Chapter

About the Notion of Inverse Problem in STEM Education

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Abstract

Inverse problems play an important role in STEM disciplines; although this concept is not well-defined in STEM education. For instance, Mason considers inversion as 'undoing', whereas Keller observes that if two problems are inverses of one another, then one of them has been studied extensively, while the other is newer and the former is called 'direct', while the latter is called 'inverse'. Groetsch observes that if y is the effect of a given cause x when a mathematical model K is posited ($Kx = y$), then, two inverse problems arise: causation (given K and y, determine x) and model identification or specification (given x and y, determine K). This last view is an adaptation of the IPO-model, taught in Computer Science. During the last 5 years, we designed and put in practice and experience based-on inverse problems and their utilization in teachers training courses. This area is strongly connected with active learning, since as Kaur observed, an effective mathematics instruction begins when the instructors take the role of designers with the aim of facilitate active learning activities. In this chapter, we reflect on these experiences to construct a wider theoretical framework for inverse problems in STEM education.

Keywords: inverse problems, STEM education, mathematical models, didactic analysis, problem posing, task enrichment, active learning

1. Introduction

There exists a wide range of learning activities, commonly included under the term active learning (AL). Among them, it should be mentioned the computer-assisted learning, project work, role play exercises, small group discussion, individualized work schemes and collaborative problem-solving.

As Lugosi and Uribe [1] observe, there is no definition of AL, used and accepted by everyone, although it is regarded in general as 'classroom practices that engage students in activities that promote higher-order thinking' [2], which includes the analysis, synthesis and evaluation of the information presented rather than receiving it passively [3].

One of the most used definitions is the one proposed by Freeman et al. [4]: 'Active learning engages students in the process of learning through activities and discussion in class, as opposed to passively listening to an expert. It emphasizes higher-order thinking and often involves group-work.'

A similar vision is given by Good & Brophy [5], who argued that 'active learning involves providing pupils with an opportunity in which they raise their own questions and use teachers and other resources to pursue self-defined goal'.

In the same way, Kyriacou [6] states that AL 'can be described as the use of learning activities where pupils are given a marked degree of ownership and control over the learning activities used, where the learning experience is open-ended rather than tightly predetermined, and where the pupil is able to actively participate and shape the learning experience'.

Independently of the definition of the concept, in the last decades an increased importance of the role of AL pedagogies was observed, which includes several reported benefits in teaching STEM disciplines [4, 7]. For instance, Freeman et al. [4] conducted a meta-analysis of 225 studies which reports the positive impact of AL pedagogies on student learning in all the STEM disciplines.

On the other hand, as Wieman [8] stated, mathematics has a different behavior, since it is a very traditional discipline in terms of its teaching and, consequently, it shows more resistance than other STEM areas in adopting teaching methods based on research results.

This is a very important issue, since as Kaur [9] mentioned, an effective mathematics instruction begins when the instructors take the role of designers, and they facilitate the AL activities. Then, in order to change teaching practices, it is necessary to start by working with prospective teachers in their formation courses.

For this reason, during the last 5 years, we designed and put in practice an experience based on inverse problems in teachers training courses.

It is important to mention that despite the essential role of inverse problems in many different disciplines, they were almost forgotten [10] and unfortunately, in the traditional Mathematics Education the situation was the same as mentioned in different works [11–13].

One of the consequences of this fact is that an elaborated theoretical framework for this theme was no developed. In an attempt, by Groetsch [11, 12], the IPO model—which is commonly taught in in Computer Science courses and textbooks [14]—was adapted for this purpose. In this article we propose to construct a wider theoretical framework for inverse problems and their utilization in Mathematics Education and so, the adaptations of the IPO model can be considered just as a starting point for this purpose.

This paper reflects on some examples previously described [14–16] and analyses data obtained in a doctoral thesis fieldwork [17], which concern several dimensions of the Didactic Analysis. Nevertheless, in this first attempt for building a theoretical framework for inverse problems and their role in Mathematics Education, we focus only on the cultural/conceptual dimension of Didactic Analysis [18]. It is important to remark that the chapter objectives, methodology and results interpretation are based on this theoretical framework.

Our main purpose is to consider this chapter as a starting point for future deeper developments about inverse problems in Mathematics Education.

2. Inverse problems in mathematics education

In a recent paper by Mason [19], the following excerpts can be read: 'the pervasive mathematical theme of inverse also known as doing and undoing … ',

'… can be used to get an answer (a doing), it is useful to consider the undoing … ' and 'The pedagogic point here is to emphasize the power of formulating and considering "undoing" or inverse problems'. These excerpts show that—at least for some researchers such as Mason—inverse problems are more or less a synonym of 'undoing'.

