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arXiv:2205.00592v1 [math.AP] 2 May 2022

LOCAL WELL-POSEDNESS OF THE CAUCHY PROBLEM FOR A $p-$ADIC NAGUMO-TYPE EQUATION

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Abstract. We introduce a new family of $p-$adic non-linear evolution equations. We establish the local well-posedness of the Cauchy problem for these equations in Sobolev-type spaces. For a certain subfamily, we show that the blow-up phenomenon occurs and provide numerical simulations showing this phenomenon.

1. Introduction

Nowadays, the theory of linear partial pseudo-differential equations for complex-valued functions over $p-$adic fields is a well-established branch of mathematical analysis, see e.g. [1]-[6], [12]-[16], [22]-[25], [27]-[33], and references therein. Meanwhile very little is known about nonlinear $p-$adic equations. We can mention some semilinear evolution equations solved using $p-$adic wavelets [1], [24], a kind of equations of reaction-diffusion type and Turing patterns studied in [31], [33], a $p-$adic analog of one of the porous medium equation [17], [22], the blow-up phenomenon studied in [4], and non-linear integro-differential equations connected with $p-$adic cellular networks [30].

In this article we introduce a new family of nonlinear evolution equations that we have named as $p-$adic Nagumo-type equations:

$$u_t = -\gamma \boldsymbol{D}_x^{\alpha} u - u^3 + (\beta+1)\, u^2 - \beta u + P(\boldsymbol{D}_x)\,(u^m)\,,\ x \in \mathbb{Q}_p^n,\ t \in [0,T]\,,$$

where $\gamma > 0$, $\beta \geq 0$, $\boldsymbol{D}_x^{\alpha}$, $\alpha > 0$, is the Taibleson operator, m is a positive integer and $P(\boldsymbol{D}_x)$ is an operator of degree δ of the form $P(\boldsymbol{D}) = \sum_{j=0}^{k} C_j \boldsymbol{D}^{\delta_j}$, where the $C_j \in \mathbb{R}$ and $\delta_k = \delta$. We establish the local well-posedness of the Cauchy problem for these equations in Sobolev-type spaces, see Theorem 1. For a certain subfamily, we show that the blow-up phenomenon occurs, see Theorem 2, and we also provide numerical simulations showing this phenomenon.

The theory of Sobolev-type spaces use here was developed in [34], see also [25], [18]. This theory is based in the theory of countably Hilbert spaces of Gel'fand-Vilenkin [8]. Some generalizations are presented in [9]-[10]. We use classical techniques of operator semigroups, see e.g. [3], [20]. The family of evolution equations studied here contains as a particular case, equations of the form

$$u_t = -\gamma \boldsymbol{D}_x^{\alpha} u - u^3 + (\beta+1)\, u^2 - \beta u, \tag{1.1}$$

2000 *Mathematics Subject Classification.* Primary 47G30, 35B44; Secondary 46E36, 32P05.

Key words and phrases. $p-$adic analysis, pseudo-differential operators, Sobolev-type spaces, blow-up phenomenon.

The third author was partially supported by the Lokenath Debnath Endowed Professorship, UTRGV.

where $x \in \mathbb{Q}_p^n$, $t \in [0,T]$, $\boldsymbol{D}_x^{\alpha}$ is the Taibleson operator, that resemble the classical Nagumo-type equations, see e.g. [21].

In [7], the authors study the equations

$$u_t = Du_{xx} - u\left(u-\kappa\right)\left(u-1\right) - \varepsilon u_x^m, \tag{1.2}$$

where $D>0$, $\kappa \in \left(0,\frac{1}{2}\right)$, $\varepsilon > 0$, $x \in \mathbb{R}$, $t>0$. They establish the local well-posedness of the Cauchy problem for these equations in standard Sobolev spaces. There are several crucial differences between (1.1) and (1.2). The operators u_{xx}, u_x^m are local while the operators $\boldsymbol{D}_x^{\alpha}$, $P(\boldsymbol{D}_x)\left(\cdot^m\right)$ are non-local. The $p-$adic heat equation $u_t = -\gamma \boldsymbol{D}_x^{\alpha} u$ has an arbitrary order of pseudo-differentiability $\alpha > 0$ in the spatial variable, while in the classical fractional heat equation $u_t = D\frac{\partial^{\mu} u}{\partial x^{\mu}}$, the degree of pseudo-differentiability $\mu \in (0,2]$. This implies that the Markov processes attached to $u_t = -\gamma \boldsymbol{D}_x^{\alpha} u$ are completely different to the ones attached to $u_t = Du_{xx}$. In other words, the diffusion mechanisms in (1.1) and (1.2) are completely different. Notice that our non-linear term involves pseudo-derivatives of arbitrary order $P(\boldsymbol{D}_x)\left(u^m\right)$, while in [7] only of first order u_x^m. Of course, the $p-$adic Sobolev spaces behave completely different from their real counterparts, but the semigroup techniques are the same in both cases, since time is a non-negative real variable.

