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Probabilistic solutions of fractional differential and partial differential equations and their Monte Carlo simulations

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Abstract

The work in this paper is four-fold. Firstly, we introduce an alternative approach to solve fractional ordinary differential equations as an expected value of a random time process. Using the latter, we present an interesting numerical approach based on Monte Carlo integration to simulate solutions of fractional ordinary and partial differential equations. Thirdly, we show that this approach allows us to find the fundamental solutions for fractional partial differential equations (PDEs), in which the fractional derivative in time is in the Caputo sense and the fractional in space one is in the Riesz-Feller sense. Lastly, using Riccati equation, we study families of fractional PDEs with variable coefficients which allow explicit solutions. Those solutions connect Lie symmetries to fractional PDEs.

Keywords: Caputo fractional derivative, Riesz-Feller fractional derivative, Riccati equation, Lie symmetries, Green functions, Monte Carlo Integration, Mittag-Leffler function.

1. Introduction

Although it was started in the second half of the eighteenth century by Leibniz, Newton and l'Hôpital, [1], fractional calculus has received great attention in the last two decades. Many physical, biological and epidemiological models have found that fractional order models could perform at least as good as well as their integer counterparts, [2, 3]. Integer order models are also appearing as special cases of the fractional order. That makes the dynamical behavior of those models richer and in some cases flexible. As advances are made in fractional calculus and fractional modeling, understanding of the physical interpretation of fractional derivatives is becoming clearer. A memory kernel with algebraic decay is the most common interpretation for the change in the dynamical behavior of the system [4, 5, 6]. Today fractional calculus is widely used in physical modeling. Examples include the nonexponential relaxation in dielectrics and ferromagnets [7], [8], the diffusion processes [9], [10] and the Hamiltonian Chaos [11] and [12].

Anomalous diffusion could be modeled using fractional order stochastic processes and their Fokker-Planck equations [13, 14, 15]. Continuous-time random walk (CTRW) is one approach used to model anomalous diffusion [16]. In particular, a CTRW with infinite-mean time to jump exhibits sub-diffusion behavior. The time to jump could be modeled by a heavy tail distribution with index β such that $0 < \beta < 1$ leading to a mean square displacement that is of order t^β depicting the short-range jump and so thus sub-diffusion. A long-range jump leads to super-diffusion, i.e., $\beta > 1$, [17].

A random time is an increasing Lévy process, which is a non-negative process with independent stationary increments; it is usually used as operational time in a physical system, or a subordinator, see e.g. [18, 19]. One of those subordinators is the $\alpha-$stable random time (or $\alpha-$stable subordinator) whose density decays algebraically like $1/t^{\alpha+1}$, as $t \to \infty$ with $0 < \alpha < 1$. That time change leads to super-diffusion when one takes $\alpha = \alpha'/2$. Another time change is using the $\beta-$inverse subordination with $0 < \beta < 1$, which results in a sub-diffusion.

The Riccati equation has played an important role in finding explicit solutions for Fisher and Burgers equations, (see [20] and [21] and references therein). Also, similarity transformations and the solutions of Riccati and Ermakov systems have been extensively applied thanks to Lie groups and Lie algebras [22], [23], [24], [25] and [26]. In this work, we show that this approach allows us to find the fundamental solutions for fractional PDEs; the fractional derivative in time is in the Caputo sense and the fractional derivative in space one is in the Riesz-Feller sense. We establish a relationship between the coefficients through the Riccati equation; we study families of fractional PDEs with variable coefficients which allow explicit solutions and find those explicit solutions. Those solutions connect Lie symmetries to fractional PDEs.

Formulating solutions of fractional differential equations and partial differential equations as expected values with respect to heavy-tail or power law distributions could be enabled using Monte-Carlo integration methods and Sampling Importance Integration to evaluate them. The mean problem is in the number of simulations or random number generations that need to be done to guarantee convergence and small standard error.

In Section 1, we will review fundamental definitions and classical results needed from the classical theory of fractional differential equations. In Section 2, we present the first main result of this work, Lemma 1, which allow us to see a fractional functions and their fractional derivatives as Wright type transformations of some functions and their derivatives. They could be also interpreted as expected values of functions in a random time process. Also, in Section 2 we present Theorem 2.1 which allow us to solve fractional ordinary differential equations through solutions of regular ordinary differential equations. In Section 3, we derive fractional green functions for some important fractional partial differential equations, like diffusion, telegraph, Schrödinger, and wave equations. In Section 4, we derive green functions of fractional partial differential equations with variable coefficients with application to Fokker-Planck equations. In Section 5, we use the integral transform or the expected value interpretation of the solutions of fractional equations to carry out Monte Carlo simulations of their solution.

1.1. **Preliminaries.** In this section we give the required background of Caputo and Riesz-Feller fractional differentiation.

Caputo Derivative. Let D^n be the Leibniz integer-order differential operator given by

$$D^n f = \frac{d^n f}{dt^n} = f^{(n)},$$

and let J^n be an integration operator of integer order given by

$$J^n f(t) = \frac{1}{n-1!} \int_0^t (t-\tau)^{n-1} f(\tau) d\tau, \tag{1.1}$$

where $n \in \mathbb{Z}^+$. Let us use $D = D^1$ for the first derivative. For fraction-order integrals, we use

$$J^{n-\beta} f(t) = \frac{1}{\Gamma(n-\beta)} \int_0^t (t-\tau)^{n-\beta-1} f(\tau) d\tau, \tag{1.2}$$

where $n-1<\beta\leq n$. Now, define the Caputo fractional differential operator D_C^{β} to be

$$D_C^{\beta}f(t)=J^{n-\beta}D^n f(t),$$

where $n-1<\beta\leq n$, for $n\in\mathbb{N}$. It is also known that

$$\lim_{\beta\to n} D_C^{\beta}f(t)=f^{(n)}(t),$$
$$\lim_{\beta\to n-1} D_C^{\beta}f(t)=f^{(n-1)}(t)-f^{(n-1)}(0) \tag{1.3}$$

for any $n\in\mathbb{N}$.

Riemann-Liouville Derivative. The Riemann-Liouville fractional differential operator D_{RL}^{β} is defined to be

$$D_{RL}^{\beta}f(t)=D^nJ^{n-\beta}f(t),$$

where $n-1<\beta\leq n$, for $n\in\mathbb{N}$. We will use $\partial_t^{\beta}F:=\dfrac{\partial^{\beta}F}{\partial t^{\beta}}$ and use $\partial_t F:=\dfrac{\partial F}{\partial t}$.

The Riemann-Liouville fractional is related to the Caputo fractional derivative through [27]:

$$D_{RL}^{\beta}f(t)=D_C^{\beta}f(t)+\sum_{k=0}^{n-1}f^{(k)}(0)\frac{t^k}{k!}.$$

While we will not discuss the Riemann-Liouville fractional derivatives in the paper, the results presented in this paper are valid for Riemann-Liouville fractional derivatives, when $f^{(k)}(0)=0$ for $k=0,1,\ldots,n-1$.

We will consider $n=1$ in this work; that is $0<\beta\leq 1$. Some of the results be extended through the remark that for $0<\beta\leq 1$, $D_C^{n+\beta-1}f(t)=D_C^{\beta}f^{(n-1)}(t)$ for $n\geq 1$. Note also that when $0<\beta\leq 1$, $D_{RL}^{\beta}f(t)=D_C^{\beta}f(t)+f(0)$.

Riesz-Feller Derivative. The Riesz-Feller fractional differential operator $D_{RF}^{\alpha,\theta}$ is defined to be [28]

$$\begin{aligned}D_{RF}^{\alpha,\theta}f(x) &= \frac{\Gamma(1+\alpha)}{\pi}\sin((\alpha+\theta)\frac{\pi}{2})\int_0^{\infty}\frac{f(x+y)-f(x)}{y^{1+\alpha}}dy\\ &\quad+\frac{\Gamma(1+\alpha)}{\pi}\sin((\alpha-\theta)\frac{\pi}{2})\int_0^{\infty}\frac{f(x-y)-f(x)}{y^{1+\alpha}}dy\end{aligned}$$

for fractional order $0<\alpha\leq 2$, and the skewness parameter $\theta\leq\min(\alpha,2-\alpha)$. The symmetric Riesz-Feller differential operator is defined at $\theta=0$ and is simply denoted by D_{RF}^{α}.

Transformations. The Laplace transform of a function $f(t)$ is defined as

$$\mathcal{L}(f)(s)=\widetilde{f}(s)=\int_0^{\infty}e^{-st}f(t)dt.$$

The inverse Laplace transform is defined by

$$\mathcal{L}^{-1}\left(\widetilde{f}\right)(t)=\frac{1}{2\pi i}\int_{\mathcal{C}}e^{st}\widetilde{f}(s)ds$$

where $\mathcal{C}$ is a contour parallel to the imaginary axis and to the right of the singularities of $\widetilde{f}$. The Laplace transform of the Caputo fractional derivative of a function is given by

$$\mathcal{L}\left(D_C^{\beta} f\right)(s) = s^{\beta}\widetilde{f}(s) - \sum_{k=0}^{n-1} s^{\beta-1-k} f^{(k)}(0). \tag{1.4}$$

The Fourier transform of a function $f(x)$ is defined as

$$\mathcal{F}(f)(y) = \widehat{f}(y) = \int_{-\infty}^{\infty} e^{ixy} f(x)dx.$$

The inverse Fourier transform is defined by

$$\mathcal{F}^{-1}\left(\widehat{f}\right)(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ixy}\widehat{f}(y)dy.$$

The Fourier transform of the Riesz-Feller fractional derivative of a function is given by

$$\mathcal{F}\left(D_{RF}^{\alpha,\theta} f\right)(y) = -\psi_{\alpha}^{\theta}(y)\widehat{f}(y), \tag{1.5}$$

where $\psi_{\alpha}^{\theta}(y) = |y|^{\alpha} e^{\frac{i\text{sign}(y)\theta\pi}{2}}$.