As Kunze and collaborators mentioned, 'this distinction is not well-defined' [20] but later, in the same paper they state that 'however, in general, a direct problem involves the identification of effects from causes' in full agreement with Mason's viewpoint.

Long time ago, Keller [21] proposed a different point of view when he says that 'We call two problems inverses of one another if the formulation of each involves all or part of the solution of the other'. After that, he argues that due to historical reasons, '… one of the two problems has been studied extensively … while the other is newer … ' and then, he states that 'In such cases, the former is called direct problem, while the latter is called the inverse problem'. So, in this original point of view, a problem is classified as direct or inverse depending on its own history, more than the inputs and outputs of the proposal.

Outside Mathematics Education, it is possible to find a wider conceptualization, for instance, Groetsch [22] says that 'The type of direct problems we have in mind is that of determining the effect y of a given cause x when a definite mathematical model K is posited: $Kx = y$' and then, he adds '… two inverse problems may be immediately posed. These are the inverse problems of causation (given K and y, determine x) and model identification (given x and y, determine K)'. Then, it is possible to outline the resolution of a conventionally so-called direct problem, as it is showed in **Figure 1**.

In a previous paper [14], it was observed that there is a strong connection with the IPO model (input-process-output), usually taught in computing and information technology courses. The IPO cycle can be represented as in **Figure 2**.

Now, as Groetsch mentioned, in the causation problems the model and the effect are well known, and the question is about the cause, so this situation can be schematized as can be observed in **Figure 3**.

In model identification problems, both cause and effect are known, and the main question consists in determining the model that gives the expected result. This situation is schematized in **Figure 4**.

It is important to remark that in other works [11, 12], the model identification inverse problems were called 'specification problems'.

Figure 1.
Scheme for an initial direct problem, following Groetsch ideas.

Figure 2.
IPO model.

Figure 3.
Scheme for causation inverse problems.

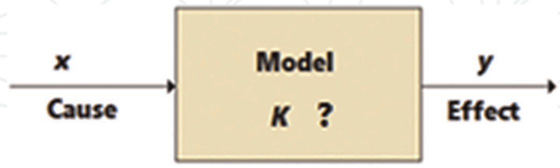


Figure 4.
Scheme for model identification inverse problems.

In a previous paper [13], three different types of inverse problems were described: causation, specification and double-inverse problems. The first two categories correspond to those described by Groetsch, whereas the last one corresponds to those proposals that can be considered at the same time, causation and specification inverse problems. As it was mentioned, the last kind of problems were called 'double inverse problems' by Martinez-Luaces [13].

An interesting example appears in a problems list which was released by PISA a few years ago [23]. One of them is the so-called 'Apartment Purchase' problem which proposes to measure the room sizes of an apartment, as a first approach to calculate the total floor area. Then, the statement adds the following: 'However, there is a more efficient method to estimate the total floor area where you only need to measure four lengths. Mark on the plan above the four lengths that are needed to estimate the total floor area of the apartment'. Since the problem statement asks to measure four lengths in order to obtain the same result of the first approach, it should be considered as a causation inverse problem. On the other hand, in order to choose the correct four lengths, the student must know what to do with those lengths, and so, implicitly it should be considered also as an inverse specification problem.

In the previously mentioned doctoral thesis research [17], it was observed that in several proposals, the student needs to solve a direct problem as a first step, and in a second step, use the obtained result as an input for solving an inverse problem. An interesting example is described in the next paragraphs.

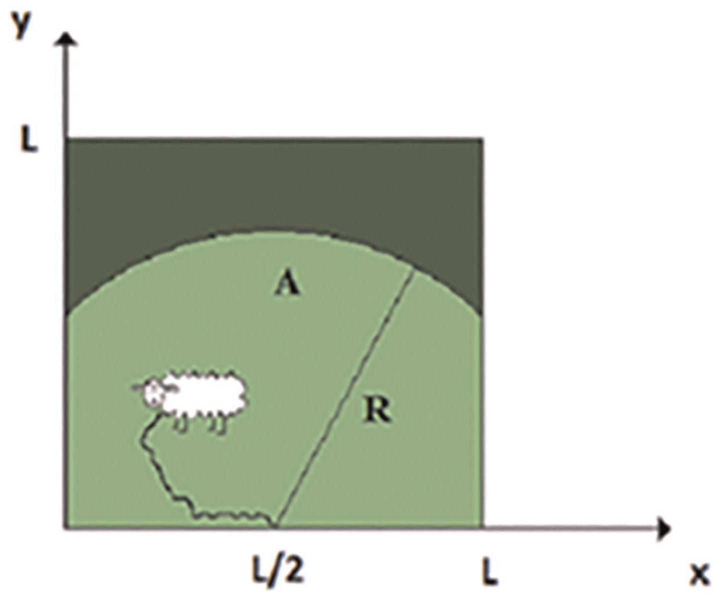
In the fieldwork of this doctoral thesis, the participants—prospective teachers of Mathematics at Secondary School—were asked to reformulate the problem proposed in **Figure 5**.

