Annali di Matematica (2005)

Digital Object Identifier (DOI) 10.1007/s10231-004-0127-3

Gian Paolo Leonardi · Simon Masnou

On the isoperimetric problem in the Heisenberg group $\mathbb{H}^n$

Received: December 17, 2003
Published online: ■ ■ ■ 2005 –

Abstract. It has been recently conjectured that, in the context of the Heisenberg group $\mathbb{H}^n$ endowed with its Carnot–Carathéodory metric and Haar measure, the isoperimetric sets (i.e., minimizers of the $\mathbb{H}$-*perimeter* among sets of constant Haar measure) could coincide with the solutions to a "restricted" isoperimetric problem within the class of sets having finite perimeter, smooth boundary, and cylindrical symmetry. In this paper, we derive new properties of these restricted isoperimetric sets, which we call *Heisenberg bubbles*. In particular, we show that their boundary has constant mean $\mathbb{H}$-curvature and, quite surprisingly, that it is foliated by the family of minimal geodesics connecting two special points. In view of a possible strategy for proving that Heisenberg bubbles are actually isoperimetric among the *whole* class of measurable subsets of $\mathbb{H}^n$, we turn our attention to the relationship between volume, perimeter, and ϵ-enlargements. In particular, we prove a Brunn–Minkowski inequality with topological exponent as well as the fact that the $\mathbb{H}$-perimeter of a bounded, open set $F \subset \mathbb{H}^n$ of class C^2 can be computed via a *generalized Minkowski content*, defined by means of any bounded set whose horizontal projection is the $2n$-dimensional unit disc. Some consequences of these properties are discussed.

Mathematics Subject Classification (2000). 28A75, 22E25, 49Q20

1. Introduction

It is well known that Euclidean balls in $\mathbb{R}^n$ are, up to negligible sets, the unique solutions to the *isoperimetric problem* in $\mathbb{R}^n$, that is, the unique minimizers of the *perimeter* among all measurable sets with the same n-dimensional Lebesgue measure. Therefore, we say that Euclidean balls are *isoperimetric sets* in $\mathbb{R}^n$.

Here we consider the isoperimetric problem in the *Heisenberg group* $\mathbb{H}^n$, where the Euclidean geometry of $\mathbb{R}^{2n+1}$ is replaced by a *sub-Riemannian* geometry induced by a certain family of *horizontal* vector fields. Recent years have seen growing interest in the study of sub-Riemannian spaces (and even more general metric measure spaces) from the viewpoint of the theory of BV functions and sets of finite perimeter, and, more generally, in the framework of geometric measure theory (see, for instance, [2–4, 14–16, 18, 22, 27]). These spaces naturally arise from

G.P. Leonardi: Dipartimento di Metodi e Modelli Matematici, Università di Padova, via Belzoni 7, 35131 Padova, Italy, e-mail: leonardi@dmsa.unipd.it

S. Masnou: Laboratoire Jacques-Louis Lions, B.C. 187, Université Pierre-et-Marie-Curie, 75252 Paris Cedex 05, France, e-mail: masnou@ann.jussieu.fr

different areas of mathematics and physics, such as harmonic analysis, control theory, nonholonomic mechanics [1,5,11,12], and, recently, from the theory of human vision [8].

Before giving the definition and discussing some properties of $\mathbb{H}^n$, let us point out the relationship between the isoperimetric problem and the *isoperimetric inequalities*. We recall that both $\mathbb{R}^n$ and $\mathbb{H}^n$ belong to the wider class of *Carnot groups*, i.e., structures of the form $(\mathbb{G}, \cdot, \delta_\lambda, d_c)$, where $(\mathbb{G}, \cdot)$ is a connected and simply connected Lie group, δ_λ is a (family of) dilation(s), and d_c is the *Carnot–Carathéodory metric* (see Section 2 for more precise definitions concerning $\mathbb{H}^n$). It is known that isoperimetric inequalities of the type

$$C\,|F|^{\frac{Q-1}{Q}} \le P_{\mathbb{G}}(F) \tag{1.1}$$

hold for all measurable $F \subset \mathbb{G}$ with $|F| < \infty$ and for some positive C depending only on $\mathbb{G}$ [18,29]. Here, $|\cdot|$ denotes the n-dimensional Lebesgue measure (the *Haar measure* of the group $\mathbb{G} \simeq \mathbb{R}^n$), Q is the *homogeneous dimension* of $\mathbb{G}$, and $P_{\mathbb{G}}(F)$ denotes the $\mathbb{G}$-perimeter defined with respect to the family of horizontal vector fields (Section 2). Since $|\cdot|$ and $P_{\mathbb{G}}$ are, respectively, Q and $(Q-1)$-homogeneous with respect to dilations δ_λ, one can write (1.1) for $|F| = 1$ and easily obtain that the best (largest) constant C that can be plugged into (1.1) is exactly the infimum of $P_{\mathbb{G}}(F)$ under the volume constraint $|F| = 1$, that is, the perimeter of any possibly existing isoperimetric set, scaled to have unit volume.

The existence of isoperimetric sets in Carnot groups was recently proved in [21], where some general properties of those sets are also carried out: more precisely, one can show that these sets are bounded, with Alhfors-regular boundary verifying a condition of "good" geometric separation (the so-called *condition B*). Moreover, at least for Carnot groups of *step* 2 and in particular for the Heisenberg group $\mathbb{H}^n$, the connectedness can also be proved as a consequence of being a *domain of isoperimetry*. Yet a more precise characterization of isoperimetric sets in a general Carnot group is still an open (and difficult) problem.

One could expect that the natural candidate isoperimetric sets in $\mathbb{H}^n$ are the balls associated to the Carnot–Carathéodory metric, as they are the counterparts of the Euclidean balls in $\mathbb{R}^n$. However, as shown in a recent work by Monti [25], these balls are *not* isoperimetric. In the particular case of the first Heisenberg group $\mathbb{H}^1$, a reasonably good approximation of an isoperimetric set can be obtained as the output of a numerical simulation, which we have performed with Brakke's Surface Evolver [6]. This simulation finds a theoretical justification in an approximation result of sets of finite $\mathbb{H}$-perimeter with polyhedral sets [23]. Starting from different polyhedra as initial configurations, the minimization of the $\mathbb{H}$-perimeter at constant volume leads, up to left translations, to a unique, apparently smooth, and convex body with an evident cylindrical symmetry (Figure 1) plus a symmetry with respect to the z-plane (recall that the points of $\mathbb{H}^n$ can be seen as the pairs $[z,t] \in \mathbb{C}^n \times \mathbb{R} \simeq \mathbb{R}^{2n+1}$). Of course, the simulation cannot guarantee that what we find is a global minimizer instead of a local one, but it surely adds credibility to the natural conjecture about the symmetries of such isoperimetric sets, which should be coherent with the symmetries of $\mathbb{H}^1$: indeed, all rotations around the t-axis, as well as the map $(x, y, t) \mapsto (x, -y, -t)$, are automorphisms of $\mathbb{H}^1$ (see [11]) preserving both the volume and the $\mathbb{H}$-perimeter.

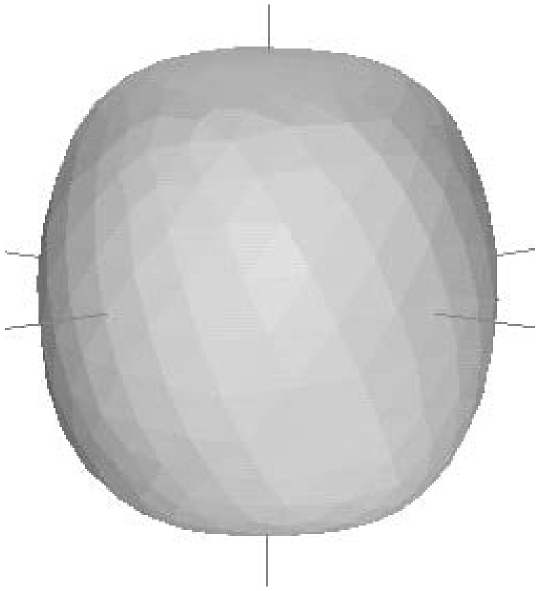

Fig. 1. Using Brakke's Surface Evolver, the minimization of the $\mathbb{H}$-perimeter at constant volume in the Heisenberg group $\mathbb{H}^1$, taking a polyhedron as a starting configuration, produces this approximate "isoperimetric" set that notably differs from the Carnot–Carathéodory ball having the same volume (which, for instance, is not convex – see the figure in [25])

Motivated by the results of our simulations and generalizing to $\mathbb{H}^n$, $n \geq 1$, we are naturally led to consider a "restricted" isoperimetric problem, that is, the minimization of the ratio $P_{\mathbb{H}}(F)/|F|^{\frac{Q}{Q-1}}$ on the subclass $\mathcal{F}$ of sets F whose boundary ∂F can be decomposed as the union $S^+ \cup S^-$, where $S^+ = \partial F \cap \{t \geq 0\}$ is the graph of some radial, smooth, and nonidentically zero function $g(z) = f(|z|)$, whereas S^- is the symmetric of S^+ with respect to the z-plane.