Mittag-Leffler Function. The Mittag-Leffler function, which generalizes the exponential function, can be written as follows,

$$E_{\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\beta k+1)}, \qquad \beta \in \mathbb{R}^+, \ \ z \in \mathbb{C}, \tag{1.6}$$

and the more general Mittag-Leffler function with two-parameters is defined to be

$$E_{\beta,\alpha}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\beta k+\alpha)}, \qquad \beta,\alpha \in \mathbb{R}^+, \ \ z \in \mathbb{C}. \tag{1.7}$$

Wright function. The Wright function is another special function of importance to fractional calculus and is defined by [29],

$$W_{\beta,\alpha}(z) = \sum_{k=0}^{\infty} \frac{z^k}{k!\Gamma(\beta k+\alpha)}, \qquad \beta > -1, \alpha \in \mathbb{C}, \ \ z \in \mathbb{C}. \tag{1.8}$$

The following Wright type function will be fundamental in the rest of this work

$$g_{\beta}(x;t) = \frac{1}{t^{\beta}} W_{-\beta,1-\beta}\left(-\frac{x}{t^{\beta}}\right), \tag{1.9}$$

which is a probability density function of the random time process $\mathcal{T}_{\beta}(t)$ for all $t > 0$ [30, 31]. It has a Laplace transform

$$\mathcal{L}\left(g_{\beta}(\cdot;t)\right)(s) = \int_0^{\infty} e^{-sx} g_{\beta}(x;t)dx = E_{\beta}\left(-st^{\beta}\right) \tag{1.10}$$

for $\Re(s) > 0$ and moments $\mathbb{E}\left[(\mathcal{T}_{\beta}(t))^k\right] = \Gamma(k+1)\frac{t^{k\beta}}{\Gamma(k\beta+1)}$ for $k \geq 1$ [32, 33]. At the same time

$$\int_0^{\infty} e^{-st} g_{\beta}(x;t)dt = s^{\beta-1}e^{-xs^{\beta}}. \tag{1.11}$$

It is shown in [30] that for $0<\beta,\alpha<1$,

(1.12) $$g_{\beta\alpha}(x;t)=\int_0^\infty g_\beta(x;s)g_\alpha(s;t)ds$$

for $x>0$. For more details about Wright function see [30, 31]. From (1.10) and (1.12), we get

$$\int_0^\infty E_\beta\left(-st^\beta\right)g_\alpha(t;r)dt=\int_0^\infty\int_0^\infty e^{-sx}g_\beta(x;t)g_\alpha(t;r)dxdt=E_{\beta\alpha}\left(-sr^{\beta\alpha}\right),$$

and from (1.11) and (1.12), we get

$$\int_0^\infty E_\alpha\left(-tx^\alpha\right)g_\beta(x;t)dt=\int_0^\infty s^{\beta-1}e^{-xs^\beta}g_\alpha(s;x)ds.$$

Lévy α-stable distribution. The Lévy α-stable distribution with stability index $0<\alpha\le 2$, $L_\alpha^\theta(x)$ has a Fourier transform given by

(1.13) $$\mathcal{F}\left(L_\alpha^\theta(\cdot)\right)(y):=\widehat{L}_\alpha^\theta(y)=e^{-\psi_\alpha^\theta(y)}.$$

The density $L_\alpha^\theta(x)$ has a fat tail proportional to $|x|^{-(1+\alpha)}$.

Define $L_\alpha^\theta(y;x)=\dfrac{1}{x^{\frac{1}{\alpha}}}L_\alpha^\theta\left(\dfrac{y}{x^{\frac{1}{\alpha}}}\right)$ for $y\in\mathbb{R}$ and $x>0$ which is a probability density function of the α-stable random process with asymmetry parameter θ, denoted by $\mathcal{L}_\alpha^\theta(x)$ for $x>0$, [34].

(1.14) $$\int_{-\infty}^\infty e^{isy}L_\alpha^\theta(y;x)dy=e^{-\psi_\alpha^\theta(s)x}.$$

For $0<\alpha<1$, and $t,x>0$,

(1.15) $$L_\alpha^{-\alpha}(t;x)=\frac{x\alpha}{t}g_\alpha(x;t).$$

See [35] for more details. Also, for $0<\alpha\le 1$

(1.16) $$L_{\beta\alpha}^{\theta\alpha}(x;t)=\int_0^\infty L_\beta^\theta(x;s)L_\alpha^{-\alpha}(s;t)ds.$$

2. Fractional Derivative as Expected Value with Respect to a Lévy Distribution

In this Section we establish the first main result of this paper.

2.1. **Caputo Fractional Derivative.**

The following lemma is one of the main results of this work for its wide applicability. A fundamental remark is that based on Lemma 1, $\mathbb{E}\left[f(\mathcal{T}_\beta(t))\right]=f_\beta(t)$, and $\mathbb{E}\left[f^{(1)}(\mathcal{T}_\beta(t))\right]$.

Lemma 1.

(1) *Let $f\in C([0,\infty))$ and f_β be a function. Hence,*

$$\widetilde{f_\beta}(s)=s^{\beta-1}\widetilde{f}\left(s^\beta\right)$$

for $s\in[0,\infty)$ if and only if

$$f_\beta(t)=\int_0^\infty f(x)g_\beta(x;t)dx$$

for $0<\beta\le 1$, and if integrals exist.

(2) *Let $f \in C^1([0,\infty))$ and f_β be a function such that*

$$\widetilde{f_\beta}(s) = s^{\beta-1}\widetilde{f}\left(s^\beta\right)$$

for $s \in [0,\infty)$ and $f_\beta(0) = f(0)$ then

$$D_C^\beta f_\beta(t) = \int_0^\infty f^{(1)}(x) g_\beta(x;t)dx$$

for $0 < \beta \leq 1$.

(3) *Let $f \in C^n([0,\infty))$ and f_β be a function such that*

$$\mathcal{L}\left(f_\beta^{(n-1)}(\cdot)\right)(s) = s^{\beta-1}\mathcal{L}\left(f^{(n-1)}(\cdot)\right)\left(s^\beta\right)$$

for $s \in [0,\infty)$ and $f_\beta^{(n-1)}(0) = f^{(n-1)}(0)$. For $0 < \beta \leq 1$, the following holds

$$D_C^{n+\beta-1} f_\beta(t) = \int_0^\infty f^{(n)}(x) g_\beta(x;t)dx$$

for $n \geq 1$.

(4) *For $0 < \alpha, \beta < 1$ and such that $f_{\beta\alpha}(0) = f_\beta(0)$*

$$D_C^{\beta\alpha} f_{\beta\alpha}(t) = D_C^\alpha\left(D_C^\beta f_\beta(s)\right).$$

Proof.

(1) To show sufficiency, we use the Laplace transform as follows:

$$\begin{aligned}\mathcal{L}\left(\int_0^\infty f(x) g_\beta(x;\cdot)dx\right)(s) &= \int_0^\infty \int_0^\infty e^{-st} f(x) g_\beta(x;t)dxdt \\ \text{by equation (1.11)} &= \int_0^\infty f(x) s^{\beta-1} e^{-xs^\beta} dx \\ &= s^{\beta-1}\widetilde{f}\left(s^\beta\right) \\ &= \widetilde{f_\beta}(s) = \mathcal{L}\left(f_\beta(\cdot)\right)(s).\end{aligned}$$

Necessity follows from the same lines.

(2) We will show that the Laplace transform of the right-hand side of equation (2) is equal to that of the left hand side which is given by (1.4). The Laplace transform of the right hand side of equation (2) is given by

$$\begin{aligned}\mathcal{L}\left(\int_0^\infty f^{(1)}(x) g_\beta(x;\cdot)dx\right)(s) &= \int_0^\infty \int_0^\infty e^{-st} f^{(1)}(x) g_\beta(x;t)dxdt \\ \text{by equation (1.11)} &= \int_0^\infty f^{(1)}(x) s^{\beta-1} e^{-xs^\beta} dx \\ &= s^{\beta-1}\left(s^\beta \widetilde{f}(s^\beta) - f(0)\right) \\ &= s^\beta \widetilde{f_\beta}(s) - s^{\beta-1} f_\beta(0) \\ &= \mathcal{L}\left(D_C^\beta f_\beta(\cdot)\right)(s).\end{aligned}$$

(3) It follows directly from part 2 and that $D_C^{n+\beta-1} f(t) = D_C^\beta f^{(n-1)}(t)$.

(4) Using equation (1.12),

$$D_C^{\beta\alpha} f_{\beta\alpha}(t) = \int_0^\infty f'(x) g_{\beta\alpha}(x;t)dx = \int_0^\infty f'(x) \int_0^\infty g_\beta(x;s) g_\alpha(s;t) ds dx,$$

and using part 2, we obtain

$$\begin{aligned} D_C^{\beta\alpha} f_{\beta\alpha}(t) &= \int_0^\infty g_\alpha(s;t) \int_0^\infty f'(x) g_\beta(x;s) dx ds \\ &= \int_0^\infty D_C^\beta f_\beta(s) g_\alpha(s;t) ds = D_C^\alpha \left(D_C^\beta f_\beta(t) \right). \end{aligned}$$

□

Theorem 1. *The linear fractional differential equation*

$$\sum_{k=1}^{n} a_k D_C^{k+\beta-1} y(t) + a_{n+1} y(t) = F_\beta(t) \tag{2.1}$$

such that $y(0) = y_0$ *and* $D_C^{\beta+i-1} y(0) = y_i$ *for* $i = 1, \ldots, n-1$ *and* $0 < \beta \le 1$, *has a solution given by*

$$y_\beta(t) = \int_0^\infty z(x) g_\beta(x;t) dx$$

where $z(t)$ *is the solution of the linear ordinary differential equation*

$$\sum_{k=1}^{n} a_k z^{(k)}(t) + a_{n+1} z(t) = F(t) \tag{2.2}$$

such that $z^{(i)}(0) = y_i$ *for* $i = 0, \ldots, n-1$ *and* $F_\beta(t) = \int_0^\infty F(x) g_\beta(x;t) dx$.