A prospective teacher proposed an inverse problem (see **Figure 6**), where the field geometry and the stake position are the same of the original statement.

In this proposal, the rope length $R = \frac{L}{3}$ is given, and the statement mentions that the sheep eats all the grass in the accessible area. The problem has several questions, the first one being about R', the rope length that allows the sheep grazing the same amount of grass. After that, other questions ask for the rope lengths corresponding to the third and for the fourth day, once again, in order to eat the same amount of grass of the first day. Finally, the author asks on which day the sheep will not have the same amount of grass to eat?

Problem: accessible area for a sheep in a square field

Let us consider a sheep that grazes in a square field, being L the side length. The sheep is tied with a rope of length R, at the point $(L/2\ ,\ 0)$ as can be observed in the following figure:



In this figure, A represents the area accessible for the sheep, $r = R/L$ is the ratio of the rope length to field side length and $f = A\,/\,L^2$ represents the fraction of the total area accessible for the animal

A possible direct problem consists in asking the students to obtain the fraction of area accesible for the sheep, f, that corresponds to a given value of the ratio r , for instance: Which is the fraction of the total area which is accessible to the sheep if the rope is three-fourths the length of the side of the field?

Figure 5.
Direct problem: Accessible area for a sheep grazing in a field.

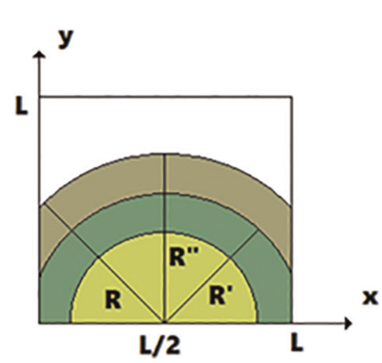


Figure 6.
Sequential inverse problem.

It is easy to observe that this proposal is as a 'mixed inverse problem' with a structure direct-inverse, since firstly it is necessary to solve a direct problem (for the first day) and then, the order of the variables involved is inverted for the following days.

Also, it is not difficult to pose an inverse problem with the opposite structure, i.e. inverse-direct, for instance: For the rope length such that the sheep can eat 50% of the grass, which is the perimeter of the accessible area?

Then, we can describe two subgroups of mixed inverse problems: direct-inverse and inverse-direct.

So, considering the type of inversion, we can describe four groups: causation problems, specification problems, double (both causation and specification) and mixed (direct-inverse or inverse-direct).

It is important to comment that posing inverse statements from an original statement is an important way of enriching mathematical problems. In fact, it is not the only possibility, since modeling can also be considered for task enrichment and of course, both can be combined in the so-called inverse modeling problems, firstly described in a book chapter [24] and a journal paper [25].

Despite the importance of these strategies, inverse problems were almost ignored. For instance, Bunge [10] remarks that 'inverse problems are so difficult and have been so discriminated that the first international congress on the subject was held as late as year 2002' and 'the treaties on the subject can be counted on the fingers of one hand'. In a certain way, this can be considered an unexpected result, since—according to Groetsch [11, 12]—the first mathematical inverse problem was posed almost five centuries ago (a ballistics inverse problem studied by Tartaglia in 1537, which consisted of determining the elevation angle θ for reaching a given shot range R).

The same situation happens in mathematics education, where inverse problems have been ignored, at least in traditional courses [11, 13].

Because of these facts, it is not surprising to observe the absence of a theoretical framework adequate to study the inversion of a statement as a task enrichment strategy in mathematics education. In order to construct this theoretical framework, firstly we need to analyze three related topics: problem posing, task enrichment and Didactic Analysis, the last one being the tool we have chosen for developing this frame.

3. Theoretical framework

As we already mentioned, there are three topics that deserve to be considered to analyze the inversion as a strategy of task enrichment in mathematics education, with'problem posing' being the first one.

3.1 Problem posing

Problem posing is a traditional research area, where important authors such as Brown and Walter [26, 27], English [28, 29], Kilpatrick [30], Silver and Cai [31] produced several milestone papers. When those authors use the term 'problem posing', they include new problems formulations but also reformulations of given statements. These problem posing activities are developed in different ways, with more or less structured formats [28, 29, 31–33].