It can be proved that this restricted isoperimetric problem admits solutions (Theorem 3.3) that we call *Heisenberg bubbles*. We believe that these are the right candidates to solve the (global) isoperimetric problem in $\mathbb{H}^n$, as suggested by our numerical results and, above all, because their intrinsic mean curvature turns out to be constant (Theorem 3.3), as happens for Euclidean balls in $\mathbb{R}^n$. Unfortunately, our belief remains conjectural because it is still unknown whether isoperimetric sets are cylindrically symmetric and have a smooth boundary. In addition, if this symmetry seems natural in $\mathbb{H}^1$ for the reasons mentioned before, it is less evident in $\mathbb{H}^n$ when $n \geq 2$, because a generic rotation around the t-axis is no longer necessarily a group automorphism.

Nevertheless, we find new properties of Heisenberg bubbles that could be of help in the search for a rigorous proof of their optimality and also for a better understanding of the geometry of $\mathbb{H}^n$ in general. The first property is the previously mentioned fact that the mean $\mathbb{H}$-curvature of the boundary of a Heisenberg bubble (following the quite natural definition proposed in [31]) is constant, and this agrees with the Euclidean case, where balls verify precisely the same property. The second one, which is indeed the more interesting and surprising one, is the fact that the boundary of any Heisenberg bubble is foliated by the (infinitely many) geodesics connecting the *north pole* and the *south pole* of the bubble. As we learned after the first redaction of this work, this quite unexpected property was observed also by Pansu (see the last few lines of [30]). It clearly has no Euclidean counterpart (recall that Euclidean geodesics connecting two given points are reduced to a single segment!) and can be checked very easily once one knows the equations of geodesics in $\mathbb{H}^n$ and the explicit solutions to the restricted isoperimetric problem.

In addition, in a recent work by Monti and Morbidelli [28], the solutions to the isoperimetric problem are completely characterized in the so-called Grushin plane, that is, the Carnot–Carathéodory space generated on $\mathbb{R}^2$ by the vector fields $X = \frac{\partial}{\partial x}$ and $Y = |x|\frac{\partial}{\partial y}$. It turns out that any isoperimetric set in the Grushin plane coincides with the 2-dimensional slice obtained cutting a symmetric Heisenberg bubble with any vertical plane containing the t-axis; moreover, the boundary of such isoperimetric sets is foliated by (two) geodesics, as happens for Heisenberg bubbles. Therefore, these results seem to confirm the conjecture that Heisenberg bubbles are the unique isoperimetric sets in $\mathbb{H}^n$.

Among the various techniques for proving that Euclidean balls are isoperimetric in $\mathbb{R}^n$, one could try to generalize first those involving a symmetrization procedure (Steiner symmetrization, Schwartz symmetrization) and the one based on the Brunn–Minkowski inequality [7,9]. However, the question of whether there exists in $\mathbb{H}^n$ a symmetrization procedure that preserves the Haar measure and does not increase the $\mathbb{H}$-perimeter is still wide open. On the other hand, the Brunn–Minkowski-based technique could be described in quite general terms as follows (we thank Zoltan Balogh for pointing out this observation to us). Let X be a space on which a binary operation $*$, a volume measure $|\cdot|$, a perimeter measure $P(\cdot)$, and a family of dilations δ_ϵ, $\epsilon > 0$ are defined, in such a way that

C1 Volume and perimeter measures are, respectively, Q and $(Q-1)$-homogeneous with respect to dilations, for some $Q > 0$;

C2 There exists a family of "regular" subsets of X that is dense (with respect to volume and perimeter) in the family of $|\cdot|$-measurable subsets of X with finite $|\cdot|$ measure;

C3 The perimeter of any regular set $F \subset X$ is finite and coincides with its *Minkowski content* $\mathcal{M}_B(F)$, defined as

$$\mathcal{M}_B(F) = \lim_{\epsilon\to 0^+} \frac{|F * \delta_\epsilon(B)| - |F|}{\epsilon}, \tag{1.2}$$

with $B \subset X$ denoting a suitable regular set whose volume and perimeter are both finite and positive, and such that $P(B) \le Q|B|$ (thus, B plays the role of the unit ball in $\mathbb{R}^n$);

C4 The *Brunn–Minkowski-type inequality*

$$|F * G|^{\frac{1}{Q}} \geq |F|^{\frac{1}{Q}} + c|G|^{\frac{1}{Q}} \tag{1.3}$$

holds for any pair of regular sets $F, G \subset X$ and for $c \in (0, 1]$ given by

$$c = \frac{P(E)}{P(B)},$$

where E is a suitable regular set satisfying $|E| = |B|$ (and playing the role of the *candidate isoperimetric set*).

It is then quite easy to prove the following.

Proposition 1.1. *If conditions C1–C4 are verified, then set E is isoperimetric, i.e., it minimizes the perimeter among all sets with the same volume.*

Proof. Take B as above and let F be a regular set with $|F| = |B|$. Using C1, C3, and C4 we deduce, for a fixed $\epsilon > 0$,

$$\frac{|F * \delta_\epsilon(B)| - |F|}{\epsilon} \geq \frac{|F|[(1 + c\epsilon)^Q - 1]}{\epsilon} \geq c\, Q|F| \geq P(E),$$

hence by taking the limit as $\epsilon \to 0$ we obtain $P(F) \geq P(E)$. Finally, by means of C2 and by an easy argument involving the dilations (needed to make sure that the approximation with regular sets can be done also by keeping the volume fixed), one obtains that E minimizes the perimeter among all measurable sets $F \subset X$ with the same volume. □

We first remark that the previous result holds when $X = \mathbb{R}^n$, $*$ equals the standard sum of vectors, $Q = n$, $|\cdot|$ and $P(\cdot)$ are the Euclidean volume and perimeter measures, δ_ϵ is the usual multiplication by a positive scalar ϵ, and both E and B coincide with the Euclidean unit ball.

It is well known that C1, C2, and C3 hold true in any Carnot group [13, 16, 27]. Concerning C4, and in the particular case of the Heisenberg group $\mathbb{H}^n$, we are able to prove the following Brunn–Minkowski inequality (Theorem 4.1):

$$|F \cdot G|^{\frac{1}{d}} \geq |F|^{\frac{1}{d}} + |G|^{\frac{1}{d}}, \quad \text{for all measurable sets } F, G \subset \mathbb{H}^n, \tag{1.4}$$

where $d = 2n + 1$ is the *topological dimension* of $\mathbb{H}^n$. Apart from some technical modifications, the proof follows the line of the classical proof for the Euclidean case, as can be found, for instance, in [10, 17].

The question now arises of whether a similar inequality could hold with a larger parameter d, and, in particular, with $d = Q = 2n + 2$ the *homogeneous dimension* of $\mathbb{H}^n$. Unfortunately, it has already been observed by Monti [26] that the inequality $|F \cdot G|^{\frac{1}{Q}} \geq |F|^{\frac{1}{Q}} + |G|^{\frac{1}{Q}}$ cannot be satisfied, since otherwise it would imply that Carnot–Carathéodory balls are isoperimetric, which is known to be false [25, 26].

We shall extend here this negative result, proving that actually for any $c \in (0, 1]$, the Brunn–Minkowski-type inequality

$$|F \cdot G|^{\frac{1}{Q}} \geq |F|^{\frac{1}{Q}} + c|G|^{\frac{1}{Q}} \tag{1.5}$$

fails to be true in general (Proposition 4.10), thus showing that the strategy à la Brunn–Minkowski mentioned above cannot be used to prove that Heisenberg bubbles are isoperimetric sets in $\mathbb{H}^n$.

The proof of Proposition 4.10 relies on an interesting fact concerning the computation of the intrinsic Minkowski content in $\mathbb{H}^n$, defined as

$$\mathcal{M}_B(F) = \lim_{\epsilon \to 0^+} \frac{|F \cdot \delta_\epsilon(B)| - |F|}{\epsilon},$$

where B denotes the unit ball with respect to the Carnot–Carathéodory distance. By [27], one knows that $\mathcal{M}_B(F) = P_{\mathbb{H}}(F)$ when F is bounded and ∂F is C^2 in the Euclidean sense. We will show in Theorem 4.7 that, given any bounded set $D \subset \mathbb{H}^n$ such that $\pi(D) = \{z \in \mathbb{C}^n : (z, t) \in D \text{ for some } t \in \mathbb{R}\}$ coincides with the $2n$-dimensional unit disc $\{|z| < 1\}$, and defining the *generalized Minkowski content* associated to D as

$$\mathcal{M}_D(F) = \lim_{\epsilon \to 0^+} \frac{|F \cdot \delta_\epsilon(D)| - |F|}{\epsilon},$$

one obtains $\mathcal{M}_D(F) = \mathcal{M}_B(F)$ for all bounded, open sets F with boundary of class C^2. This implies, for instance, that the Minkowski content (and hence the $\mathbb{H}$-perimeter) of a regular set F can be computed by ϵ-enlarging F with a flat horizontal disc of radius ϵ as well as with the δ_ϵ-scaled copy of a "tall" cylinder (Corollary 4.8). This somehow clarifies the "horizontal" nature of both $\mathbb{H}$-perimeter and Minkowski content in $\mathbb{H}^n$.