Remark. The function $F(x)$ could be found using Laplace transform and Lemma 1 part a.

$$\widetilde{F}_\beta(s) = s^{\beta-1} \widetilde{F}\left(s^\beta\right).$$

Proof. Taking a $\beta-$Wright type transformation on both sides of equation (2.2), we obtain

$$\sum_{k=1}^{n} a_k \int_0^\infty z^{(k)}(x) g_\beta(x;t) dx + a_{n+1} \int_0^\infty z(x) g_\beta(x;t) dx = \int_0^\infty F(x) g_\beta(x;t) dx,$$

and applying Lemma 1 we obtain

$$\sum_{k=1}^{n} a_k D_C^{k+\beta-1} y_\beta(t) + a_{n+1} D_C^{\beta-1} y_\beta(t) = F_\beta(t)$$

as we wanted. □

In the following, let $0 < \beta \le 1$. The best method to find

$$f_\beta(t) = \int_0^\infty f(x) g_\beta(x;t) dx$$

is by using Taylor's expansion of $f(t)$ about 0 giving

$$f_\beta(t) = \sum_{n=0}^{\infty} f^{(n)}(0) \frac{t^{\beta n}}{\Gamma(n\beta + 1)}.$$

That relationship can be used to find many fractional analogue functions that can be found in the literature. It will be also seen through the following examples.

Example 1 (Fractional velocity). *The solution of the FDE $D_C^\beta y_\beta(t) = c$ with $y_\beta(0) = y_0$, where c is a real-valued constant is given by*

$$y_\beta(t) = \int_0^\infty (y_0 + x) g_\beta(x; t) dx = y_0 + \frac{t^\beta}{\Gamma(\beta + 1)}$$

since $y(t) = y_0 + t$ solves $Dy(t) = c$, with $y(0) = y_0$. That is, $y_\beta(t) = \mathbb{E}[y_0 + \mathcal{T}_\beta(t)]$.

Example 2 (Fractional growth/decay models). *The solution of the FDE $D_C^\beta y(t) = \lambda y(t)$ with $y(0) = y_0$ where λ is a real-valued constant is given by*

$$y(t) = \int_0^\infty y_0 e^{\lambda x} g_\beta(x; t) dx = y_0 E_\beta \left(\lambda t^\beta \right)$$

since $z(x) = y_0 e^{\lambda x}$ solves $Dz(x) = \lambda z(x)$, with $z(0) = y_0$. That is, $y_\beta(t) = \mathbb{E}[y_0 e^{\lambda \mathcal{T}_\beta(t)}]$. See section 5 for graphical representation.

Example 3 (Fractional oscillations). *The solution of the FDE $D_C^{\beta+1} y(t) = -\omega^2 y(t)$ with $y(0) = 0$, where ω is a real-valued constant is given by*

$$y(t) = \int_0^\infty \sin(\omega x) g_\beta(x; t) dx = \sin_\beta(\omega t)$$

since $z(x) = \sin(\omega x)$ solves $D^2 z(x) = -\omega^2 z(x)$, with $z(0) = 0$. That is, $y_\beta(t) = \mathbb{E}[\sin(\omega \mathcal{T}_\beta(t))]$.

The fractional analogue of the sine function could be found using [36]

$$\sin_\beta(t) = \sum_{n=0}^{\infty} (-1)^n \frac{t^{(2n+1)\beta}}{\Gamma((2n+1)\beta + 1)} = t^\beta E_{2\beta, \beta+1}\left(-t^{2\beta}\right).$$

Similarly,

$$\cos_\beta(t) = \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n\beta}}{\Gamma(2n\beta + 1)} = E_{2\beta}\left(-t^{2\beta}\right).$$

It is not hard then to see that $D_C^\beta \cos_\beta(t) = -\sin_\beta(t)$ and $D_C^\beta \sin_\beta(t) = \cos_\beta(t)$.

3. Green functions for fractional partial differential equations

In the following we will use the notation ${}_C D_t^\beta$ and ${}_{RF} D_x^\alpha$ for Caputo and Riesz-Feller derivatives, respectively, to identify the variable with respect to which the derivatives are calculated.

3.1. **The fractional diffusion equation.** In this example, using the approach of the previous section, we find the Green function for the fractional diffusion equation

$$\begin{cases} {}_C D_t^{\beta} u(x,t) = {}_{RF}D_x^{\alpha,\theta} u(x,t) & (x,t) \in \mathbb{R} \times [0,\infty) \\ u(x,0) = f(x) & x \in \mathbb{R} \end{cases} \tag{3.1}$$

with $0 < \beta \leq 1$, $0 < \alpha \leq 2$ and $u(\pm\infty, t) = 0$, $t > 0$. Applying the Fourier transform to x, we obtain

$$\begin{aligned} {}_C D_t^{\beta} \widehat{u}(k,t) &= -\psi_\alpha^\theta(k)\widehat{u}(k,t) \\ \widehat{u}(k,0) &= \widehat{f}(k). \end{aligned}$$

Solving this ordinary differential equation using Lemma 1, we obtain

$$\widehat{u}(k,t) = \int_0^\infty \widehat{f}(k) e^{-\psi_\alpha^\theta(k)x} g_\beta(x;t)dx = \widehat{f}(k) E_\beta\left(-\psi_\alpha^\theta(k)t^\beta\right),$$

see example 2.1. Using the Inverse Fourier Theorem and the convolution theorem we get

$$u(x,t) = \int_{-\infty}^{\infty} G_{\alpha,\beta}^\theta(x-y,t) f(y)dy$$

where the Green function is given by

$$G_{\alpha,\beta}^\theta(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx} E_\beta\left(-\psi_\alpha^\theta(k)t^\beta\right)dk.$$

In the particular case of $\theta = 0$,

$$G_{\alpha,\beta}^0(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx} E_\beta\left(-|k|^\alpha t^\beta\right)dk. \tag{3.2}$$

For the case $\beta = 1, \alpha = 2$, we obtain the classical heat equation

$$\begin{cases} \partial_t u(x,t) = \partial_x^2 u(x,t) & (x,t) \in \mathbb{R} \times [0,\infty) \\ u(x,0) = f(x). & x \in \mathbb{R} \end{cases}$$

Since $E_1(\lambda t) = e^{\lambda t}$, we obtain for $x \in \mathbb{R}$ and $t > 0$

$$G_{1,2}^0(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx - tk^2} dk = \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}}$$

or $L_2^0(x,t)$, by using standard formulas from Fourier transform. This result agrees with the Green function of the standard heat equation, see for example [37].

Corollary 1. *The solution of the fractional heat equation* (3.1) *could be written as*

$$u_{\alpha,\beta}(x,t) = \int_0^\infty \int_0^\infty L_2^0(x,\tau) L_{\alpha/2}^{-\alpha/2}(\tau,s) g_\beta(s,t) d\tau ds, \tag{3.3}$$

where $L_2^0(x,\tau)$ is the solution of the classical heat equation.

Proof. As was shown in [28], it could be shown that the solution in (3.2) could be rewritten as

$$u_{\alpha,\beta}(x,t) = \int_0^\infty L_\alpha^0(x,s) g_\beta(s,t) ds,$$

but for $0 < \alpha/2 \leq 1$

$$L^0_\alpha(x,s) = \int_0^\infty L^0_2(x,\tau) L^{-\alpha/2}_{\alpha/2}(\tau,s) d\tau, \tag{3.4}$$

by equation (1.16). Then the result follows. □

Based on this corollary, $u_{\alpha,\beta}(x,t) = \mathbb{E}\left[L^0_2(x, \mathcal{L}^{-\alpha/2}_{\alpha/2}(\mathcal{T}_\beta(t)))\right]$ where $\mathcal{L}^{-\alpha/2}_{\alpha/2}(t)$ is the $\alpha/2-$Lévy process with $\theta = -\alpha/2$ and $\mathcal{T}_\beta(t)$ is the random $\beta-$time process. See section 5 for graphical representation.

The moments of $X_{\alpha,\beta}(t) \sim u_{\alpha,\beta}(\cdot,t)$ are given by [34],

$$\mathbb{E}(|X_{\alpha,\beta}(t)|^s) = t^{\frac{s\beta}{\alpha}} \frac{\Gamma(1-\frac{s}{\alpha})\Gamma(1+\frac{s}{\alpha})\Gamma(1+s)}{\Gamma(1-\frac{s}{2})\Gamma(1+\frac{s}{2})\Gamma(1+\frac{s\beta}{\alpha})} \tag{3.5}$$

for $-\min(1,\alpha) < \mathcal{R}(s) < \alpha$. That formula shows different regimes of diffusion, subdiffusion and superdiffusion based on whether $\frac{\alpha}{2} = \beta$, $\frac{\alpha}{2} > \beta$, or $\frac{\alpha}{2} < \beta$, respectively.

Moreover, Equation (3.3) shows that the $\alpha - \beta$ fractional process $\{X_{\alpha,\beta}(t), t \geq 0\}$ is equivalent in distribution to the subordinated process $\{B(\mathcal{L}^{-\alpha/2}_{\alpha/2}(\mathcal{T}_\beta(t))), t \geq 0\}$, where $\{B(t), t \geq 0\}$ is a Brownian motion. That relationship postulates that Lévy flights are random dilation, with probability distribution $L^{-\alpha/2}_{\alpha/2}$, of the standard deviation or the time parameter in the Brownian motion. That dilation results in an expand in the range of possible displacement by magnitude beyond the regular tails of the standard Gaussian distribution. The process $\{B(\mathcal{L}^{-\alpha/2}_{\alpha/2}(t)), t \geq 0\}$ was introduced in [38], and used in [39] to model stock price differences. See also more about those subordinated processes in [40].