An interesting case takes place when students need to pose a different problem as part of the solving process of the given one [34]. This situation already appeared in the works of Polya [35], who proposes this possibility as a possible strategy which consists of establishing variants, such as discarding one or more of the original statement conditions.

It is important to remark that problem reformulation is not always a strategy for problem-solving. In fact, in some works problem invention starts from an experience or a situation previously given [32, 33].

It is also possible to combine the last previous approaches, for instance, the students are asked to solve a certain problem, where one of the conditions was changed, and for this reason, it should be considered as a new problem [32].

Brown and Walter [26, 27] proposed a different approach for obtaining new problems, in a strategy that they called 'What if not?'. Their approach consists of changing the problem conditions or restrictions to create a new statement. This strategy may lead to interesting causation/specification inverse problems.

In the work of Stoyanova [36], problem-posing activities are classified into three groups: free, semi-structured and structured. In free problem posing activities, there are no restrictions. In the case of the semi-structured activities, problem posing is based on a previous experience and/or some quantitative information. Finally, in structured problem posing, the original statement problem is reformulated (or some conditions are changed) in order to obtain a new one. In our research a direct problem was given to the participants, and they were asked to propose an inverse reformulation. As a consequence, our fieldwork constitutes an example of structured situation, if the previous classification is followed [36].

3.2 Task enrichment

In our research, problem posing is carried out for tasks enrichment as the final goal. It is important to remark that Lester and Cai [37] stated '... teachers can develop worthwhile mathematical tasks by simply modifying problems from the textbooks'.

Other authors, such as Santos and Barmby [38], observed that 'the question of what is meant by enrichment has been an ongoing question for researchers'. Moreover, Barbe [39] says 'an aura of vagueness and confusion seems to surround the term' and 40 years after that, Feng [40] concluded that 'no overall consensus has yet been reached on the definition and nature of enrichment'.

Taking into account the previous comments, what is a rich task that consists more of a description than a precise/exact term definition. An example of this approach is given by Grootenboer's works [41], since he makes a description of 'the key aspects of rich mathematical tasks', like the following:

- Academic and intellectual quality.
- Extended engagement.
- Group work.
- Attention to the diversity, achieved through multiple solution pathways and entry points.
- Connectedness.
- Multi-representational'

After discussions with several teachers, Clarke and Clarke [42] suggested their own list of characteristics for 'rich assessment tasks'. Some of the selected characteristics are the following:

- 'Address several outcomes in only one task.
- Allow all the students to make a start.
- Different approaches and methods can be successfully utilized.
- Encourage students to reveal their own understanding of what they have learned.
- Allow students to make connections between the concepts they have learned.
- Authentically represents the forms in which mathematical knowledge and skills will be used in the future.
- Allow students to transfer knowledge from a well-known context to a new, less familiar one'

As it was mentioned, this list is about 'rich assessment tasks', although it can also be used for 'rich tasks' characterization. Obviously, those characterizations are not a definition of rich tasks, but they reasonably describe what can be expected in task enrichment activities.

As it was mentioned, in our research we consider inversion as another tool for task enrichment, so there is a strong connection between Section 2 and sub-Section 3.2 of this work.

Finally, as Bonwell and Eison [2] remarked 'Though the term "active learning" has never been precisely defined in educational literature, some general characteristics are commonly associated with the use of strategies promoting active learning in the classroom' and particularly, they mentioned the following ones:

- 'Students are involved in more than listening.
- Less emphasis is placed on transmitting information and more on developing students' skills.
- Students are involved in higher-order thinking (analysis, synthesis, evaluation).
- Students are engaged in activities (e.g., reading, discussing, writing).
- Greater emphasis is placed on students' exploration of their own attitudes and values'.

It is easy to observe a strong connection with the characteristics of rich tasks described by Santos and Barmby [38] and Clarke and Clarke [42], and most of these characteristics are present in our fieldwork, described below, in Section 4.

3.3 Didactic analysis

This sub-section is devoted to briefly describe the characteristics of the Didactic Analysis. The didactic Analysis considers 'the structure, levels, dimensions, categories, and components to perform and organize the didactic school mathematics content analysis and how it makes possible the design, implementation and evaluation of teaching and learning activities, corresponding to any specific mathematics subject' [18, 43, 44].

'This description gives rise to a cyclical structure, where the information obtained in each analysis will be essential for a new implementation of the didactic analysis' [45].

The system of components, categories and contents for the didactic analysis of the mathematics school curriculum is briefly summarized in **Figure 7**. **Figure 7** is a remake of Table 2 of a previous paper written by one of the co-authors of the chapter[18].

