169

[21] Prange RE, Girvin SM, editors. The Quantum Hall Effect. Berlin: Springer; 1987

[22] Ong NP. In: Ginsberg DM, editor. *Physical Properties of High Temperature Superconductors*. Vol. 2. Singapore: World Scientific; 1990

[23] Li XL, Ford GW, O'Connell RF. Physical Review A. 1990;**41**:5287

[24] Li XL, Ford GW, O'Connell RF. Physical Review A. 1990;**42**:4519

[25] Hong TM, Wheatley JM. Physical Review B. 1991;**42**:6492; **43** 5702

[26] Li XL, O'Connell RF. Physica A. 1996;**224**:639-668

[27] Li XL, Ford GW, O'Connell RF. Physical Review E. 1996;**53**:3359

[28] Dattagupta S, Singh J. Pramana. 1996;**47**:211

[29] Dattagupta S, Singh J. Physical Review Letters. 1997;**79**:961

[30] Kumar J. Physica A. 2014;**393**: 182-206

[31] Ford GW, O'Connell RF. Physical Review D. 2001;**64**:105020

[32] Kumar J. Physica Scripta. 2014;**89**: 095701