3.2. **The fractional Telegraph equation.** In this example, we consider the fractional telegraph equation

$$\begin{cases} {}_C D^{\beta+1}_t u(x,t) + {}_C D^{\beta}_t u(x,t) = c^2\, \partial^2_x u(x,t) & (x,t) \in \mathbb{R} \times [0,\infty) \\ u(x,0) = \delta(x) & x \in \mathbb{R} \\ \partial_t u(x,0) = 0 & x \in \mathbb{R} \end{cases} \tag{3.6}$$

with $0 < \beta \leq 1$, $c > 0$ and $u(\pm\infty, t) = 0$, $t > 0$. Notice that in this equation the higher time derivative is of order $\beta + 1$ in contrast to other fractional telegraph models that consider an order of 2β for example in [41, 42, 43, 44]. Note that $2\beta < \beta + 1$ for $0 < \beta < 1$. The solution of the integer Telegraph equation $\beta = 1$ is given by

$$u_1(x,t) = \begin{cases} \dfrac{1}{2c} e^{-\frac{t}{2}} I_0(\frac{\sqrt{c^2t^2 - x^2}}{2c}), \text{ for } |x| < ct, \\ 0, \text{ otherwise,} \end{cases}$$

where I_0 is the modified Bessel function (see below). Thus, the solution of the fractional telegraph equation (3.6) is given by

$$u_\beta(x,t) = \frac{1}{2c} \int_{|x|/c}^\infty e^{-\frac{s}{2}} I_0(\frac{\sqrt{c^2s^2 - x^2}}{2c}) g_\beta(s,t) ds.$$

3.3. **The time and space fractional Schrödinger equation.** Let $0 < \beta, \alpha \leq 1$. The solution of the time and space-fractional Schrödinger equation

$$\begin{cases} {}_0D_t^{\beta}\phi_{\beta,\alpha}(x,t) = ih_x D_{\theta}^{\alpha+1}\phi_{\beta,\alpha}(x,t) - ic_x D_{\theta}^{\alpha}\phi_{\beta,\alpha} & (x,t) \in \mathbb{R}\times[0,\infty) \\ \phi_{\beta,\alpha}(x,0) = f(x) & x \in \mathbb{R} \end{cases}$$

is as follows. The Schrödinger Fourier transform is

$$\begin{cases} {}_0D_t^{\beta}\widehat{\phi_{\beta,\alpha}}(x,t) = -i\widehat{\phi_{\beta,\alpha}}(k,t)\left[h\psi_{\alpha+1}^{\theta}(k) - c\psi_{\alpha}^{\theta}(k)\right] \\ \widehat{\phi_{\beta,\alpha}}(k,0) = F(k) \end{cases}$$

Then, applying lemma 3, the solution is given by $\widehat{\phi_{\beta,\alpha}}(k,t) = \int_0^{\infty} z(x)g_{\beta}(x;t)dx$,where the function $z(x) = F(k)\exp\left(-ix\left[h\psi_{\alpha+1}^{\theta}(k) - c\psi_{\alpha}^{\theta}(k)\right]\right)$ and solves the last Fractional PDE initial problem. Now, setting the new solution we have that

$$\widehat{\phi_{\beta,\alpha}}(k,t) = F(k)E_{\beta}\left(-it^{\beta}\left[h\psi_{\alpha+1}^{\theta}(k) - c\psi_{\alpha}^{\theta}(k)\right]\right).$$

Next, applying the inverse Fourier transform and convolution theorems lead us to

$$\phi(x,t) = \mathcal{F}^{-1}\left\{F(k)E_{\beta}\left(-it^{\beta}\left[h\psi_{\alpha+1}^{\theta}(k) - c\psi_{\alpha}^{\theta}(k)\right]\right)\right\} = \int_{-\infty}^{\infty} G(x-y,t)f(y)dy$$

where the Green function is given by

$$G(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp(ikx)E_{\beta}\left(-it^{\beta}\left[h\psi_{\alpha+1}^{\theta}(k) - c\psi_{\alpha}^{\theta}(k)\right]\right)dk$$

In the particular case $\theta = 0$, and $\beta = \alpha = 1$ it results to

$$G(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp\left(-ithk^2 + itc|k| + ixk\right)dk.$$

3.4. **The linear time and space fractional Schrödinger equation.** Let $0 < \beta, \alpha \leq 1$. The solution of the time and space-fractional linear Schrödinger equation

$$\begin{cases} {}_0D_t^{\beta}\phi_{\beta,\alpha}(x,t) = ih_x D_{\theta}^{\alpha+1}\phi_{\beta,\alpha}(x,t) - ic\phi_{\beta,\alpha} & (x,t) \in \mathbb{R}\times[0,\infty) \\ \phi_{\beta,\alpha}(x,0) = f(x) & x \in \mathbb{R} \end{cases}$$

is as follows. The linear Schrödinger Fourier transform is

$$\begin{cases} {}_0D_t^{\beta}\widehat{\phi_{\beta,\alpha}}(x,t) = -i\widehat{\phi_{\beta,\alpha}}(k,t)\left[h\psi_{\alpha+1}^{\theta}(k) + ic\right] \\ \widehat{\phi_{\beta,\alpha}}(k,0) = F(k) \end{cases}$$

Then, applying lemma 3, the solution is given by $\widehat{\phi_{\beta,\alpha}}(k,t) = \int_0^\infty z(x)g_\beta(x;t)dx$ where the function $z(t) = F(k)\exp\left(-it\left[h\psi^\theta_{\alpha+1}(k)+c\right]\right)$ and solves the last Fractional PDE initial problem. Now, setting the new solution we have that

$$\widehat{\phi_{\beta,\alpha}}(k,t) = F(k)E_\beta\left(-it^\beta\left[h\psi^\theta_{\alpha+1}(k)+c\right]\right).$$

Next, applying the Fourier transform inverse and convolution theorems lead us to

$$\phi(x,t) = \mathcal{F}^{-1}\left\{F(k)E_\beta\left(-it^\beta\left[h\psi^\theta_{\alpha+1}(k)+c\right]\right)\right\} = \int_{-\infty}^{\infty} G(x-y,t)f(y)dy$$

where the Green function is given by

$$G(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp(ikx)E_\beta\left(-it^\beta\left[h\psi^\theta_{\alpha+1}(k)+c\right]\right)dk$$

In the particular case $\theta = 0$, and $\beta = \alpha = 1$ it results to

$$G(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp(-ithk^2 + ixk - itc).$$

Applying the Gaussian integration generalization we get,

$$G(x,t) = \sqrt{\frac{1}{4i\pi th}}\exp\left(\frac{-x^2}{4ith} - itc\right).$$

4. Green functions for fractional PDEs with variable coefficients

In this section we use Lie symmetry group methods and Lemma 1 to introduce families of fractional partial differential equations with variabe coefficients exhibiting explicit solutions. More specifically, we show that solutions for FDE with variable coefficients of the form

$$ {}_CD_t^\beta u(x,t) = \sigma(x)u_{xx}(x,t) + \mu(x)u_x(x,t), 0 < \beta \leq 1. \tag{4.1}$$

can be expressed as a Wright type transformation for PDEs that we define as

$$u_\beta(x,t) = \int_0^\infty u(x,s)g_\beta(s,t)ds, \tag{4.2}$$

where where $u(x,t)$ is the solution of the associated standard PDE

$$\frac{\partial u(x,t)}{\partial t} = \sigma(x)u_{xx}(x,t) + \mu(x)u_x(x,t). \tag{4.3}$$

Theorem 2. *Let's consider the fractional partial differential equation with a variable coefficient of the form (4.1). The solution is given by the Wright-type transformation define by (4.2) of the analogous PDE of the form (4.3) which admits a Green function. Furthermore,*

$$\lim_{\beta\to 1^{-1}} u_\beta(x,t) = u(x,t).$$

Proof. Taking the $\beta-$ Wright type transformation on both sides of (4.1), we get

$$\int_0^\infty \frac{\partial u(x,s)}{\partial s} g_\beta(s;t)ds = \sigma(x)\int_0^\infty u_{xx}(x,s)g_\beta(s;t)ds + \mu(x)\int_0^\infty u_x(x,s)g_\beta(s;t)ds.$$

By Lemma 1, we obtain

$$ {}_C D_t^\beta u_\beta(x,t) = \sigma(x)\frac{\partial^2}{\partial x^2}\int_0^\infty u(x,s)g_\beta(s;t)ds + \mu(x)\frac{\partial}{\partial x}\int_0^\infty u(x,s)g_\beta(s;t)ds.$$

Finally, we obtain

$$ {}_C D_t^\beta u_\beta(x,t) = \sigma(x)\frac{\partial^2 u_\beta(x,t)}{\partial x^2} + \mu(x)\frac{\partial u_\beta(x,t)}{\partial x}.$$

For $0 < v < 1$, we recall that

$$\lim_{v\to 1^-} W_{-v,1-v}(-z) = \delta(x-1),$$

therefore, we get

$$\lim_{v\to 1^-} g_v(s;t) = \lim_{v\to 1^-} t^{-v}W_{-v,1-v}(-st^{-v}) = \frac{1}{t}\delta(st^{-1}-1) = \delta(s-t)$$

Therefore, if we assume $u(x,t) = \int_0^\infty G(x,y,t)f(y)dy$, u has a Green function, using Fubini-Tonelli's theorem and assuming f is a suitable function, then

$$\begin{aligned}\lim_{\beta\to 1^-} u_\beta(x,t) &= \lim_{\beta\to 1^-}\int_0^\infty g_\beta(s;t)u(x,s)ds = \lim_{\beta\to 1^-}\int_0^\infty g_\beta(s;t)\int_0^\infty G(x,y,s)f(y)dyds \\ &= \int_0^\infty \delta(s-t)G(x,y,s)f(y)dy = \int_0^\infty G(x,y,t)f(y)dy.\end{aligned}$$

□

Next, we will present the following families, exhibiting explicit solutions thanks to the Theorem above.