In our research, didactic analysis was the main tool for the design of the fieldwork and also for the prospective teachers'productions analysis. However, due to length constraints, in this chapter, we analyze only the conceptual content analysis of the proposals, so our focus is placed on the meanings analysis (i.e. the second column of **Figure 7**).

DIDACTICAL ANALYSIS CATEGORIES			
Dimensions			
Cognitive Content Analysis	**Conceptual Content Analysis**	**Instruction Content Analysis**	**Evaluative Content Analysis**
Study object			
Intentionality and learning conditions of school mathematics	Meanings of school mathematics contents	Planning and implementation of mathematics teaching	Evaluation and decision making from learning achievements
Organizers or categories used to perform the content analysis of the curricular dimensions			
1. Learning expectations	1. Conceptual structure	1. Tasks and sequences	1. Modalities and design
2. Limitations	2. Representation systems	2. Classroom work organization	2. Intervention and decision making
3. Opportunities to learn	3. Senses and modes of use	3. Materials and resources	3. Quality indicators
Components of the organizers to analyze didactically school mathematics			
1. Objectives, competencies	1. Formal and Cognitive functionality, emotional, moral, and ethical attitudes	1. Task variables and their functions	1. Functions, regulations, and moments
2. Errors, difficulties, blockages	2. Symbolic, graphic, numerical representations	2. Complexity, creativity, and organization	2. Criteria, instruments, and performance
3. Conditions, demands, challenges	3. Terms, contexts, phenomena, modes of use	3. Characteristics, types, and uses	3. Strategic and comparative studies
Synthesis of main contents			
Task learning structure and coherence	Priority of meanings for teaching and learning	Teaching organization through units	Quality of the achieved learning
Mathematics didactic content obtained as synthesis of elements encompassed by didactical analysis			

Figure 7.
Components, categories and contents for the didactic analysis.

It is important to comment that the meaning concept in this work based on the so-called semantic triangle, which corresponds to an interpretation of Frege's works [46–48]. Following this frame, the mathematical content analysis can be organized into three categories: the contents' structure, the representation systems and the senses or modes of use.

The conceptual structure takes into account the relations of the concepts and the procedures which are involved in the studied content, paying attention to the mathematical structure that includes them.

Representation systems are related to the different forms of 'representing a mathematical content, which can be expressed through signs, graphics, symbols, rules, relationships, conventions, along with their translations into other concepts and conversions according to different procedures'.

Finally, the sense considers the modes of use of the content, which includes phenomena, situations and contexts that give meaning to the mathematical content.

4. Research objectives and methodology of a related study

The general objective of this work consists of characterizing and identifying prospective teachers' strategies to propose inverse problems and reflect on those experiences, in order to clarify and construct a wider theoretical framework for inverse problems in STEM education.

The specific objectives related to the general objective are:

- Characterization of the statements corresponding to the inverse problems reformulations proposed by the participants.

- Characterization of the complexity in terms of the resolution process, corresponding to the inverse problems proposed by the prospective teachers.

- Organize the prospective teachers' proposals and describe and explain the different groups obtained.

For this purpose, the fieldwork of this research was carried out with two groups of prospective teachers at the University of Granada, Spain, during the academic year 2018–2019. The first group of prospective teachers (Group A) had 32 students and the second one (Group B) consisted of 33 students.

It is important to mention that both groups' professors (Moreno in Group A) and Ruiz-Hidalgo in Group B) collaborated with this research, which was developed in two sessions. In the first one, a brief explanation about inverse problems was given to the participants, and also a few examples were discussed. As a homework of this first session, an inverse reformulation of a given problem was requested. That problem was about a swimming pool filled with water, and the reformulation should be considered as a task enrichment of the original statement to be utilized in Secondary School courses.

In the second session, the prospective teachers' reformulations corresponding to the swimming pool problem were analyzed with the whole group. After the swimming pool reformulations discussion, a new problem (the sheep problem, **Figure 5**) was provided for reformulation to the participants as a new homework. There were several differences with the first homework, since the participants were asked to solve the given (direct) problem and then propose an inverse reformulation and solve their own proposal.

It is important to remark that only a few reformulations were discarded because they corresponded to ill-posed problems, or the reformulation cannot be considered an inverse one.

It should be mentioned that several participants proposed two or even three inverse reformulations. Then, there is not a single proposal for each participant, and not all the proposals were considered for this study. As a consequence, the reformulations were coded by responses rather than coding by author.

For the classification of the prospective teachers' reformulations corresponding to the sheep problem, several criteria were taken into account. One of the most important elements analyzed was the kind of inversion, but also possible changes in the geometry, the use or not of external variables, among others criteria, were considered.