The paper is organized as follows. In Section 2 we present the basic definitions and facts about $\mathbb{H}^n$, while in Section 3 we show the announced properties of Heisenberg bubbles (Theorem 3.3). In the final Section 4 we prove the Brunn–Minkowski inequality (Theorem 4.1) and the result on the equivalence between generalized Minkowski contents (Theorem 4.7), then discuss the relevant consequences mentioned above, and in particular the failure of a direct application of the Brunn–Minkowski theory to the isoperimetric problem in $\mathbb{H}^n$.

2. Notations and main facts about $\mathbb{H}^n$

The Heisenberg group $\mathbb{H}^n$ can be identified with $\mathbb{C}^n \times \mathbb{R} \simeq \mathbb{R}^{2n+1}$, and we shall frequently denote its elements by $P = [z, t]$, where $z \in \mathbb{C}^n$ and $t \in \mathbb{R}$. We will also sometimes identify $z = x + iy$ with the $2n$-tuple (x, y), where $x, y \in \mathbb{R}^n$. As with any Carnot group, the algebraic and metric structure of $\mathbb{H}^n$ can be completely derived via the exponential map from its tangent, stratified Lie algebra $\mathcal{G}$ generated by the following family of *horizontal* vector fields: for $i = 1, \ldots, n$, define

$$X_i(P) = \partial_{x_i} + 2y_i\partial_t,$$
$$Y_i(P) = \partial_{y_i} - 2x_i\partial_t,$$

where $P = [x + iy, t]$. Note that the stratification is nontrivial, since $\mathcal{G} = H \oplus V$, where $H = \text{span}\{X_1, \ldots, X_n, Y_1, \ldots, Y_n\}$ is the so-called horizontal subbundle and $V = \text{span}\{[X_i, Y_i]\} = \text{span}\{\partial_t\}$ is the center of the algebra (as usual, $[X, Y] = XY - YX$ denotes the commutator of the two fields X and Y). The resulting group operation on $\mathbb{H}^n$ is

$$P \cdot P' = [z, t] \cdot [z', t'] = [z + z', t + t' + 2\,\text{Im}(\sum_{i=1}^{n} z_i \cdot \bar{z}_i')]. \tag{2.1}$$

Thanks to (2.1), one defines a family of *left translations* on $\mathbb{H}^n$ as the group automorphisms $\tau_P : \mathbb{H}^n \to \mathbb{H}^n$ that associate to any $Q \in \mathbb{H}^n$ the point $\tau_P(Q) = P \cdot Q$. There is also a family of intrinsic *dilations* on $\mathbb{H}^n$, given by $\delta_\lambda([z, t]) = [\lambda z, \lambda^2 t]$, with $\lambda > 0$.

To complete the Carnot structure, we define the *Carnot–Carathéodory metric* as follows. We say that an absolutely continuous curve $\gamma : [0, T] \to \mathbb{H}^n$ is a subunit curve if there exist $2n$ measurable functions $h_1, \ldots, h_{2n} : [0, T] \to \mathbb{R}$ such that $\sum_{j=1}^{2n} h_j^2(s) \leq 1$ and

$$\dot{\gamma}(s) = \sum_{i=1}^{n} h_i(s) X_i(\gamma(s)) + h_{i+n}(s) Y_i(\gamma(s))$$

for a.e. $s \in [0, T]$. By Chow's theorem, any two points P and Q in $\mathbb{H}^n$ can be joined by a subunit curve. Then the Carnot–Carathéodory distance between P and Q is

$$\begin{aligned} d_c(P, Q) = \inf\{T > 0;\ &\text{there exists a subunit curve } \gamma : [0, T] \to \mathbb{H}^n \\ &\text{such that } \gamma(0) = P,\ \gamma(T) = Q\}. \end{aligned}$$

It is worth noting that the distance d_c is coherent with the group structure and the dilations: indeed,

$$d_c(\tau_P(Q), \tau_P(W)) = d_c(Q, W) \qquad \text{and} \qquad d_c(\delta_\lambda(P), \delta_\lambda(Q)) = \lambda d_c(P, Q)$$

for all $P, Q, W \in \mathbb{H}^n$ and $\lambda > 0$. Given P and Q as above, there always exists a subunit curve joining P and Q of length $d_c(P, Q)$ (i.e., a minimal geodesic). We recall here the equations for geodesics of unit length starting from $[0, 0]$, since all other geodesics can be recovered by left translations and dilations [19,24,25]. Let $s \in [0, 1]$ be the time-length parameter and $\phi \in [-2\pi, 2\pi]$, and let $A_i, B_i \in \mathbb{R}$ such that $\sum_{i=1}^{n} A_i^2 + B_i^2 = 1$: then the set of equations

$$\begin{cases} x_i(s) = \dfrac{A_i(1 - \cos(\phi s)) + B_i \sin(\phi s)}{\phi}, & i = 1, \ldots, n, \\ y_i(s) = \dfrac{-B_i(1 - \cos(\phi s)) + A_i \sin(\phi s)}{\phi}, & i = 1, \ldots, n, \\ t(s) = 2\,\dfrac{\phi s - \sin(\phi s)}{\phi^2} \end{cases} \tag{2.2}$$

defines a geodesic $\gamma(s)$ connecting $[0,0]$ with the point $[x+iy, t]$ whose coordinates are

$$\begin{cases} x_i = x_i(1) = \dfrac{A_i(1-\cos\phi) + B_i \sin\phi}{\phi}, & i = 1,\dots,n, \\ y_i = y_i(1) = \dfrac{-B_i(1-\cos\phi) + A_i \sin\phi}{\phi}, & i = 1,\dots,n, \\ t = t(1) = 2\dfrac{\phi - \sin\phi}{\phi^2} \end{cases} \tag{2.3}$$

(of course, this gives a parameterization of the boundary of the Carnot–Carathéodory ball with unit radius). Finally, the structure $(\mathbb{H}^n, \cdot, \delta_\lambda, d_c)$ provides an example of the *Carnot group*, as mentioned in the introduction.

It is not difficult to check that the $(2n+1)$-dimensional Lebesgue measure on $\mathbb{H}^n \simeq \mathbb{R}^{2n+1}$ is the Haar measure of the group, invariant under left translations and $(2n+2)$-homogeneous with respect to dilations (this degree of homogeneity coincides with the so-called *homogeneous dimension* of $\mathbb{H}^n$, henceforth denoted as Q). As a consequence, the topological dimension of $\mathbb{H}^n$ $(d = 2n+1)$ is strictly less than its Hausdorff dimension $(Q = 2n+2)$.

We now define the sets with finite $\mathbb{H}$-perimeter (for more details, the reader may refer to [14]). If Ω is an open subset of $\mathbb{H}^n$ and $F \subset \mathbb{H}^n$ is measurable, we set

$$P_{\mathbb{H}}(F,\Omega) = \sup\left(\int_{\Omega\cap F} \mathrm{div}_{\mathbb{H}}\,\phi \, d\mathcal{L}^{2n+1}\,;\ \phi \in C_0^1(\Omega,\mathbb{R}^{2n}),\ \|\phi\|_\infty \le 1\right),$$

where

$$\mathrm{div}_{\mathbb{H}}\,\phi = \sum_{i=1}^{n} X_i\phi_i + Y_i\phi_{i+n}$$

for any $\phi = (\phi_1,\dots,\phi_{2n}) \in C_0^1(\Omega,\mathbb{R}^{2n})$. Here $C_0^1(\Omega,\mathbb{R}^{2n})$ denotes the space of $\mathbb{R}^{2n}$-valued functions of class C^1 (in the Euclidean sense) with compact support in Ω. Of course, F will be said to have a finite $\mathbb{H}$-perimeter in Ω if and only if $P_{\mathbb{H}}(F,\Omega) < \infty$. As for the notation, we will write $P_{\mathbb{H}}(F)$ instead of $P_{\mathbb{H}}(F,\mathbb{H}^n)$. Among the various properties of the $\mathbb{H}$-perimeter, we just recall the invariance with respect to left translations and the $(Q-1)$-homogeneity with respect to dilations. It is also worth recalling that the definition of $\mathbb{H}$-perimeter is closely related to that of $BV_{\mathbb{H}}$ space, hence it can be useful to define the horizontal (distributional) gradient of a function $f : \mathbb{H}^n \to \mathbb{R}$:

$$\nabla_{\mathbb{H}} f = (X_1 f, \dots, X_n f, Y_1 f, \dots, Y_n f).$$

If F has Lipschitz boundary in Ω (in the Euclidean sense), we have the following *integral representation* of the $\mathbb{H}$-perimeter, as a consequence of Green's formulae:

$$P_{\mathbb{H}}(F,\Omega) = \int_{w\in\partial F\cap\Omega} |C(w)\cdot\nu_F(w)|\,\partial\mathcal{H}^{2n}(w), \tag{2.4}$$

where $\mathcal{H}^{2n}$ is the Euclidean $(2n)$-dimensional Hausdorff measure, $\nu_F(w)$ is the Euclidean normal vector to ∂F at w, and $C(w)$ is the $(2n \times 2n+1)$-matrix whose rows are given by the components of the vector fields $X_i(w)$ and $Y_i(w)$, i.e.,

$$C(w) = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 2y_1 \\ 0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 & 2y_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 & 2y_n \\ 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & -2x_1 \\ 0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 0 & -2x_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 & -2x_n \end{bmatrix}, \qquad w = [x+iy, t]. \tag{2.5}$$