4.1. **Solutions for a fractional Fokker-Planck equation with a forcing function.** Let's recall that the Langevin stochastic differential equation is given by

$$dX(t) = f(X(t))dt + \sqrt{2D}dW(t) \qquad (4.4)$$

such that $X(0) = x_0$, where $\dot{W}(t)$ is a white noise. The solution process $X(t)$ of the Langevin equation (4.4) is a stationary Markov process and is completely specified by finding the probability density $p(x,t|x_0) \geq 0$, which satisfies the Fokker-Planck equation

$$\begin{aligned}\frac{\partial p}{\partial t} &= D\frac{\partial^2 p}{\partial x^2} + \frac{\partial}{\partial x}[f(x)p] \\ p(x,0|x_0) &= \delta(x-x_0).\end{aligned}$$

Applying the Theorem of this Section, we obtain the following Corollaries.

Corollary 2. *If f satisfies the Riccati equation of the form*

$$2f'(x) - f^2(x) + \beta^2 x^2 - \gamma + \frac{16v^2 - 1}{x^2} = 0$$

and β, γ and v are constants, then the family of the fractional Fokker-Planck equation

$$ {}_C D_t^{\beta} u(x,t) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial}{\partial x}\left[f(x)u(x,t)\right] \tag{4.5}$$

where $f(x) = -f(-x)$ (f is an odd function) admits an explicit solution of the form (4.2) where u is given by

$$u(x,t) = \int_0^{\infty} p(x,t|y)\varphi(y)dy, \tag{4.6}$$

p is given by

$$\begin{aligned} p(x,t|x_0) &= F\left(\frac{x}{\sqrt{4\sinh^2(\beta t)}}\right)\frac{e^{\gamma t/4}}{\left(4\sinh^2(\beta t)\right)^{1/4}} \\ &\quad \times \exp\left[-\left(\frac{\beta}{4}\coth(\beta t)x^2 + \frac{1}{2}\int f(x)dx\right) + \frac{x_0^2\beta}{2(1-e^{2\beta t})}\right], \end{aligned}$$

and

$$F(z) = \begin{cases} z^{1/2}\left[A_1 I_{2v}(kz) + A_2 I_{-2v}(kz)\right], & \text{for } z > 0 \\ |z|^{1/2}\left[B_1 K_{2v}(k|z|) + B_2 I_{2v}(k|z|)\right], & \text{for } x < 0 \end{cases}$$

where $k = \beta x_0$ and A_1, A_2, B_1 and B_2 are arbitrary constants be determined by boundary and continuity conditions and if α is a real number

$$\begin{aligned} I_{\alpha}(z) &= \sum_{m=0}^{\infty} \frac{(z/2)^{2m+\alpha}}{m!\Gamma(m+\alpha+1)}, \\ K_{\alpha}(z) &= \frac{\pi}{2}\frac{I_{-\alpha}(z) - I_{\alpha}(z)}{\sin(\alpha\pi)}. \end{aligned}$$

Proof. This Corollary is a direct consequence of the Theorem of this section and the similarity solution presented in [22] using the method of Lie group symmetries for the equation

$$\frac{\partial p}{\partial t} = \frac{\partial^2 p}{\partial x^2} + \frac{\partial}{\partial x}\left[f(x)p\right] \tag{4.7}$$

$$p(x,0|x_0) = \phi(x - x_0) \tag{4.8}$$

when f is odd. □

Corollary 3. *A particular case of interest is obtained if*

$$f(x) = ax + \frac{b}{x}, \text{ such that } a > 0 \text{ and } -\infty < b < 1.$$

The transition probability density is

$$p(x,t|x_0) = a x_0^{1/2}\left(\frac{x}{x_0}\right)^{-\frac{b}{2}} z^{\frac{1}{2}} I_{-\left(\frac{1}{2}+\frac{b}{2}\right)}(kz)$$

$$\times \frac{e^{\gamma t/4}}{\left(4\sinh^2(at)\right)^{1/4}} \exp\left(-\left(\frac{a}{4}\coth(at)\right)^2 - \frac{ax^2}{4}\right)$$

for $x \geq 0$.

Example 4. *Let's consider the standard fractional Fokker-Planck equation of the form*

$$ {}_C D_t^\beta u(x,t) = \frac{\partial^2 u}{\partial x^2} + x\frac{\partial u}{\partial x} + u \tag{4.9}$$

$$ u(x,0) = f(x). \tag{4.10}$$

By the Theorem of this Section, we obtain (4.2) *where* $p(x,t)$ *satisfies the classical Fokker-Planck equation (FPE)*

$$\frac{\partial p}{\partial t} = \frac{\partial^2 p}{\partial x^2} + x\frac{\partial p}{\partial x} + p.$$

It is possible to find the Green function of FPE, see for example [37]. *Therefore we obtain explicit expression for the solution*

$$u(x,t) = \int_0^\infty \int_0^\infty \frac{\exp\left[-\frac{\left(x-e^{-s}y\right)^2}{2(1-e^{-2s})}\right]}{\sqrt{2\pi\left(1-e^{-2s}\right)}} g_\beta(s;t) ds f(y) dy.$$

Corollary 4. *The family of fractional PDEs with space variable coefficients of the form*

$$ {}_C D_t^\beta u(x,t) = x u_{xx}(x,t) + f(x) u_x(x,t) \tag{4.11}$$

$$ u(x,0) = \varphi(x) \tag{4.12}$$

where f *satisfies the Riccati equation of the form*

$$xf' - f + \frac{1}{2}f^2 = Ax^{\frac{3}{2}} + Cx - \frac{3}{8}$$

admits an explicit solution of the form (4.2) *where* u *is given by* (4.6) *and* $p(x,y,t)$ *is the inverse Laplace transform of*

$$U_\lambda(x,t) = \sqrt{\frac{\sqrt{x}\,(1+\lambda t)}{\sqrt{x}\,(1+\lambda t) - A\lambda\frac{t^3}{12}}} \exp[S(x,y,t)]$$
$$\times \exp\left[-\frac{1}{2}\left(F(x) - F\left(\frac{(12(1+\lambda t)\sqrt{x} - A\lambda t^3)^2}{144\,(1+\lambda t)^4}\right)\right)\right]$$

where $F'(x) = \frac{f(x)}{x}$,

$$S(\lambda,x,t) = -\frac{\lambda(x + Ct^2/2)}{1+\lambda t} - \frac{2At^2\sqrt{x}(3+\lambda t)}{3(1+\lambda t)^2} + \frac{A^2 t^4(2\lambda t(3+\lambda t/2) - 3)}{108(1+\lambda t)^3}$$

for $\lambda \geq 0$.

Proof. This Corollary is a direct consequence of the Theorem of this section and the similarity solution presented in Theorem 6.1 on [45] for

$$\frac{\partial u}{\partial t} = x\frac{\partial^2 u}{\partial x^2} + f(x)\frac{\partial u}{\partial x} \tag{4.13}$$

$$ u(x,0) = \varphi(x) \tag{4.14}$$

when f is an odd function. □

Corollary 5. *The family of fractional PDEs with space variable coefficients of the form*

$$ {}_CD_t^\beta u(x,t) = x u_{xx}(x,t) + f(x) u_x(x,t) \tag{4.15} $$

$$ u(x,0) = \varphi(x) \tag{4.16} $$

where f satisfies the Riccati equation of the form

$$ xf' - f + \frac{1}{2} f^2 = Ax + B $$

admits an explicit solution of the form (4.2) where u is given by (4.6) and $p(x, y.t)$ is the inverse Laplace transform of

$$ U_\lambda(x,t) = \exp\left[-\frac{\lambda(x + At^2/2)}{1+\lambda t} - \frac{1}{2}\left(F(x) - F\left(\frac{x}{(1+\lambda t)^2} \right) \right) \right] $$

where $F'(x) = \frac{f(x)}{x}$ and for $\lambda \geq 0$.

Proof. This Corollary is a direct consequence of the Theorem of this section and the similarity solution presented in Theorem 4.1 on [45] for

$$ \frac{\partial u}{\partial t} = x \frac{\partial^2 u}{\partial x^2} + f(x) \frac{\partial u}{\partial x} \tag{4.17} $$

$$ u(x,0) = \varphi(x) \tag{4.18} $$

when f is odd. □

Example 5.

$$ {}_CD_t^\beta u(x,t) = x u_{xx}(x,t) + \left(\frac{1+3\sqrt{x}}{2\left(1+\sqrt{x}\right)} \right) u_x(x,t) \tag{4.19} $$

$$ u(x,0) = \varphi(x) \tag{4.20} $$

has a solution of the form (4.2) where $u(x,t) = \int_0^\infty G(x,y,t)\varphi(y)dy$ and

$$ G(x,y,t) = \frac{\cos\left(\frac{2\sqrt{xy}}{t}\right)}{\sqrt{\pi y t}\left(1+\sqrt{x}\right)} \left(1 + \sqrt{y} \tanh\left(\frac{2\sqrt{xy}}{t} \right) \right) \exp\left[-\frac{x+y}{t} \right]. $$

Example 6.

$$ {}_CD_t^\beta u(x,t) = x u_{xx}(x,t) + \left(\frac{1}{2} + \sqrt{x} \coth\left(\sqrt{x}\right) \right) u_x(x,t) \tag{4.21} $$

$$ u(x,0) = \varphi(x) \tag{4.22} $$

has a solution of the form (4.2) where $u(x,t) = \int_0^\infty G(x,y,t)\varphi(y)dy$ and

$$ G(x,y,t) = \frac{\sinh\left(\frac{2\sqrt{xy}}{t}\right)}{\sqrt{\pi y t}} \frac{\sinh\left(\sqrt{y}\right)}{\sinh\left(\sqrt{x}\right)} \exp\left[-\frac{x+y}{t} - \frac{1}{4} t \right]. $$