As mentioned, firstly, inverse problems were classified in terms of the type of inversion (i.e. causation, specification, etc.). Nevertheless, there are several variants that deserve to be considered, for instance, a general inversion of the given function can be proposed, or an inversion just for a particular value and even an interval inversion, among other possibilities. A similar situation happens with specification inverse problems, where the prospective teacher proposal may ask about parameters interpretation, graphical representations or even a more creative solution without using integrals. Lastly, double and mixed inverse problems were also found among the proposals.

A similar analysis can be made for the item 'changes in geometry', since they can include changes in the shape of the field and/or the stake position as well as other obstacles, such as fences, not included in the original problem.

Considering all these variables, a first classification was made, in order to summarize the information about the reformulations (and their authors) in a table which is partially shown in **Figure 8**.

For this first classification (see **Figure 8**), a list of the variables considered are as follows:

- Type of inversion, as discussed in Section 2.
- Inputs of the proposal (particular or general values, an integral, etc.).
- Outputs of the problem (values, croquis, a process, etc.).
- Changes in geometry (field, stake, fences, etc.).
- Other elements (additional variables and/or change of context).
- Solution (geometrical, numerical, etc.).
- Difficulty (trivial, low, medium and high).

Finally, considering this classification, it was possible to observe three groups of problems, which will be deeply analyzed in the following section.

5. Results

In the previous analysis, we used the following nomenclature: firstly, the capital letters 'PT' indicates that the participant is a prospective teacher. Secondly, the letters are followed by a two-digit number (which corresponds to the student number), and

Group B Sheep Problem	Direct/ Inverse	Type of Inversion	Geometry and Dimensions	Other elements (fences, costs, etc.)	Difficulty	Solution	Input	Output	Comments
PT1	Inverse	Several pointwise inversions	Same Geometry No numbers	No	Greater	Incorrect and incomplete solution	f	r	Incomplete solutions for both direct and inverse problems
PT2	Inverse	Sketch of the region	Same Geometry No numbers	Change of context: cobblestone town area	Conceptual problem	From the integral the curve and the boundary terms are obtained	f (R, L)	Sketch	Gives f (R, L) as an integral
PT3	Inverse	Sketch of the region	Same Geometry No numbers	No	Conceptual problem	From the integral the curve and the boundary terms are obtained	f (R, L)	Sketch	Similar to PT2, but keep the original context
PT4	Inverse	Sketch of the region	Same Geometry No numbers	No	Conceptual problem	Gives the integral and the change of variable. Reverts the substitution and obtains the region	A as integral with its change of variable	Sketch	Gives A as an integral after the change of variable
PT5	Inverse	Tends to ask for the sketch of the region	Same Geometry No numbers	Change of context: irrigation and fertilize part of the field	Conceptual problem	Gives f as a quotient of the integral and L^2 and from that obtains the region	Gives f as a quotient of the integral and L^2	Sketch	Sketch is not requested explicitly, but it arises from the solution

Figure 8.
Example of a first classification of the proposals.

finally, in some cases there is a small letter, which refers to the number of reformulation considered, if it is applicable. As an example, the code PT23c corresponds to a proposal authored by 23rd prospective teacher, and it can be observed that this participant proposed at least three reformulations (here, the one considered is the third).

As it was mentioned, the analysis of the proposals allowed us to identify three groups of problems.

- A first group of proposals related to procedural knowledge (skills).
- A second group that involves graphic representations.
- A third group of proposals related to conceptual and deep knowledge (reasoning and strategies).

5.1 The first group of problems

The first group of problems is formed by prospective teachers' proposals based on the procedural content knowledge. An interesting example is given by the production PT31a, which considers a square field $[0, L] \times [0, L]$, where the sheep is tied at $(L/2, 0)$. The situation is sketched in **Figure 9**.

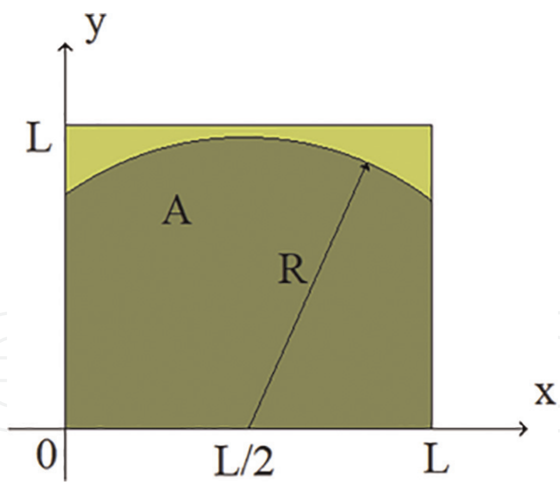


Figure 9.
Sketch corresponding to the proposal coded as PT31a.