Going back to the isoperimetric problem, we now recall the theorem proved in [21] about the existence of isoperimetric sets in any Carnot group, here specialized to the context of $\mathbb{H}^n$:

Theorem 2.1 (Leonardi-Rigot). *For all $v > 0$, there exists a measurable set $E \subset \mathbb{H}^n$ that is isoperimetric, that is, it verifies $|E| = v$ and minimizes the $\mathbb{H}$-perimeter among all measurable $F \subset \mathbb{H}^n$ such that $|F| = v$. Moreover, E is open, bounded, and connected, and its boundary is Alhfors-regular and verifies condition B. In addition, it is a domain of isoperimetry, i.e., a relative isoperimetric inequality holds on E.*

This result is, however, not sufficient to completely identify isoperimetric sets since the recovered properties are too generic.

3. Heisenberg bubbles

The purpose of this section is to describe some interesting properties of Heisenberg bubbles. We first show that they are isoperimetric sets within a particular class of sets of finite perimeter, which we now define:

Definition 3.1. *We call $\mathcal{F}$ the class of sets F of finite perimeter whose boundary ∂F can be written, up to left translations, as $\partial^+ F \cup \partial^- F$, with $\partial^+ F$ and $\partial^- F$ being the graphs of, respectively, $g(z)$ and $-g(z)$, where $g0028z) = f(|z|)$ is a smooth, nonnegative, radial function defined on some $2n$-ball $D_r \subset \mathbb{C}^n \simeq \mathbb{R}^{2n}$ of radius r centered at 0, and such that $g = 0$ on ∂D_r.*

As mentioned in the introduction, we shall also prove that, as happens for Euclidean balls in $\mathbb{R}^n$, Heisenberg bubbles have constant mean $\mathbb{H}$-curvature, following a definition that has been proposed by Pauls in [31] and that we recall:

Definition 3.2. *Let Ω be an open subset of $\mathbb{R}^{2n}$ and $g : \Omega \to \mathbb{R}$ a smooth function. Define the function $F(x, y, t) = g(x, y) - t$ whose zero level set is precisely the graph of g in $\mathbb{R}^{2n+1} \simeq \mathbb{H}^n$. Then, the quantity*

$$H_{cc}(g) = -\operatorname{div}_{\mathbb{H}} \frac{\nabla_{\mathbb{H}} F}{|\nabla_{\mathbb{H}} F|}$$

is called the mean $\mathbb{H}$-curvature of the graph of g.

We now can state our main result about Heisenberg bubbles.

Theorem 3.3. *There exists, up to dilations and left translations, a unique solution E to the isoperimetric problem within the restricted class $\mathcal{F}$, with the following properties:*

1. *The mean $\mathbb{H}$-curvature of ∂E is constant;*
2. *∂E coincides with the union of all infinite geodesics connecting the north pole N and the south pole S of E (Figure 2).*

In the sequel, we shall call E a Heisenberg bubble.

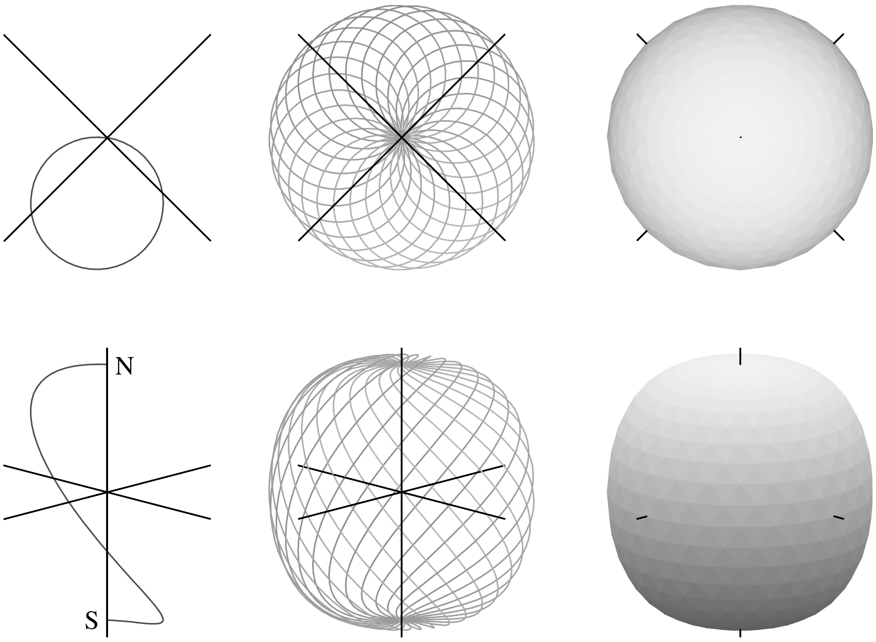

Fig. 2. *Left* to *right*: one of the geodesics connecting S and N, few geodesics, and, finally, the total geodesic envelope corresponding to the boundary of a Heisenberg bubble in $\mathbb{H}^1 \simeq \mathbb{R}^3$. Figures on *top* are views from above

Proof. The first part of the statement, that is, the existence of a solution to the isoperimetric problem in $\mathcal{F}$, is somehow a known fact (see, for instance, [26]), but we give the proof for the sake of completeness. Using the integral representation (2.4), we can compute the isoperimetric ratio of any $F \in \mathcal{F}$ from its associated function $f(\rho)$, with $\rho \in [0, r]$. Indeed, we first compute the $\mathbb{H}$-perimeter of F by using (2.4) and the fact that $\partial^+ F$ and $\partial^- F$ give the same contribution to the whole perimeter (a consequence of the radial symmetry of f):

$$P_{\mathbb{H}}(F) = 2 \int_{D_r} |C \cdot \nu| \sqrt{1 + |\nabla g|^2}\, dx\, dy = 4n\omega_{2n} \int_0^r \sqrt{4\rho^2 + f'(\rho)^2}\, \rho^{2n-1}\, d\rho,$$

where ω_{2n} denotes the volume of the unit ball in $\mathbb{R}^{2n}$. Then, the volume of F in terms of f is

$$|F| = 4n\omega_{2n} \int_0^r f(\rho)\rho^{2n-1}\, d\rho.$$

At this point, by computing the Euler equation of the functional $C_I(f) = P_{\mathbb{H}}(F)/|F|^{\frac{Q-1}{Q}}$ one obtains after some calculations

$$\left(\frac{\rho^{2n-1} f'(\rho)}{\sqrt{4\rho^2 + f'(\rho)^2}}\right)' = -\frac{(Q-1)P_{\mathbb{H}}(F)}{Q|F|}\rho^{2n-1},$$

and hence, by a first integration,

$$\frac{f'(\rho)}{\sqrt{4\rho^2 + f'(\rho)^2}} = -\frac{\lambda_n}{2}\rho, \tag{3.1}$$

with $\lambda_n = \frac{(Q-1)P_{\mathbb{H}}(F)}{nQ|F|}$. The assumed smoothness of ∂F implies that $f'(0) = 0$, and we obtain by integration the following solution to (3.1):

$$f(\rho) = f(0) + \frac{2}{\lambda_n^2}\left(\lambda_n\rho\sqrt{1 - \left(\frac{\lambda_n\rho}{2}\right)^2} + 2\arccos\left(\frac{\lambda_n\rho}{2}\right) - \pi\right),$$

whence we infer that ρ must be less than or equal to $r = \frac{2}{\lambda_n}$; moreover, if we ask that $f(r) = 0$, then we obtain $f(0) = \frac{2\pi}{\lambda_n^2}$, and the unique solution to (3.1) having this property can be written as

$$f(\rho) = \frac{2}{\lambda_n^2}\left(\lambda_n\rho\sqrt{1 - \left(\frac{\lambda_n\rho}{2}\right)^2} + 2\arccos\left(\frac{\lambda_n\rho}{2}\right)\right). \tag{3.2}$$

This function uniquely determines a solution to the isoperimetric problem in the class $\mathcal{F}$.