The article is organized as follows. In section 2, we review some basic aspects of the $p-$adic analysis and fix the notation. In section 3, we present some technical results about Sobolev-type spaces and $p-$adic pseudo-differential operators. In section 4, we show the local well-posedness of the $p-$adic Nagumo-type equations, see Theorem 1. In section 5, we show a subfamily of $p-$adic Nagumo-type equations whose solutions blow-up in finite time, see Theorem 2. In section 6, we present a numerical simulation showing the blow-up phenomenon.

2. $p-$Adic Analysis: Essential Ideas

In this section, we collect some basic results on $p-$adic analysis that we use through the article. For a detailed exposition the reader may consult [1], [14], [26], [29].

2.1. The field of $p-$adic numbers.

Along this article p will denote a prime number. The field of $p-$adic numbers $\mathbb{Q}_p$ is defined as the completion of the field of rational numbers $\mathbb{Q}$ with respect to the $p-$adic norm $|\cdot|_p$, which is defined as

$$|x|_p = \begin{cases} 0 & \text{if} \quad x=0 \\ \\ p^{-\gamma} & \text{if} \quad x = p^{\gamma}\frac{a}{b}, \end{cases}$$

where a and b are integers coprime with p. The integer $\gamma := ord(x)$, with $ord(0) := +\infty$, is called the *$p-$adic order of x.*

Any $p-$adic number $x \neq 0$ has a unique expansion of the form

$$x = p^{ord(x)} \sum_{j=0}^{\infty} x_j p^j,$$

where $x_j \in \{0, \ldots, p-1\}$ and $x_0 \neq 0$. By using this expansion, we define *the fractional part of* $x \in \mathbb{Q}_p$, denoted $\{x\}_p$, as the rational number

$$\{x\}_p = \begin{cases} 0 & \text{if} \quad x = 0 \text{ or } ord(x) \geq 0 \\ \\ p^{ord(x)} \sum_{j=0}^{-ord_p(x)-1} x_j p^j & \text{if} \quad ord(x) < 0. \end{cases}$$

2.2. **Topology of $\mathbb{Q}_p^n$.** For $r \in \mathbb{Z}$, denote by $B_r^n(a) = \{x \in \mathbb{Q}_p^n; ||x-a||_p \leq p^r\}$ *the ball of radius p^r with center at $a = (a_1, \ldots, a_n) \in \mathbb{Q}_p^n$*, and take $B_r^n(0) := B_r^n$. Note that $B_r^n(a) = B_r(a_1) \times \cdots \times B_r(a_n)$, where $B_r(a_i) := \{x_i \in \mathbb{Q}_p; |x_i - a_i|_p \leq p^r\}$ is the one-dimensional ball of radius p^r with center at $a_i \in \mathbb{Q}_p$. The ball B_0^n equals the product of n copies of $B_0 = \mathbb{Z}_p$, *the ring of $p-$adic integers*. We also denote by $S_r^n(a) = \{x \in \mathbb{Q}_p^n; ||x-a||_p = p^r\}$ *the sphere of radius p^r with center at* $a = (a_1, \ldots, a_n) \in \mathbb{Q}_p^n$, and take $S_r^n(0) := S_r^n$. We notice that $S_0^1 = \mathbb{Z}_p^\times$ (the group of units of $\mathbb{Z}_p$), but $\left(\mathbb{Z}_p^\times\right)^n \subsetneq S_0^n$. The balls and spheres are both open and closed subsets in $\mathbb{Q}_p^n$. In addition, two balls in $\mathbb{Q}_p^n$ are either disjoint or one is contained in the other.