The author mentions that $r = \frac{R}{L}$ represents the ratio of the rope length to field side length, and he/she gives the following data: $L = 20m.$ and $f = \frac{A}{L^2} = 0.877$ with A being the area accessible for the sheep. The required output of this proposal is the radius R.

In his/her solution, the prospective teacher divides the area A into three parts (a circle sector and two right triangles) as in **Figure 10**.

Since the circumference is given by $(x - L/2)^2 + y^2 = R^2$ for $x = 0$ it is easy to obtain $h = \sqrt{R^2 - (L/2)^2} = \sqrt{R^2 - \frac{1}{4}L^2}$, then each triangle has an area equal to $A(T) = \frac{1}{2}\frac{L}{2}\sqrt{R^2 - \frac{1}{4}L^2} = \frac{1}{4}L^2\sqrt{r^2 - \frac{1}{4}}$. Moreover, the angle θ in those triangles can be obtained as $\theta = \cos^{-1}\left(\frac{L/2}{R}\right)$, and then the angle of the circle sector can be written as: $\alpha = \pi - 2\cos^{-1}\left(\frac{L/2}{R}\right)$. So, the circle sector area is $A(CS) = \frac{\alpha}{2\pi}\pi R^2 = \left[\frac{\pi}{2} - \cos^{-1}\left(\frac{L}{2R}\right)\right]R^2$, which can be written as $A(CS) = \left[\frac{\pi}{2} - \cos^{-1}\left(\frac{1}{2r}\right)\right]R^2$, and then a non-linear equation is obtained: $f(r) = \frac{1}{2}\sqrt{r^2 - \frac{1}{4}} + \left[\frac{\pi}{2} - \cos^{-1}\left(\frac{1}{2r}\right)\right]r^2 = 0.877$.

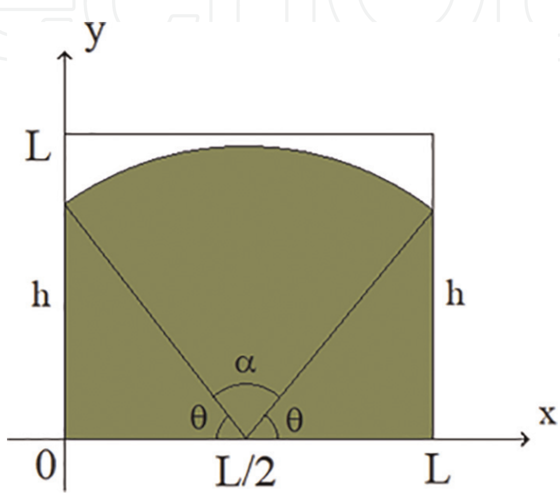


Figure 10.
Sketch corresponding to the solution of the proposal coded as PT31a.

The prospective teacher solves this non-linear equation in r by using Bisection Method, resulting in $r = 0.925$ and finally $R = 18.5$ m.

As it can be observed, the prospective teacher solved his/her proposal without using integrals. As a final remark, it should be mentioned that the problem solution mainly requires procedural knowledge.

5.2 The second group of proposals

This second group is very homogeneous and contains only proposals based on the use of different representation systems. The production PT03 gives an interesting example. This proposal utilizes the same geometry of the given direct problem and adds a different input. The participant proposes that e $f = \frac{1}{L^2}\int_0^L \sqrt{R^2 - (x - L/2)^2}\, dx$, gives the fraction of area where the sheep may graze, where $R,\ L > 0$. A sketch of the situation descripted in the previous statement is the required output of this reformulation.

In his/her solution, the author observes that $y = \sqrt{R^2 - (x - L/2)^2}$, is the equation of a semi-circumference centred in the point $\left(\frac{L}{2}, 0\right)$ with a radius R and L^2 is the area of a square with a field side length L.

From these observations, the requested region can be obtained, and the corresponding sketch is the same as in **Figure 9**.

5.3 The third group of inverse problems

The conceptual content knowledge is fundamental for solving the third group of proposals. For instance, the production coded as PT23c is a representant of this third group. In his/her proposal, the geometry of the field remains the same and the participant asks for the stake position and radius R to maximize the accessible area for the sheep with an extra condition: the animal should not reach the boundary of the field.

Obviously, the stake should be in the centre of the field, i.e. at the point $\left(\frac{L}{2}, \frac{L}{2}\right)$ and the radius proposed by the author is $R = L/2$, as illustrated in **Figure 11**.

It should be mentioned that the proposed solution—sketched in **Figure 11**—is a supreme, not a maximum. The other important remark is that the solution basically requires conceptual knowledge and no geometrical, analytical and/or numerical procedures are necessary to solve it.