Property 1 can be checked by direct computation, using (3.2) and the definition of the mean $\mathbb{H}$-curvature. Indeed, taking $\lambda_n = 2$ and $\rho^2 = \sum_{i=1}^n x_i^2 + y_i^2$ as before, one obtains

$$\begin{aligned}
H_{cc}(g) &= \sum_{i=1}^n \partial_{x_i}\left(\frac{-\frac{f'(\rho)}{\rho}x_i + 2y_i}{\sqrt{4\rho^2 + f'^2(\rho)}}\right) + \partial_{y_i}\left(\frac{-\frac{f'(\rho)}{\rho}y_i - 2x_i}{\sqrt{4\rho^2 + f'^2(\rho)}}\right) \\
&= \sum_{i=1}^n \frac{2\rho^2(1-\rho^2) + x_i(\partial_{y_i}\rho)\sqrt{1-\rho^2} - y_i(\partial_{x_i}\rho)\sqrt{1-\rho^2}}{\rho^2(1-\rho^2)} \\
&= \sum_{i=1}^n \frac{2\rho^2(1-\rho^2)}{\rho^2(1-\rho^2)} \\
&= 2n.
\end{aligned}$$

In particular, one sees that $H_{cc}(g)$ is given exactly by the multiplier $n\lambda_n = \frac{(Q-1)P_{\mathbb{H}}(F)}{Q|F|}$, as happens in the Euclidean case.

Property 2 is verified, for example, by computing the equation of the surface given by the union of all geodesics between $[0, 0]$ and $[0, \frac{1}{\pi}]$ (the latter point lying on the t-axis at distance $d_c = 1$ from the origin), since the general case can be recovered by scaling:

$$\begin{cases} x_i(s) = \dfrac{1}{2\pi}\left(A_i(1-\cos(2\pi s)) + B_i \sin(2\pi s)\right) \\ y_i(s) = \dfrac{1}{2\pi}\left(-B_i(1-\cos(2\pi s)) + A_i \sin(2\pi s)\right) \quad s \in [0,1],\ \sum_i A_i^2 + B_i^2 = 1. \\ t(s) = \dfrac{1}{\pi}\left(s - \dfrac{\sin(2\pi s)}{2\pi}\right) \end{cases} \tag{3.3}$$

Then, by left-translating the union of geodesics by the element $[0, -\frac{1}{2\pi}]$ (in this case, the left translation coincides with the Euclidean one) and by expressing t as a function of $\rho = \sqrt{\sum_i x_i^2 + y_i^2}$, one gets

$$t(\rho) = \frac{\rho}{\pi}\sqrt{1-\pi^2\rho^2} + \frac{1}{2\pi^2}\arccos(2\pi^2\rho^2 - 1),$$

which, thanks to the identity

$$\arccos(2\pi^2\rho^2 - 1) = 2\arccos(\pi\rho),$$

becomes

$$t(\rho) = \frac{\rho}{\pi}\sqrt{1-\pi^2\rho^2} + \frac{1}{\pi^2}\arccos(\pi\rho). \tag{3.4}$$

It is now easy to check that (3.4) corresponds to (3.2) when $\lambda_n = 2\pi$. □

Remark 3.4. The boundary of any Heisenberg bubble E is, *up to left translations*, the union of $\partial^+ E$ and $\partial^- E$ (Definition 3.1). By scaling, any point of $\partial^+ E$ is of the general form $(ax, ay, a^2 f(\sqrt{x^2+y^2}))$, where $\sum_i x_i^2 + y_i^2 \le 1$ and f is the function defined on $[0, 1]$ by

$$f(\rho) = \rho\sqrt{1-\rho^2} + \arccos(\rho). \tag{3.5}$$

The parameter a will be called the *horizontal radius* of E.

4. Brunn–Minkowski inequality and Minkowski content

The first result of this section establishes in $\mathbb{H}^n$ the analog of the well-known Brunn–Minkowski inequality in $\mathbb{R}^{2n+1}$ (see, for instance, [10,17]). Here, of course, the Euclidean sum is replaced by the noncommutative group operation. By suitably adapting the classical proof, we are able to prove the following.

Theorem 4.1 (Brunn–Minkowski inequality). *Let F, G be two nonempty measurable subsets of $\mathbb{H}^n$. Then*

$$|F \cdot G|^{\frac{1}{d}} \geq |F|^{\frac{1}{d}} + |G|^{\frac{1}{d}}, \tag{4.1}$$

where $d = 2n + 1$ is the topological dimension of $\mathbb{H}^n$.

Proof. Inequality (4.1) will be proved in three steps.

Step 1. We suppose that F and G are d-rectangles, i.e., that we may write $F = Q \times I$ and $G = Q' \times I'$, with $Q = Q_1 \times \cdots \times Q_{2n}$ and $Q' = Q'_1 \times \cdots \times Q'_{2n}$, where Q_i, Q'_i, I, I' are bounded, measurable subsets of $\mathbb{R}$ with positive $\mathcal{L}^1$ measure. Of course, we can think of Q, Q' as subsets of $\mathbb{C}^n$. Therefore, we have

$$F \cdot G = \{(z + z', t + t' + 2\,\mathrm{Im}(z\overline{z'})) : \; z \in Q, z' \in Q', t \in I, t' \in I'\};$$

hence, setting $w = z + z'$, we obtain the equivalent representation

$$F \cdot G = \{(w, t + t' + 2\,\mathrm{Im}(z\overline{w})) : \; w \in Q + Q', z \in Q \cap (w - Q'), t \in I, t' \in I'\}. \tag{4.2}$$

Define $h : Q + Q' \to [0, +\infty)$ as $h(w) = \mathcal{L}^1(\{t : \; (w, t) \in F \cdot G\})$. By (4.2) one sees immediately that

$$h(w) = \mathcal{L}^1 \left(\bigcup_{z \in Q \cap (w - Q')} 2\,\mathrm{Im}(z\overline{w}) + I + I' \right) \geq \mathcal{L}^1(I + I') \geq \mathcal{L}^1(I) + \mathcal{L}^1(I'). \tag{4.3}$$

Now set $q_i = \mathcal{L}^1(Q_i)$, $q'_i = \mathcal{L}^1(Q'_i)$, $\tau = \mathcal{L}^1(I)$, and $\tau' = \mathcal{L}^1(I')$. Define, for $i = 1, \ldots, 2n$, the positive numbers

$$u_i = \frac{q_i}{q_i + q'_i}, \qquad v_i = \frac{q'_i}{q_i + q'_i}$$

and

$$u_d = u_{2n+1} = \frac{\tau}{\tau + \tau'}, \qquad v_d = v_{2n+1} = \frac{\tau'}{\tau + \tau'}.$$

By the well-known geometric/arithmetic mean inequality, we infer

$$\prod_{i=1}^{d} u_i^{\frac{1}{d}} + \prod_{i=1}^{d} v_i^{\frac{1}{d}} \leq \sum_{i=1}^{d} \frac{u_i + v_i}{d} = 1;$$

then, thanks also to (4.3) and to Fubini's theorem, it follows that

$$\begin{aligned} |F|^{\frac{1}{d}} + |G|^{\frac{1}{d}} &= \left(\tau \prod_{i=1}^{2n} q_i \right)^{\frac{1}{d}} + \left(\tau' \prod_{i=1}^{2n} q'_i \right)^{\frac{1}{d}} \\ &\leq \left((\tau + \tau') \prod_{i=1}^{2n} (q_i + q'_i) \right)^{\frac{1}{d}} \leq \left(\int_{Q+Q'} h(w)\, dw \right)^{\frac{1}{d}} \\ &= |F \cdot G|^{\frac{1}{d}}. \end{aligned}$$

Step 2. Suppose now that $F = F_1 \cup \cdots \cup F_m$ and $G = G_1 \cup \cdots \cup G_k$, where $F_s = Q^s \times I^s$, $G_r = P^r \times J^r$ are d-rectangles with the property that $Q^s \cap Q^i = P^r \cap P^j = \emptyset$ for $s \neq i$ and $r \neq j$. Moreover, we suppose that Q^s and P^r are open cells of some orthogonal lattice in $\mathbb{R}^{d-1}$. We proceed by induction on $m + k$ as follows. If $m = k = 1$, then F and G are d-rectangles, and therefore (4.1) holds by the previous step. Suppose now that (4.1) is verified whenever $m + k \leq s$ for some $s \geq 2$; then we prove that it must be verified also if $m + k = s + 1$. Indeed, we face in general the following alternative:

(1) Both F and G are d-rectangles;
(2) Either F or G is not a d-rectangle.