Theorem 3. *A solution for*

$$ {}_CD_t^{\beta\alpha} u_{\beta\alpha}(x,t) = \sigma(x)\frac{\partial^2 u_{\beta\alpha}(x,t)}{\partial x^2} + \mu(x)\frac{\partial u_{\beta\alpha}(x,t)}{\partial x} \tag{4.23}$$

can be obtained as an $\alpha-$Wright type transformation of u_β, where u_β is a solution for

$$ {}_CD_t^{\beta} u_{\beta}(x,t) = \sigma(x)\frac{\partial^2 u_{\beta}(x,t)}{\partial x^2} + \mu(x)\frac{\partial u_{\beta}(x,t)}{\partial x}.$$

Proof. Let's consider

$$ {}_CD_t^{\beta} u_{\beta}(x,t) = \sigma(x)\frac{\partial^2 u_{\beta}(x,t)}{\partial x^2} + \mu(x)\frac{\partial u_{\beta}(x,t)}{\partial x}.$$

By Lemma 1, it can be written as

$$\begin{aligned}\int_0^\infty \frac{\partial u(x,w)}{\partial w} g_\beta(w;s)dw &= \sigma(x)\frac{\partial^2}{\partial x^2}\int_0^\infty u(x,w)g_\beta(w;s)dw \\ &+ \mu(x)\frac{\partial}{\partial x}\int_0^\infty u(x,w)g_\beta(w;s)dw.\end{aligned}$$

Taking a α-Wright type transformation on both sides of the equation we obtain,

$$\begin{aligned}\int_0^\infty\int_0^\infty \frac{\partial u(x,w)}{\partial w} g_\beta(w;s)dw g_\alpha(s;t)ds &= \int_0^\infty \sigma(x)\frac{\partial^2}{\partial x^2}\int_0^\infty u(x,w)g_\beta(w;s)dw g_\alpha(s;t)ds \\ &+ \int_0^\infty \mu(x)\frac{\partial}{\partial x}\int_0^\infty u(x,w)g_\beta(w;s)dw g_\alpha(s;t)ds\end{aligned}$$

and using (1.12) we obtain

$$\begin{aligned}\int_0^\infty \frac{\partial u(x,w)}{\partial w} g_{\beta\alpha}(w;t)dw &= \sigma(x)\frac{\partial^2}{\partial x^2}\int_0^\infty u(x,w)g_{\beta\alpha}(w;t)dw \\ &+ \mu(x)\frac{\partial}{\partial x}\int_0^\infty u(x,w)g_{\beta\alpha}(w;t)dw,\end{aligned}$$

which completes the proof. □

5. Numerical Simulations

We use Monte Carlo integration to simulate the solutions of factional differential equations and partial differential equations given their solutions. As expected, simulations of heavy tail distributions require the use of a large amount of random numbers for adequate coverage. We first present the lemma by M. Kanter [46] to generate random numbers distributed as the Lévy α-stable distribution with stability index $0 < \alpha \leq 2$, $L_\alpha^\theta(x)$, defined by equation (1.13). See also [47].

Lemma 2 (Lemma 4.1 [46])**.** *Let $\alpha \in (0,1)$ and let $L_\alpha^{-\alpha}(x)$ as defined in (1.13). Then for $x \geq 0$*

$$L_\alpha^{-\alpha}(x) = \frac{1}{\pi}\left(\frac{\alpha}{1-\alpha}\right)\left(\frac{1}{x}\right)^{(1-\alpha)^{-1}}\int_0^\pi a(\varphi)\exp\left(-\left(\frac{1}{x}\right)^{\alpha/(1-\alpha)}\right)d\varphi \tag{5.1}$$

where

$$a(\varphi) = \left(\frac{\sin(\alpha\varphi)}{\sin(\varphi)}\right)^{(1-\alpha)^{-1}}\left(\frac{\sin((1-\alpha)\varphi)}{\sin(\alpha\varphi)}\right). \tag{5.2}$$

Corollary 6 (Corollary 4.1 [46]). *Let U_1 and U_2 be independent random variables where U_1 is uniformly distributed on $[0,1]$, and U_2 is uniformly distributed on $[0,\pi]$. Then for $\alpha \in (0,1)$, $L_\alpha^{-\alpha}(x)$ is the density of $(-a(U_2)/\log(U_1))^{(1-\alpha)/\alpha}$ where a is given by equation (5.2).*

Corollary 7. *If $X \sim L_\beta^{-\beta}(\cdot)$, then $g_\beta(\cdot,t)$ is the probability density function of $Y = \dfrac{t^\beta}{X^\beta}$.*

Proof. For $t>0$, the probability density function of Y is given by

$$
\begin{aligned}
f_Y(y) &= L_\beta^{-\beta}\left(\frac{t}{y^{1/\beta}}\right)\left|\frac{dx}{dy}\right| \\
&= \frac{t}{\beta y}\frac{1}{y^{1/\beta}}L_\beta^{-\beta}\left(\frac{t}{y^{1/\beta}}\right) \qquad (5.3)\\
&= \frac{t}{\beta y}L_\beta^{-\beta}(t,y) \\
&= g_\beta(y,t).
\end{aligned}
$$

□

Corollary 8. *If $X \sim L_\alpha^{-\alpha}(\cdot)$, then $L_\alpha^{-\alpha}(\cdot,t)$ is the probability density function of $Y = Xt^{1/\alpha}$.*

Proof. Similar proof as above. □

By using the above corollaries, a powerful computational approach is performed using g_β instead of simulating the Wright function. The codes are done on Python using numba and matplotlib libraries.

Example 7 (Fractional growth/decay models). *The solution of the Fractional differential equation*

$$
\begin{cases} D_C^\beta y(t) = -y(t) \\ y(0) = 1 \end{cases} \qquad (5.4)
$$

has a solution given by $y(t) = \int_0^\infty e^{-x} g_\beta(x;t)dx = E_\beta(-t^\beta)$.

As a conclusion, the numerical solution converges to the actual solution when the random number values of g_β are considerably high. Convergence is, however, good starting from 10^4 randomly generated values from g_β.

Example 8. *Let's consider the standard fractional Fokker-Planck equation of the form*

$$
{}_C D_t^\beta p_\beta(x,t) = \frac{\partial^2 p_\beta}{\partial x^2} + x\frac{\partial p_\beta}{\partial x} + p_\beta \qquad (5.5)
$$

$$
p(x,0) = f(x). \qquad (5.6)
$$

We obtained a solution in the form

$$
p_\beta(x,t) = \int_0^\infty \int_0^\infty \frac{\exp\left[-\frac{\left(x-e^{-s}y\right)^2}{2(1-e^{-2s})}\right]}{\sqrt{2\pi\left(1-e^{-2s}\right)}} g_\beta(s;t)ds f(y)dy. \qquad (5.7)
$$

See Figure 5.2 for numerical simulation of the solution and see Figure 5.3 for the mean absolute error for $N=10^6$ with respect to $N=10^7$.

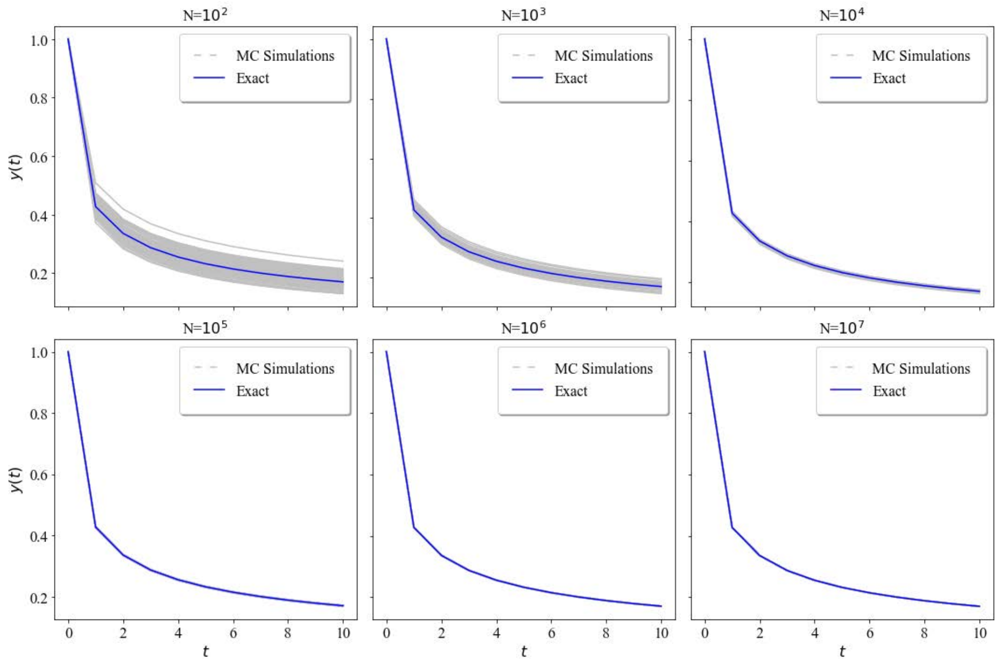
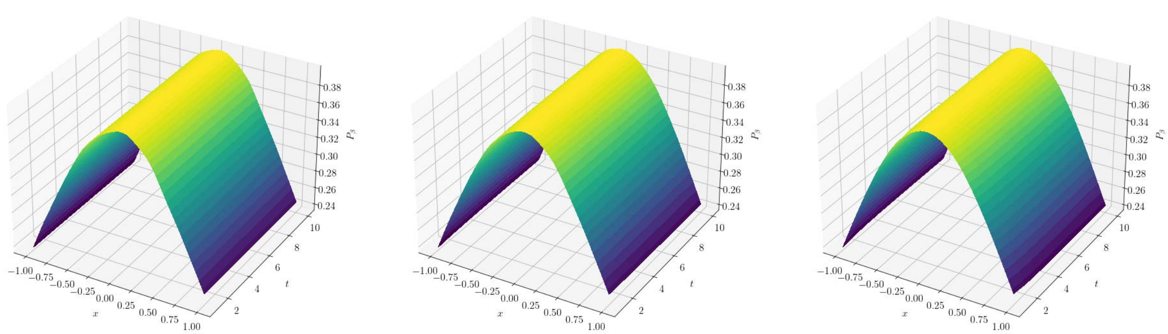


FIGURE 5.1. The Monte Carlo integration repeated 100 times of simulations of the solution of equation (5.4) for each $\beta = 0.5$ compared to the exact solution.