As a topological space $\left(\mathbb{Q}_p^n, ||\cdot||_p\right)$ is totally disconnected, i.e. the only connected subsets of $\mathbb{Q}_p^n$ are the empty set and the points. A subset of $\mathbb{Q}_p^n$ is compact if and only if it is closed and bounded in $\mathbb{Q}_p^n$, see e.g. [29, Section 1.3], or [1, Section 1.8]. The balls and spheres are compact subsets. Thus $\left(\mathbb{Q}_p^n, ||\cdot||_p\right)$ is a locally compact topological space.

Since $(\mathbb{Q}_p^n, +)$ is a locally compact topological group, there exists a Haar measure $d^n x$, which is invariant under translations, i.e. $d^n(x+a) = d^n x$. If we normalize this measure by the condition $\int_{\mathbb{Z}_p^n} dx = 1$, then $d^n x$ is unique.

Notation 1. *We will use $\Omega\left(p^{-r}||x-a||_p\right)$ to denote the characteristic function of the ball $B_r^n(a)$. For more general sets, we will use the notation 1_A for the characteristic function of a set A.*

2.3. **The Bruhat-Schwartz space.** A complex-valued function φ defined on $\mathbb{Q}_p^n$ is *called locally constant* if for any $x \in \mathbb{Q}_p^n$ there exist an integer $l(x) \in \mathbb{Z}$ such that

$$\varphi(x + x') = \varphi(x) \text{ for any } x' \in B_{l(x)}^n. \tag{2.1}$$

A function $\varphi : \mathbb{Q}_p^n \to \mathbb{C}$ is called a *Bruhat-Schwartz function (or a test function)* if it is locally constant with compact support. Any test function can be represented as a linear combination, with complex coefficients, of characteristic functions of balls. The $\mathbb{C}-$vector space of Bruhat-Schwartz functions is denoted by $\mathcal{D}(\mathbb{Q}_p^n) := \mathcal{D}$. We denote by $\mathcal{D}_{\mathbb{R}}(\mathbb{Q}_p^n) := \mathcal{D}_{\mathbb{R}}$ the $\mathbb{R}-$vector space of Bruhat-Schwartz functions. For $\varphi \in \mathcal{D}(\mathbb{Q}_p^n)$, the largest number $l = l(\varphi)$ satisfying (2.1) is called *the exponent of local constancy (or the parameter of constancy) of* φ.

We denote by $\mathcal{D}_m^l(\mathbb{Q}_p^n)$ the finite-dimensional space of test functions from $\mathcal{D}(\mathbb{Q}_p^n)$ having supports in the ball B_m^n and with parameters of constancy $\geq l$. We now define a topology on $\mathcal{D}$ as follows. We say that a sequence $\{\varphi_j\}_{j\in\mathbb{N}}$ of functions in $\mathcal{D}$ converges to zero, if the two following conditions hold:

(1) there are two fixed integers k_0 and m_0 such that each $\varphi_j \in \mathcal{D}_{m_0}^{k_0}$;

(2) $\varphi_j \to 0$ uniformly.

$\mathcal{D}$ endowed with the above topology becomes a topological vector space.

2.4. L^ρ **spaces.** Given $\rho \in [1,\infty)$, we denote by $L^\rho := L^\rho\left(\mathbb{Q}_p^n\right) := L^\rho\left(\mathbb{Q}_p^n, d^n x\right)$, the $\mathbb{C}-$vector space of all the complex-valued functions g satisfying

$$\int_{\mathbb{Q}_p^n} |g(x)|^\rho \, d^n x < \infty.$$

The corresponding $\mathbb{R}-$vector spaces are denoted as $L_{\mathbb{R}}^\rho := L_{\mathbb{R}}^\rho\left(\mathbb{Q}_p^n\right) = L_{\mathbb{R}}^\rho\left(\mathbb{Q}_p^n, d^n x\right)$, $1 \leq \rho < \infty$.

If U is an open subset of $\mathbb{Q}_p^n$, $\mathcal{D}(U)$ denotes the space of test functions with supports contained in U, then $\mathcal{D}(U)$ is dense in

$$L^\rho(U) = \left\{ \varphi : U \to \mathbb{C}; \|\varphi\|_\rho = \left\{ \int_U |\varphi(x)|^\rho \, d^n x \right\}^{\frac{1}{\rho}} < \infty \right\},$$

where $d^n x$ is the normalized Haar measure on $\left(\mathbb{Q}_p^n, +\right)$, for $1 \leq \rho < \infty$, see e.g. [1, Section 4.3]. We denote by $L_{\mathbb{R}}^\rho(U)$ the real counterpart of $L^\rho(U)$.