If (1) holds, then we conclude as in the previous step. Otherwise, if (2) holds, then, without loss of generality, we suppose that F is not a d-rectangle. This implies the existence of a vertical hyperplane of equation $x_i = a$ ($i \in \{1, \ldots, 2n\}$) such that both sets $F^+ = F \cap \{x_i > a\}$ and $F^- = F \cap \{x_i < a\}$ contain at least a d-rectangle of the decomposition of F and thus are unions of a number of d-rectangles strictly less than m. Now, choose $b \in \mathbb{R}$ in such a way that, defining $G^+ = G \cap \{x_i > b\}$ and $G^- = G \cap \{x_i < b\}$, one obtains

$$\frac{|G^{\pm}|}{|G|} = \frac{|F^{\pm}|}{|F|}.$$

It is easy to see that $F^+ \cdot G^+$ and $F^- \cdot G^-$ are necessarily disjoint (indeed, they are separated by the vertical hyperplane $x_i = a + b$), as well as the fact that

$$F \cdot G \supseteq (F^+ \cdot G^+) \cup (F^- \cdot G^-);$$

therefore by the inductive hypothesis we conclude

$$\begin{aligned} |F \cdot G| &\geq |F^+ \cdot G^+| + |F^- \cdot G^-| \\ &\geq (|F^+|^{\frac{1}{d}} + |G^+|^{\frac{1}{d}})^d + (|F^-|^{\frac{1}{d}} + |G^-|^{\frac{1}{d}})^d \\ &= (|G^+| + |G^-|) \left(1 + \left(\frac{|F|}{|G|}\right)^{\frac{1}{d}}\right)^d \\ &= (|F|^{\frac{1}{d}} + |G|^{\frac{1}{d}})^d, \end{aligned}$$

that is, (4.1) is proved for such F and G.

Step 3. The general case follows by approximation: one fixes $\epsilon > 0$ and takes F, G measurable with a finite Lebesgue measure and such that $F \cdot G$ also has a finite measure (otherwise the conclusion is trivial), then chooses O open set containing $F \cdot G$ and such that $|O \setminus F \cdot G| < \epsilon$ by Borel regularity. Since the $\cdot$ operation is continuous, we can find two open sets $F' \supset F$ and $G' \supset G$ such that $|F' \setminus F| < \epsilon$, $|G' \setminus G| < \epsilon$, and $F' \cdot G' \subset O$. Then, we approximate the two open sets F', G' from inside, by means of sets $R_F \subset F'$ and $R_G \subset G'$, which are finite unions of d-rectangles constructed on a dyadic subdivision of the horizontal coordinate space (so that step 2 is still applicable) and in order to have $|F' \setminus R_F| < \epsilon$ and $|G' \setminus R_G| < \epsilon$. The conclusion follows by applying step 2 to the pair R_F, R_G and by letting $\epsilon \to 0$. □

Remark 4.2. The right side of (4.1) is obviously symmetric in F and G, and thus we could write more precisely that

$$\min\{|F \cdot G|, |G \cdot F|\}^{\frac{1}{d}} \geq |F|^{\frac{1}{d}} + |G|^{\frac{1}{d}}.$$

It is also worth observing that, in general, $|F \cdot G|$ can be different from $|G \cdot F|$, as explained in the following example. Fix a parameter $\epsilon \geq 0$ and take

$$F_\epsilon = C_F^1(\epsilon) \cup C_F^2(\epsilon), \qquad G_\epsilon = C_G^1(\epsilon) \cup C_G^2(\epsilon),$$

where

$$\begin{aligned}
C_F^1(\epsilon) &= \left\{(z,t) \in \mathbb{H}^1 : \ |z - i| \leq \epsilon, \ t \in \left(-\frac{1}{2}, \frac{1}{2}\right)\right\}, \\
C_F^2(\epsilon) &= \left\{(z,t) \in \mathbb{H}^1 : \ \left|z - \frac{i}{2}\right| \leq \epsilon, \ t \in (0,1)\right\}, \\
C_G^1(\epsilon) &= \left\{(z,t) \in \mathbb{H}^1 : \ |z - 1| \leq \epsilon, \ t \in \left(-\frac{1}{2}, \frac{1}{2}\right)\right\}, \\
C_G^2(\epsilon) &= \left\{(z,t) \in \mathbb{H}^1 : \ \left|z - \left(1 + \frac{i}{2}\right)\right| \leq \epsilon, \ t \in (0,1)\right\}.
\end{aligned}$$

Hence, F_ϵ and G_ϵ are defined as unions of pairs of vertical cylinders with a circular section of radius ϵ. For ϵ small enough, these cylinders are disjoint and, moreover, the sets $C_F^1(\epsilon) \cdot C_G^2(\epsilon)$ and $C_F^2(\epsilon) \cdot C_G^1(\epsilon)$, as well as $C_G^1(\epsilon) \cdot C_F^2(\epsilon)$ and $C_G^2(\epsilon) \cdot C_F^1(\epsilon)$, are pairwise disjoint. On the other hand, it can be checked that $C_F^1(\epsilon) \cdot C_G^1(\epsilon)$ and $C_F^2(\epsilon) \cdot C_G^2(\epsilon)$ overlap significantly, in contrast to $C_G^1(\epsilon) \cdot C_F^1(\epsilon)$ and $C_G^2(\epsilon) \cdot C_F^2(\epsilon)$, due to the noncommutativity of the group operation. It follows that $|F_\epsilon \cdot G_\epsilon| < |G_\epsilon \cdot F_\epsilon|$ (one may first do the much simpler computation for the "limit" case $\epsilon = 0$ and then extend to $\epsilon > 0$ small).

Remark 4.3. The fact that inequality (4.1) holds with exponent $d = 2n + 1$ does not prevent the same inequality from holding with a larger exponent (at least in principle). It is actually easy to verify that, as soon as (4.1) holds for a certain exponent d, it holds for any exponent $d' \in (0, d)$. Indeed, suppose $|F| \geq |G| > 0$ without loss of generality, and rewrite (4.1) as

$$|F \cdot G| \geq |F| \left(1 + \left(\frac{|G|}{|F|}\right)^{\frac{1}{d}}\right)^d;$$

then observe that, for all $x \in (0, 1]$, the function $m(t) = \left(1 + x^{\frac{1}{t}}\right)^t$ is nondecreasing in $t > 0$; hence one obtains

$$|F \cdot G| \geq |F| \left(1 + \left(\frac{|G|}{|F|}\right)^{\frac{1}{d'}}\right)^{d'}$$

for all $d' \in (0, d)$, as desired.

We already mentioned in the introduction that another way of computing the $\mathbb{H}$-perimeter, at least for a suitable subclass of measurable sets, is provided by the *Minkowski content*, defined as the following limit (if it exists):

$$\mathcal{M}_B(F) = \lim_{\epsilon \to 0^+} \frac{|F \cdot B_\epsilon| - |F|}{\epsilon}. \tag{4.4}$$

Indeed, as a particular case of a more general result of [27], we have the following.

Theorem 4.4 (Monti-Serra Cassano). *Let $F \subset \mathbb{H}^n$ be a bounded, open set of class C^2. Then the limit in* (4.4) *exists finite, and one has*

$$\mathcal{M}_B(F) = P_{\mathbb{H}}(F).$$

We now consider the following generalization of (4.4):

Definition 4.5. *Given $D \subset \mathbb{H}^n$, the generalized Minkowski content associated to D is defined as*

$$\mathcal{M}_D(F) = \lim_{\epsilon \to 0^+} \frac{|F \cdot \delta_\epsilon(D)| - |F|}{\epsilon}$$

whenever the limit exists.

Before going further, let us define the *horizontal projection*.

Definition 4.6. *Given $(z,t) \in \mathbb{H}^n$, we define $\pi((z,t)) = (z,0)$. For any set $A \subset \mathbb{H}^n$, we write $\pi(A) = \{\pi(x) : x \in A\}$.*

As we will see in the sequel, the $\mathbb{H}$-perimeter of a bounded, open set F with ∂F of class C^2 must coincide not only with $\mathcal{M}_B(F)$ (the Minkowski content associated to the Carnot–Carathéodory distance), as stated by Theorem 4.4, but also with $\mathcal{M}_D(F)$, for all bounded sets D such that their horizontal projection $\pi(D)$ coincides with the unit disc in $\mathbb{R}^{2n}$ or, in other words, $\pi(D) = \pi(B)$ (here B denotes the Carnot–Carathéodory ball of radius 1). This is a direct consequence of Theorem 4.7 below and can be understood by simply observing that the ϵ-neighbourhood of F is built by "adding" the set $\delta_\epsilon(D)$ to F (in the sense of group multiplication) and that the scaling factor of the anisotropic dilation produces a horizontal scaling of factor ϵ and a vertical scaling of factor ϵ^2, which says somehow that the "vertical" shape of D is "less important" than its "horizontal" shape when ϵ is small.