(a) *(b)* *(c)*

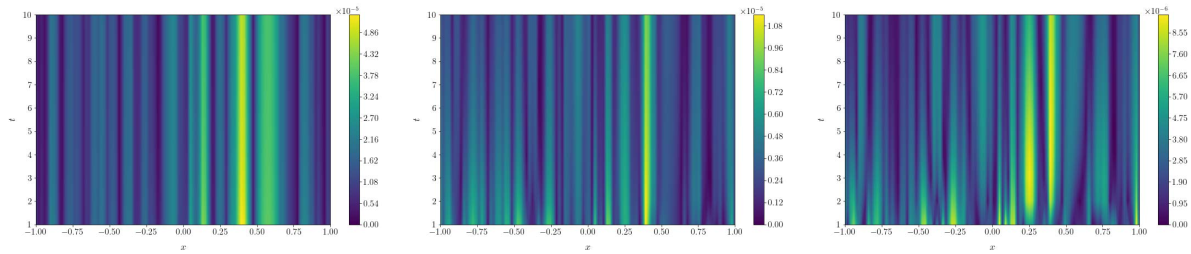
FIGURE 5.2. Fokker Plank Monte Carlo integration simulation of solution in equation (5.7) using a number of 10^7 random values for $\beta = 0.1$, 0.5 and 0.9, respectively.

Example 9. *The solution of the fractional heat/diffusion equation*

$$u_{\alpha,\beta}(x,t) = \int_0^\infty \int_0^\infty L_2^0(x,\tau) L_{\alpha/2}^{-\alpha/2}(\tau,s) g_\beta(s,t) d\tau ds \tag{5.8}$$

could be numerically computed for each x and t using a large number of simulations from $g_\beta(s,t)$, and for each s simulate a large number of τ are simulated from $L_2^0(x,\tau)$ $L_{\alpha/2}^{-\alpha/2}(\tau,s)$. Then the values of $L_2^0(x,\tau)$ are averaged up over all values of τ and then over all values of s. See Figure 5.4.

A similar procedure could be done to simulate random numbers from $u_{\alpha,\beta}(x,t)$, but after simulating one s from $g_\beta(s,t)$, and then one τ from $L_{\alpha/2}^{-\alpha/2}(\tau,s)$, then we simulate one x from $L_2^0(x,\tau)$.



(a) (b) (c)

FIGURE 5.3. Mean error in Fokker Plank Monte Carlo integration simulation of solution in equation (5.7) over time $t \in [1, 10]$. The mean absolute error between using a number of 10^7 and 10^6 random values from g_β for $\beta = 0.1$, 0.5 and 0.9 respectively.

Linear ordinary differential equations and partial differential equations could be also solved numerically. Solutions then get interpolated at the values generated by $g_\beta(\cdot, t)$ and averaged at each value of t.

Example 10. *The solution of the fractional differential equation*

$$D_C^{\beta+2}y + 5D_C^{\beta+1}y + D_C^{\beta}y + 2y = E_\beta(-t^\beta) \tag{5.9}$$

could be numerically computed for each t by numerically solving

$$y^{(3)} + 5y^{(2)} + y^{(1)} + 2y = \exp(-t) \tag{5.10}$$

at N randomly generated s values from $g_\beta(s, t)$ and then average the values up for each t. See Figure 5.5. We used the Runge-Kutta method of hybrid order 4 and 5 from minimum generated s to maximum generated s and then interpolated the solution at the rest of the randomly generated s values.

6. Conclusion

In this work, we have shown an alternative way to find the fundamental solutions for fractional partial differential equations (PDEs). Indeed, Riccati equation have allow us to study families of fractional diffusion PDEs with variable coefficients which allow explicit solutions. Those solutions connect Lie symmetries to fractional PDEs. We expect similar results for fracational dispersive equations. These results will appear in another work.

In our approach, we have taken advantage of Lie symmetries applied to fractional diffusion PDEs with variable coefficients. We predict that Feynman path integrals can play a similar role with fractional dispersive equations.

We conjecture that a general solution similar to that in equation (3.3) can be shown to hold for a larger class of fractional partial differential equations. A conjecture for which an interesting consequence that the Lévy flight could be a result of the wide expanse allowed for the normal diffusion to make a jump over.

Monte-Carlo integration of solutions of ordinary differential equations and partial differential equations with respect to heavy-tailed distributions is a new approach that can prove valuable for other general fractional equations. Evaluating such solutions take small amount of time thanks to

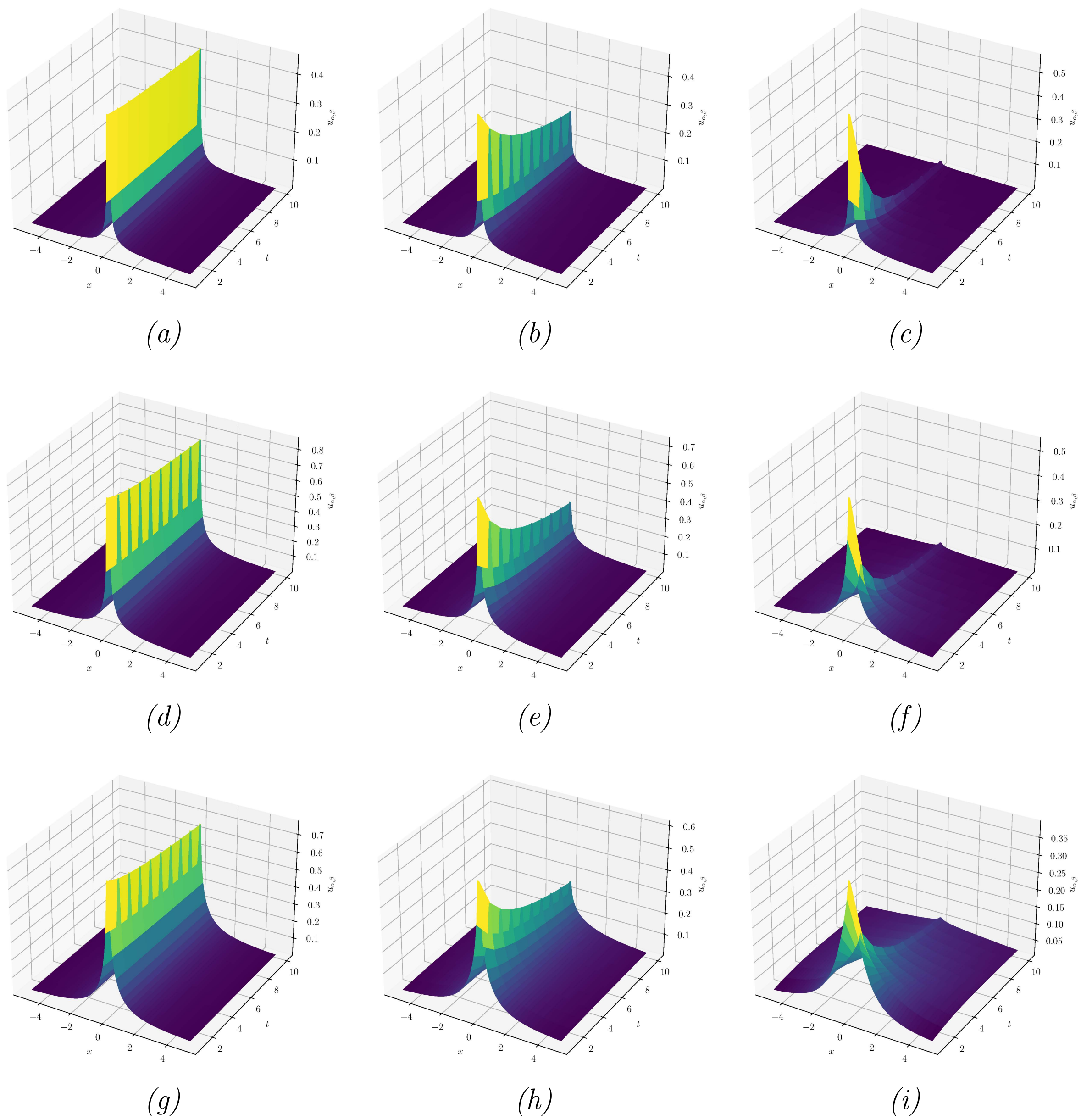
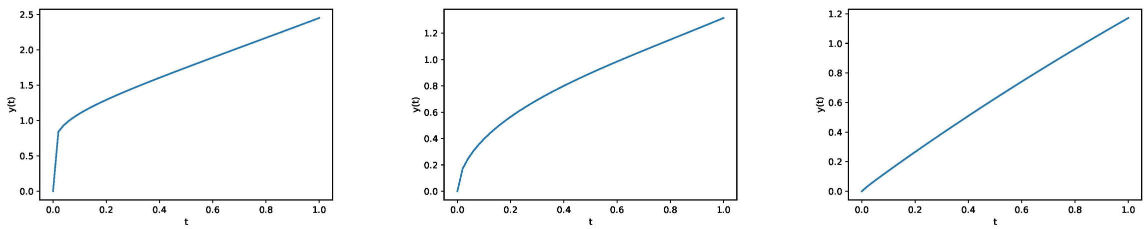


FIGURE 5.4. Simulation of the solution of fractional heat equation $u_{\alpha,\beta}$ of solution in equation (5.8) for $\beta = 0.1$ in (a), (d) and (g), $\beta = 0.5$ in (b), (e), and (h) and $\beta = 0.9$ in (c), (f), and (i) and $\alpha = 0.2$ in (a), (b) and (c), 0.6 in (d), (e) and (f) and 1 in (g), (h) and (i) for $(x, t) \in [-5, 5] \times [1, 10]$. The Monte Carlo integration was performed using a number of 10^7 random values from g_β.

the availability of fast computing devices. A general formula like equation (3.3) can make numerical solutions easier.