2.5. **The Fourier transform.** Set $\chi_p(y) = \exp(2\pi i\{y\}_p)$ for $y \in \mathbb{Q}_p$. The map $\chi_p(\cdot)$ is an additive character on $\mathbb{Q}_p$, i.e. a continuous map from $(\mathbb{Q}_p, +)$ into S (the unit circle considered as multiplicative group) satisfying $\chi_p(x_0 + x_1) = \chi_p(x_0)\chi_p(x_1)$, $x_0, x_1 \in \mathbb{Q}_p$. The additive characters of $\mathbb{Q}_p$ form an Abelian group which is isomorphic to $(\mathbb{Q}_p, +)$. The isomorphism is given by $\kappa \to \chi_p(\kappa x)$, see e.g. [1, Section 2.3].

Given $\xi = (\xi_1, \dots, \xi_n)$ and $y = (x_1, \dots, x_n) \in \mathbb{Q}_p^n$, we set $\xi \cdot x := \sum_{j=1}^n \xi_j x_j$. The Fourier transform of $\varphi \in \mathcal{D}(\mathbb{Q}_p^n)$ is defined as

$$(\mathcal{F}\varphi)(\xi) = \int_{\mathbb{Q}_p^n} \chi_p(\xi \cdot x)\varphi(x) d^n x \quad \text{for } \xi \in \mathbb{Q}_p^n,$$

where $d^n x$ is the normalized Haar measure on $\mathbb{Q}_p^n$. The Fourier transform is a linear isomorphism from $\mathcal{D}(\mathbb{Q}_p^n)$ onto itself satisfying

$$(\mathcal{F}(\mathcal{F}\varphi))(\xi) = \varphi(-\xi), \tag{2.2}$$

see e.g. [1, Section 4.8]. We will also use the notation $\mathcal{F}_{x\to\xi}\varphi$ and $\widehat{\varphi}$ for the Fourier transform of φ.

The Fourier transform extends to L^2. If $f \in L^2$, its Fourier transform is defined as

$$(\mathcal{F}f)(\xi) = \lim_{k\to\infty} \int_{||x||_p \leq p^k} \chi_p(\xi \cdot x) f(x) d^n x, \quad \text{for } \xi \in \mathbb{Q}_p^n,$$

where the limit is taken in L^2. We recall that the Fourier transform is unitary on L^2, i.e. $||f||_2 = ||\mathcal{F}f||_2$ for $f \in L^2$ and that (2.2) is also valid in L^2, see e.g. [26, Chapter III, Section 2].

2.6. **Distributions.** The $\mathbb{C}-$vector space $\mathcal{D}'\left(\mathbb{Q}_p^n\right) := \mathcal{D}'$ of all continuous linear functionals on $\mathcal{D}(\mathbb{Q}_p^n)$ is called the *Bruhat-Schwartz space of distributions*. Every linear functional on $\mathcal{D}$ is continuous, i.e. $\mathcal{D}'$ agrees with the algebraic dual of $\mathcal{D}$, see e.g. [29, Chapter 1, VI.3, Lemma]. We denote by $\mathcal{D}_{\mathbb{R}}'\left(\mathbb{Q}_p^n\right) := \mathcal{D}_{\mathbb{R}}'$ the dual space of $\mathcal{D}_{\mathbb{R}}$.

We endow $\mathcal{D}'$ with the weak topology, i.e. a sequence $\{T_j\}_{j\in\mathbb{N}}$ in $\mathcal{D}'$ converges to T if $\lim_{j\to\infty} T_j(\varphi) = T(\varphi)$ for any $\varphi \in \mathcal{D}$. The map

$$\begin{array}{ccc} \mathcal{D}' \times \mathcal{D} & \to & \mathbb{C} \\ (T,\varphi) & \to & T(\varphi) \end{array}$$

is a bilinear form which is continuous in T and φ separately. We call this map the pairing between $\mathcal{D}'$ and $\mathcal{D}$. From now on we will use (T,φ) instead of $T(\varphi)$.