6. Discussion

In this first attempt to construct a theoretical framework for inversion as a task enrichment strategy, we focused only in the cultural/conceptual dimension of the didactic analysis. Although we have concentrated on this single dimension, we have observed that there several different ways to classify inverse problems according to the type of inversion, input/output, type and difficulty of the solution and external variables/context changes, among others.

As it was mentioned, considering the type of inversion—which is related to the inputs and outputs of the proposal—we can consider four types of inverse problems:

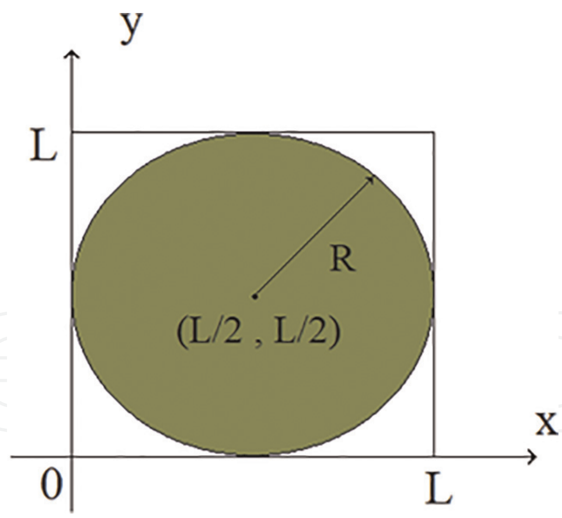


Figure 11.
Sketch corresponding to the solution of the proposal coded as PT23c.

- Causation inverse problems.
- Specification inverse problems.
- Double or dual inverse problems.
- Mixed (divided into two subgroups: direct-inverse and inverse-direct).

Nevertheless, this is not the only possible classification. In fact, this classification does not explain the results obtained in Section 5, where three groups of proposals were observed. Then, in order to explain the previous result, another classification based on the semantic triangle should be considered. Then, we can also classify inverse problems into three groups:

- Conceptual inverse problems.
- Procedural inverse problems.
- Representational inverse problems.

These classifications fit well with the three groups observed in Section 5 and at same time, have a stronger connection with the framework utilized (the didactic analysis).

Another option is to classify inverse problems in pure or combined with another task enrichment strategy, such as modeling. This fact was already observed in previous papers [24, 25] where inverse modeling problems were studied as a different group from other inverse problems.

It should be remarked that in the fieldwork, it was observed that several participants decided to add other variables to the original problem and/or make changes in the context. The external variables can be physical, chemical, biological and economical. One proposal asked for the amount of herbicide and other asked for the amount of fertilizer for the grass, which are chemical external variables. Other participant asked for the velocity of a sheep, which is a physical variable, other one asked for the cost of a fence per unit of length (economical variable) and finally, there was a proposal which requested about the kilograms of grass the sheep can eat per day (i.e. a biological parameter).

About the context changes, for instance, one of the participants (PT22b) considers a goalkeeper that throws the ball in a handball field and which can cover a certain area (instead of a field where a sheep is grazing). Another one (PT27) proposed considering a bush fire without wind. This fire evolves covering concentric circles and the proposal asks for the point where the fire started, knowing that $R = L$ and f is 95.60%. It must be remarked that PT22b and PT27 are just a couple of examples where the context is different from the original proposal.

So, the inclusion or not of external variables and/or the context changes can be used as criteria for a different classification into four groups:

- Inverse problems with the same variables and context.
- Inverse problems where external variables were added.
- Inverse problems in a different context.
- Inverse problems with external variables and a different context.

Another option considers the inputs and outputs of the proposal. These inputs and/or outputs can be particular values (like in PT33, where the area accessible for the sheep is $A = 433.36\,m^2$) or general ones (like in PT27, where $R = L$), a croquis (like PT03, where a sketch of the region where the sheep may graze is requested) or even a process (like PT34, who asks for a criterion that allows distinguishing among different geometries in terms of the grazing area).

Lastly, it should be mentioned that the problem may be classified by analyzing its possible solutions. Then, the proposals can be classified in terms of the difficulty (trivial problems, low, medium and high difficulty) and the mathematics branches involved (analytical, trigonometrical, algebraic and numerical solutions, among others). These possibilities constitute more 'classical' options for classifying the prospective teachers' productions.

As a final comment, it is important to observe that these different criteria do not contradict previous works of other authors such as Groetsch [22] or Mason [19], or our own previous works [14–16]. On the contrary, we believe that they complement those previous works, giving a broader vision of inversion as a strategy for task enrichment.

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