Theorem 4.7. *Let D_1, D_2 be two bounded subsets of $\mathbb{H}^n$ for which $\pi(D_1) = \pi(D_2)$. Then $\mathcal{M}_{D_1}(F) = \mathcal{M}_{D_2}(F)$ for all open, bounded sets F of class C^2.*

Proof. Suppose first that, for some $h > 0$,

$$D_i \subset C_h = \{(z,t) : z \in \pi(D_1),\ |t| < h\}, \qquad i = 1, 2. \tag{4.5}$$

Now, it is sufficient to prove that, given F as above, there exists a constant $C > 0$, depending only on F and h, such that

$$|(F \cdot \delta_\epsilon(D_1)) \setminus (F \cdot \delta_\epsilon(D_2))| \le C\epsilon^2 \tag{4.6}$$

for ϵ small enough. Indeed, by exchanging the role of D_1 and D_2 in (4.6), one gets

$$\big||F \cdot \delta_\epsilon(D_1)| - |F \cdot \delta_\epsilon(D_2)|\big| \leq C\epsilon^2,$$

which in turn gives the conclusion. We shall prove a local version of (4.6): more precisely, for a fixed $\delta > 0$, we consider the Euclidean, open ball B_δ centered at 0 with radius δ, then we take the open covering $\{\tau_{p_j}(B_\delta)\}_{j=1}^k$ of ∂F, obtained by suitably choosing points $p_1, \dots, p_k$ in ∂F. Thanks to the regularity and boundedness of ∂F, we shall prove that, for all $\eta > 0$, there exist $\delta > 0$ and points $p_1, \dots, p_k \in \partial F$ (k depends on δ) such that, setting $E_j = \tau_{-p_j}(F)$, the surface

$$S_j = \partial E_j \cap B_{4\delta}$$

is "almost flat," that is, there exists a unit vector $v^j \in \mathbb{R}^{2n+1}$ such that, denoting by $n^j(q)$ the Euclidean outer normal to ∂E_j at q, one has

$$\langle n^j(q), v^j \rangle > 1 - \eta \tag{4.7}$$

for all $q \in S_j$. Then, we only need to prove the local estimate

$$|(E_j \cdot \delta_\epsilon(D_1)) \setminus (E_j \cdot \delta_\epsilon(D_2)) \cap B_\delta| \leq C\epsilon^2 \tag{4.8}$$

for all $j = 1, \dots, k$, which implies (4.6) by the following argument: $\{\tau_{p_j}(B_\delta)\}_{j=1}^k \cup \{F\}$ covers $F \cdot \delta_\epsilon(D_i)$, provided ϵ is small enough, hence thanks to the invariance of the Lebesgue measure under left translations one obtains

$$|(F \cdot \delta_\epsilon(D_1)) \setminus (F \cdot \delta_\epsilon(D_2))| \leq \sum_{j=1}^k |(E_j \cdot \delta_\epsilon(D_1)) \setminus (E_j \cdot \delta_\epsilon(D_2)) \cap B_\delta| \leq kC\epsilon^2,$$

and since k depends only on η (that will be later fixed), (4.6) follows.

The proof is now split into two parts.

Part I. Suppose that $|v^j_{2n+1}| > 2\sqrt{2\eta}$ ($\eta > 0$ to be later chosen); then, by (4.7) we get $|n^j_{2n+1}(q)| > \sqrt{2\eta}$. This means that S_j defined above coincides with a portion of the graph of a Lipschitz function $f : B'_{4\delta} \subset \mathbb{R}^{2n} \to \mathbb{R}$ of class C^2 and Lipschitz constant $\leq \frac{1}{\sqrt{2\eta}}$ (here B'_r is the ball of radius r in $\mathbb{R}^{2n}$, centered at 0). We can also suppose without loss of generality that

$$E_j \cap B_{4\delta} = sgr(f) \cap B_{4\delta},$$

where $sgr(f)$ denotes the subgraph of f.

We fix $i \in \{1, 2\}$, $j \in \{1, \dots, k\}$, and set $E = E_j$, $D = D_i$ for more simplicity, then claim that $E \cdot \delta_\epsilon(D) \cap B_\delta$ still coincides with the subgraph of some function for ϵ small enough. Indeed, fix $q = (z, t) \in (E \cdot \delta_\epsilon(D)) \cap B_\delta$, then choose $e \in E$ such that $q \in e \cdot \delta_\epsilon(D)$. Thanks to (4.5), we can choose ϵ so small that e belongs to $E \cap B_{2\delta}$ (indeed, if $q = e \cdot \delta_\epsilon(d)$ for some $d \in D \subset C_h$, then clearly $e = q \cdot \delta_\epsilon(-d)$), hence if $q' = (z', t') \in B_\delta$ is such that $z' = z$ and $t' < t$, then $q' \in e' \cdot \delta_\epsilon(D)$, where $e' = \tau_{(0,t'-t)}(e)$. Now, by $t - t' < 2\delta$ and the fact that E is a subgraph in $B_{4\delta}$, it follows that e' belongs to $E \cap B_{4\delta}$, hence $q' \in E \cdot \delta_\epsilon(D)$, and this proves our claim.

We define

$$\Delta_\epsilon = (E \cdot \delta_\epsilon(D_1) \ \setminus \ E \cdot \delta_\epsilon(D_2)) \cap B_\delta$$

and take $q_1 = (z_1, t_1) \in \Delta_\epsilon$, then find $e \in E$ such that $q_1 \in e \cdot \delta_\epsilon(D_1) \cap B_\delta$. Therefore, by (4.5) and $\pi(D_1) = \pi(D_2)$, there exists $q_2 \in e \cdot \delta_\epsilon(D_2)$ such that $q_2 = (z_2, t_2)$ with $z_2 = z_1$ and $|t_2 - t_1| \leq 2h\epsilon^2$. This shows that the 1-dimensional section of Δ_ϵ defined for all $z \in \mathbb{C}^n$ by

$$\Delta_\epsilon(z) = \{t \in \mathbb{R} : \ (z, t) \in \Delta_\epsilon\}$$

is necessarily an interval of length at most $2h\epsilon^2$, owing to the fact that Δ_ϵ is a difference of subgraphs. Thus, by Fubini's theorem, we get

$$|\Delta_\epsilon| \leq |B'_\delta| \cdot 2h\epsilon^2 = C\epsilon^2,$$

as desired.

Part II. Suppose now that $|v^j_{2n+1}| \leq 2\sqrt{2\eta}$; then, reasoning as in Part I, we obtain that $|n^j_{2n+1}(q)| \leq 3\sqrt{2\eta}$; thus we can see S_j as part of the graph of a Lipschitz function $f : B''_{4\delta} \subset \Pi \to \mathbb{R}$ of class C^2 and Lipschitz constant $L_\eta \to 0$ as $\eta \to 0$ (here B''_r is the ball of radius r on the "vertical" hyperplane Π passing through 0 and orthogonal to $\pi(v^j)$).

As before, we can prove that $(E \cdot \delta_\epsilon(D_i)) \cap B_\delta$ is a subgraph and, without losing generality, we suppose that the vertical hyperplane Π coincides with $x_1 = 0$. Fix $q \in E \cdot \delta_\epsilon(D) \cap B_\delta$, with $q = (x_1, x_2, \ldots, x_{2n+1})$, and take $q' \in B_\delta$ such that $q' = (x'_1, x'_2, \ldots, x'_{2n+1})$, $x'_i = x_i$ for all $i > 1$ and $x'_1 < x_1$. Clearly, there exists $e \in E \cap B_{2\delta}$ such that $q \in e \cdot \delta_\epsilon(D)$, provided ϵ is small enough. The element $v = (x^v_1, \ldots, x^v_{2n+1}) \in \mathbb{H}^n$ such that $\tau_v(q) = v \cdot q = q'$ is defined by

$$\begin{aligned} x^v_1 &= x'_1 - x_1, \\ x^v_i &= 0 \quad \forall\, i = 2, \ldots, 2n, \\ x^v_{2n+1} &= 2x_2\, x^v_1. \end{aligned}$$

Now, write $e = (y_1, \ldots, y_{2n+1})$ and define $e' = (y'_1, \ldots, y'_{2n+1}) = \tau_v(e)$, then observe that

$$\begin{aligned} y'_1 - y_1 &= x'_1 - x_1, \\ y'_i - y_i &= 0 \qquad \forall\, i = 2, \ldots, 2n, \\ y'_{2n+1} - y_{2n+1} &= 2(x_2 - y_2)(y'_1 - y_1), \\ |x_2 - y_2| &< 3\delta, \end{aligned}$$

hence

$$|y'_{2n+1} - y_{2n+1}| \leq 6\delta |y'_1 - y_1|.$$

If we take $\delta < \frac{1}{2}$, then $e' \in B_{4\delta}$, and if we choose η so small that $L_\eta < 1/4$, we obtain $e' \in E$, too. This proves that $q' \in E \cdot \delta_\epsilon(D)$, that is, $E \cdot \delta_\epsilon(D)$ is a subgraph in B_δ, as claimed.