REFERENCES

[1] A. Consiglio and F. Mainardi, "On the evolution of fractional diffusive waves," *Ricerche di Matematica*, 2019.



(a) *(b)* *(c)*

FIGURE 5.5. Monte Carlo simulation of the solution of fractional differential equation (5.9) for (a) $\beta = 0.1$, (b) $\beta = 0.5$ and (c) $\beta = 0.9$. The Monte Carlo integration was performed using a number of 10^7 random values from g_β.

[2] M. R. Islam, A. Peace, D. Medina, and T. Oraby, "Integer versus fractional order SEIR deterministic and stochastic models of measles," *International Journal of Environmental Research and Public Health*, 2020.

[3] R. Almeida, A. B. da Cruz, N. Martins, and M. T. T. Monteiro, "An epidemiological mseir model described by the caputo fractional derivative," *International Journal of Dynamics and Control*, pp. 1–14, 2018.

[4] M. Stynes, "Fractional-order derivatives defined by continuous kernels are too restrictive," *Appl. Math. Lett.*, vol. 85, pp. 22–26, nov 2018.

[5] K. Diethelm, R. Garrappa, A. Giusti, and M. Stynes, "Why Fractional Derivatives with Nonsingular Kernels Should Not Be Used," *Fractional Calculus and Applied Analysis 2020 23:3*, vol. 23, pp. 610–634, jul 2020.

[6] Dokuyucu and A. Mustafa, "On the Fractional Derivative and Integral Operators," *Fractional Order Analysis*, pp. 1–41, sep 2020.

[7] R. R. Nigmatullin and Y. E. Ryabov, "Renewal processes of mittag-leffler and wright type," *Fiz. Tverd. Tela.*, vol. 39, p. 101, 1997.

[8] K. Weron and A. Klauser, "A 2000 probabilistic basis for the cole-cole relaxation law ferroelectrics," *Current Developments in Mathematics*, vol. 236, p. 59â€“69, 2000.

[9] R. Metzler, E. Barkai, and J. Klafter, "Anomalous diffusion and relaxation close to thermal equilibrium: A fractional fokker-planck equation approach," *Phys. Rev. Lett.*, vol. 82, p. 3563., 1999.

[10] R. Metzler and J. Klafter, "The random walk's guide to anomalous diffusion: a fractional dynamics approach," *Phys. Rep.*, vol. 339, p. 1., 2000.

[11] M. F. Shlesinger and G. M. Zaslavsky *J. Klafter. Nature*, vol. 363, p. 31., 1993.

[12] A. I. Saichev and G. M. Zaslavsky, "Fractional kinetic equations: solutions and applications," *Chaos*, vol. 7, p. 753, 1997.

[13] J. Klafter, A. Blumen, and M. Shlesinger, "Stochastic pathway to anomalous diffusion," *Physical Review A*, vol. 35, pp. 3081–3085, apr 1987.

[14] I. Sokolov and J. Klafter, "From diffusion to anomalous diffusion: A century after Einstein's Brownian motion," *Chaos*, vol. 15, no. 2, pp. 1–18, 2005.

[15] V.Yanovsky, A.Chechkin, D. Schertzer, and A.Tur, "Lévy anomalous diffusion and fractional Fokkerâ€“Planck equation," *Physica A: Statistical Mechanics and its Applications*, vol. 282, pp. 13–34, jul 2000.

[16] R. Metzler and J. Klafter, "The random walk's guide to anomalous diffusion: a fractional dynamics approach," *Physics Reports*, vol. 339, pp. 1–77, dec 2000.

[17] R. Metzler, J. H. Jeon, A. Cherstvy, and E. Barkai, "Anomalous diffusion models and their properties: non-stationarity, non-ergodicity, and ageing at the centenary of single particle tracking," *Physical Chemistry Chemical Physics*, vol. 16, pp. 24128–24164, oct 2014.

[18] E. Orsingher and F. Polito, "Fractional pure birth processes," *Bernoulli*, vol. 16, no. 3, pp. 858–881, 2010.

[19] E. Orsingher, C. Ricciuti, and B. Toaldo, "Population models at stochastic times," *Advances in Applied Probability*, vol. 48, no. 2, pp. 481–498, 2016.

[20] S. Eule and R. Friedrich, "A note on the forced burgers equation," *Physics Letters A: General, Atomic and Solid State Physics*, vol. 351, pp. 234 – 241., 2006.

[21] Z. Feng, "Traveling wave behavior for a generalized fisher equation," *Chaos Solitons Fractals*, vol. 38, p. 481â€"488., 2008.

[22] G. W. Bluman, "Similarity solutions of the one dimensional fokker planck equation," *Int. J. Non-Lin. Mech.*, vol. 6, pp. 143–153., 1971.

[23] G. Bluman and S. Kumei, "Symmetries and differential equations," *Springer-Verlag*, p. New York., 1989.

[24] G. Bluman and S. Anco, "Symmetry and integration methods for differential equations," *Applied Mathematical Sciences, Springer-Verlag*, p. New York., 2002.

[25] G. Bluman and J. D. Cole, "The general similarity solution of the heat equation," *J. Math. Mech.*, vol. 18, p. 1025â€"1042, 1969.

[26] P. Olver, "Application of lie groups to differential equations," *Springer-Verlag*, p. New York., 1986.

[27] I. Podlubny, *Fractional Differential Equations: An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of Their Solution and Some of Their Applications*. Academic Press, 1999.

[28] F. Mainardi, Y. Luchko, and G. Pagnini, "The fundamental solution of the space-time fractional diffusion equation," *An International Journal for Theory and Applications*, vol. 4 No 2, pp. 153 – 192., 2001.

[29] F. Mainardi, R. Glorenflo, and A. Vivoli, "Renewal processes of mittag-leffler and wright type," *Fractional Calculus and Applied Analysis*, vol. 8 No 1, pp. 7 – 38., 2001.

[30] F. Mainardi and A. Consiglio, "The wright functions of the second kind in mathematical physics," *Mathematics 2020*, vol. 8 No 6, p. 884., 2020.

[31] R. Gorenflo, "Mittag-leffler waiting time, power laws,rarefaction, continuous time random walk, diffusion limit," 2010.

[32] A. Piryatinska, "Inference for the levy models and their applications in medicine and statistical physics," *Department of Statistics, Case Western Reserve University*, 2005.

[33] K. K. Kataria and P. Vellaisamy, "On the convolution of mittagâ€"leffler distributions and its applications to fractional point processes," *Stochastic Analysis and Applications*, vol. 37, no. 1, pp. 115–122, 2019.

[34] F. Mainardi and G. Pagnini, "Mellin-barnes integrals for stable distributions and their convolutions," *Fractional Calculus and Applied Calculus*, vol. 11, no. 4, 2008.

[35] F. Mainardi, A. Mura, and G. Pagnini, "The m-wright function in time-fractional diffusion processes: A tutorial survey," *International Journal of Differential Equations*, vol. 2010, 2009.

[36] F. Mainardi and E. Masina, "On modifications of the exponential integral with the mittag-leffler function," *Fractional Calculus and Applied Analysis*, vol. 21, p. 1156â€"1169, Oct 2018.

[37] E. Suazo, S. K. Suslov, and J. M. Vega, "The riccati differential equation and a diffusion-type equation," *New York J. Math*, vol. 17a, pp. 225 – 224., 2011.

[38] B. W. Huff, " The Strict Subordination of Differential Processes," *Sankhy Ä: The Indian Journal of Statistics, Series A*, vol. 31, no. 4, pp. 403–412, 1969.

[39] B. Mandelbrot and H. M. . Taylor, "On the Distribution of Stock Price Differences," *Operations Research*, vol. 15, no. 6, pp. 1057–1062, 1967.

[40] O. E. Barndorff-Nielsen and A. Shiryaev, *Change of Time and Change of Measure*. Advanced Series on Statistical Science & Applied Probability, WORLD SCIENTIFIC, 2010.

[41] E. Orsingher and L. Beghin, "Time-fractional telegraph equations and telegraph processes with brownian time," *Probability Theory and Related Fields*, vol. 128, pp. 141–160, jan 2004.

[42] F. Huang, "Analytical solution for the time-fractional telegraph equation," *Journal of Applied Mathematics*, vol. 2009, 2009.

[43] J. Chen, F. Liu, and V. Anh, "Analytical solution for the time-fractional telegraph equation by the method of separating variables," *Journal of Mathematical Analysis and Applications*, vol. 338, pp. 1364–1377, feb 2008.

[44] R. C. Cascaval, E. C. Eckstein, C. Frota, and J. Goldstein, "Fractional telegraph equations," *Journal of Mathematical Analysis and Applications*, vol. 276, pp. 145–159, dec 2002.

[45] M. Craddock, "Symmetry group methods for fundamental solutions," *J. Differential Equations*, vol. 207, pp. 285–302., 2004.

[46] M. Kanter, "Stable densities under change of scale and total variation inequalities," *The annals of Probability*, vol. 3, No. 4,, pp. 697 – 707., 1975.

[47] D. Cahoy, "Estimation and Simulation for the M-Wright Function," *Communications in Statistics-Theory and Methods*, vol. 41, no. 8, pp. 1466–1477, 2012.