Let Δ_ϵ be defined as in Part I, and let $q_1 \in \Delta_\epsilon$. For ϵ small enough, we find $e_1 \in E \cap B_{2\delta}$ such that $q_1 \in e_1 \cdot \delta_\epsilon(D_1)$, and then there exists at least one point $q_0 \in e_1 \cdot \delta_\epsilon(D_2)$ such that $\pi(q_0) = \pi(q_1)$; moreover, if we denote by t_1 and t_0 the $(2n+1)$th coordinate of, respectively, q_1 and q_0, we necessarily have that $|t_1 - t_0| \leq 2h\epsilon^2$. Reasoning as before, we can find $v \in \mathbb{H}^n$ such that the corresponding translation τ_v maps q_0 onto a certain point q_2 with the property that the coordinates of q_2 and q_1 are the same except the first ones, denoted by x_1^2 and x_1^1 respectively, and satisfying $x_1^1 - x_1^2 = |t_1 - t_0|$. Again, it is not difficult to see that $e_2 = \tau_v(e_1) \in E$, provided ϵ and η are chosen small enough. Thus, we conclude as in Part I that $|\Delta_\epsilon| \leq C\epsilon^2$, and the proof is now completed. □

Corollary 4.8. *Let B denote the Carnot–Carathéodory ball of radius* 1 *and let D be a bounded set such that $\pi(D) = \pi(B)$. Then $\mathcal{M}_D(F) = P_{\mathbb{H}}(F)$ for all bounded, open sets F of class C^2.*

Proof. It is an immediate consequence of Theorems 4.7 and 4.4. □

Remark 4.9. Another relevant consequence of Theorem 4.7 is that $\mathcal{M}_{C_0}(F) = \mathcal{M}_B(F)$, where $C_0 = \{(z,0) : \ |z| < 1\}$ is the flat $2n$-dimensional unit disc centered at 0. This provides a simpler way of computing the Minkowski content (it is, of course, much easier to compute ϵ-enlargements by left-translating a flat disc of radius ϵ instead of a Carnot–Carathéodory ball).

As stated in the introduction, we are now in a position to prove that, for any $c \in (0, 1]$ the inequality

$$|F \cdot G|^{\frac{1}{Q}} \geq |F|^{\frac{1}{Q}} + c|G|^{\frac{1}{Q}}, \quad F, G \subset \mathbb{H}^n \text{ measurable}, \tag{4.9}$$

is false in general.

Proposition 4.10. *The Brunn–Minkowski-type inequality* (4.9) *cannot hold for any pair (F, G) of measurable subsets of $\mathbb{H}^n$.*

Proof. Take $C_h = \{(z,t) : \ |z| < 1, \ |t| < h\}$ and let F be an open, bounded set of class C^2. Then, $P_{\mathbb{H}}(F)$ is finite and, by Corollary 4.8, we have $\mathcal{M}_{C_h}(F) = P_{\mathbb{H}}(F)$ for all $h > 0$. Therefore, if (4.9) were satisfied with $G = C_h$, we would get

$$|F| \frac{\left(1 + c\left(\frac{|C_h|}{|F|}\right)^{\frac{1}{Q}} \epsilon\right)^Q - 1}{\epsilon} \leq \frac{|F \cdot \delta_\epsilon(C_h)| - |F|}{\epsilon},$$

hence by taking the limit as $\epsilon \to 0$ we would obtain

$$|F| c Q \left(\frac{|C_h|}{|F|}\right)^{\frac{1}{Q}} \leq P_{\mathbb{H}}(F) \qquad \forall\, h > 0,$$

which is clearly false in general, because $|C_h| \to +\infty$ as $h \to +\infty$. □

Acknowledgements. We warmly thank Luigi Ambrosio, Zoltan Balogh, Yuxin Ge, and Roberto Monti for many fruitful discussions.

References

1. Agrachev, A.A., Sachkov, Yu.L.: Control Theory from the Geometric Viewpoint. Lecture notes S.I.S.S.A. (2002)
2. Ambrosio, L.: Some fine properties of sets of finite perimeter in Ahlfors regular metric measure spaces. Adv. Math. **159**, 51–67 (2001)
3. Ambrosio, L., Kirchheim, B.: Rectifiable sets in metric and Banach spaces. Math. Ann. **318**, 527–555 (2000)
4. Ambrosio, L., Kirchheim, B.: Currents in metric spaces. Acta Math. **185**, 1–80 (2000)
5. Bellaïche, A.: The tangent space in sub-Riemannian geometry. In: Sub-Riemannian geometry, pp. 1–78. Basel: Birkhäuser 1996
6. Brakke, K.A.: The surface evolver. Exp. Math. **1**, 141–165 (1992)
7. Burago, Yu.D., Zalgaller, V.A.: Geometric inequalities. Translated from the Russian by A.B. Sosinskiĭ. Springer Series in Soviet Mathematics. Berlin Heidelberg New York: Springer 1988
8. Citti, G., Sarti, A.: A cortical based model of perceptual completion in the roto-translation space. In: Birindelli, I., Gutierez, C., Lanconelli, E. (eds.): Proceedings of the workshop on second order subelliptic equations and applications, Cortona, June 16–22, 2003. Lecture Notes of Seminario Interdisciplinare di Matematica, **3**, 145–161 (2004)
9. De Giorgi, E.: Sulla proprietà isoperimetrica dell'ipersfera, nella classe degli insiemi aventi frontiera orientata di misura finita. Atti Accad. Naz. Lincei, Mem. Cl. Sci. Fis. Mat. Nat. Sez. I (8) **5**, 33–44 (1958)
10. Federer, H.: Geometric Measure Theory. Berlin Heidelberg New York: Springer 1969
11. Folland, G.B.: Harmonic Analysis in Phase Space. Princeton, NJ: Princeton University Press 1989
12. Folland, G.B., Stein, E.M.: Hardy Spaces on Homogeneous Groups. Princeton, NJ: Princeton University Press 1982
13. Franchi, B., Serapioni, R., Serra Cassano, F.: Meyers-Serrin type theorems and relaxation of variational integrals depending on vector fields. Houston J. Math. **22**, 859–889 (1996)
14. Franchi, B., Serapioni, R., Serra Cassano, F.: Rectifiability and perimeter in the Heisenberg group. Math. Ann. **321**, 479–531 (2001)
15. Franchi, B., Serapioni, R., Serra Cassano, F.: Rectifiability and perimeter in step 2 groups. Math. Bohem. **127**, 219–228 (2002)
16. Franchi, B., Serapioni, R., Serra Cassano, F.: Regular hypersurfaces, Intrinsic perimeter and implicit function theorem in Carnot groups. Commun. Anal. Geom. **11**(5), 909–944 (2003)
17. Gardner, R.J.: The Brunn–Minkowski inequality. Bull. Am. Math. Soc. **39**, 355–405 (2002)
18. Garofalo, N., Nhieu, D.-M.: Isoperimetric and Sobolev inequalities for Carnot-Carathéodory spaces and the existence of minimal surfaces. Commun. Pure Appl. Math. **49**, 1081–1144 (1996)
19. Gaveau, B.: Principe de moindre action, propagation de la chaleur et estimées sous-elliptiques sur certains groupes nilpotents. Acta Math. **139**, 95–153 (1977)
20. Gromov, M.: Carnot-Carathéodory spaces seen from within. In: Sub-Riemannian Geometry. pp. 79–323. Basel: Birkhäuser 1996
21. Leonardi, G.P., Rigot, S.: Isoperimetric sets on Carnot groups. Houston J. Math. **29**, 609–637 (2003)
22. Magnani, V.: Differentiability and area formula on stratified Lie groups. Houston J. Math. **27**, 297–323 (2001)
23. Montefalcone, F.: Sets of finite perimeter associated with vector fields and polyhedral approximation. Rend. Mat. Acc. Lincei **14**, 270–295 (2003)
24. Montgomery, R.: A tour of subriemannian geometry. Math. Surveys and Monographs, vol. 91, pp. xx + 259. Providence, RI: American Mathematical Society 2002

25. Monti, R.: Some properties of Carnot-Carathéodory balls in the Heisenberg group. Atti Accad. Naz. Lincei, Cl. Sci. Fis. Mat. Nat. IX. Ser., Rend. Lincei, Mat. Appl. **11**, 155–167 (2001)
26. Monti, R.: Brunn-Minkowski and isoperimetric inequality in the Heisenberg group. Ann. Acad. Sci. Fenn. Math. **28**, 99–109 (2003)
27. Monti, R., Serra Cassano, F.: Surfaces measures in Carnot-Carathéodory spaces. Calc. Var. **13**, 339–376 (2001)
28. Monti, R., Morbidelli, D.: Isoperimetric inequality in the Grushin plane. J. Geom. Anal. **14**, 355–368 (2004)
29. Pansu, P.: Une inégalité isopérimétrique sur le groupe de Heisenberg. C. R. Acad. Sci. Paris Sér. I Math. **295**, 127–130 (1982)
30. Pansu, P.: An isoperimetric inequality on the Heisenberg group. Conference on differential geometry on homogeneous spaces (Turin 1983). Rend. Semin. Mat., Torino, Special Issue, 159–174 (1984)
31. Pauls, S.: Minimal surfaces in the Heisenberg group. Geom. Dedicata **104**, 201–231 (2004)