Every f in L^1_{loc} defines a distribution $f \in \mathcal{D}'\left(\mathbb{Q}_p^n\right)$ by the formula

$$(f,\varphi) = \int_{\mathbb{Q}_p^n} f(x)\,\varphi(x)\,d^n x.$$

Such distributions are called *regular distributions*. Notice that for $f \in L^2_{\mathbb{R}}$, $(f,\varphi) = \langle f,\varphi\rangle$, where $\langle\cdot,\cdot\rangle$ denotes the scalar product in $L^2_{\mathbb{R}}$.

2.7. **The Fourier transform of a distribution.** The Fourier transform $\mathcal{F}[T]$ of a distribution $T \in \mathcal{D}'\left(\mathbb{Q}_p^n\right)$ is defined by

$$(\mathcal{F}[T],\varphi) = (T,\mathcal{F}[\varphi]) \text{ for all } \varphi \in \mathcal{D}(\mathbb{Q}_p^n).$$

The Fourier transform $T \to \mathcal{F}[T]$ is a linear (and continuous) isomorphism from $\mathcal{D}'\left(\mathbb{Q}_p^n\right)$ onto $\mathcal{D}'\left(\mathbb{Q}_p^n\right)$. Furthermore, $T = \mathcal{F}[\mathcal{F}[T](-\xi)]$.

3. Sobolev-Type Spaces

The Sobolev-type spaces used here were introduce in [34], [25]. We follow here closely the presentation given in [18, Sections 10.1, 10.2].

We set $[\xi]_p := \max\left\{1, \|\xi\|_p\right\}$ for $\xi = (\xi_1,\ldots,\xi_n) \in \mathbb{Q}_p^n$. Given $\varphi, \varrho \in \mathcal{D}(\mathbb{Q}_p^n)$ and $s \in \mathbb{R}$, we define the scalar product:

$$\langle \varphi, \varrho\rangle_s = \int_{\mathbb{Q}_p^n} [\xi]_p^s\, \widehat{\varphi}(\xi)\overline{\widehat{\varrho}(\xi)} d^n\xi,$$

where the bar denotes the complex conjugate. We also set $\|\varphi\|_s^2 = \langle\varphi,\varphi\rangle_s$, and denote by $\mathcal{H}_s := \mathcal{H}_s(\mathbb{Q}_p^n,\mathbb{C}) = \mathcal{H}_s(\mathbb{C})$ the completion of $\mathcal{D}(\mathbb{Q}_p^n)$ with respect to $\langle\cdot,\cdot\rangle_s$. Notice that if $r,s \in \mathbb{R}$, with $r \le s$, then $\|\cdot\|_r \le \|\cdot\|_s$ and $\mathcal{H}_s \hookrightarrow \mathcal{H}_r$ (continuous embedding). In particular,

$$\cdots \supset \mathcal{H}_{-2} \supset \mathcal{H}_{-1} \supset \mathcal{H}_0 \supset \mathcal{H}_1 \supset \mathcal{H}_2 \cdots,$$

where $\mathcal{H}_0 = L^2$. We set

$$\mathcal{H}_\infty(\mathbb{Q}_p^n,\mathbb{C}) = \mathcal{H}_\infty := \bigcap_{s\in\mathbb{N}} \mathcal{H}_s.$$

Since $\mathcal{H}_{[s]+1} \subseteq \mathcal{H}_s \subseteq \mathcal{H}_{[s]}$ for $s \in \mathbb{R}_+$, where $[\cdot]$ is the integer part function, then $\mathcal{H}_\infty = \bigcap_{s\in\mathbb{R}_+}\mathcal{H}_s$. With the topology induced by the family of seminorms $\{\|\cdot\|_l\}_{l\in\mathbb{N}}$, $\mathcal{H}_\infty$ becomes a locally convex space, which is metrizable. Indeed,

$$d(f,g) := \max_{l\in\mathbb{N}}\left\{2^{-l}\frac{\|f-g\|_l}{1+\|f-g\|_l}\right\}, \text{ with } f,g \in \mathcal{H}_\infty,$$

is a metric for the topology of $\mathcal{H}_\infty$ considered as a convex topological space. The metric space $(\mathcal{H}_\infty, d)$ is the completion of the metric space $(\mathcal{D}(\mathbb{Q}_p^n), d)$, cf. [18, Lemma 10.4]. Furthermore, $\mathcal{H}_\infty \subset L^\infty \cap C^{\text{unif}} \cap L^1 \cap L^2$, and $\mathcal{H}_\infty(\mathbb{Q}_p^n, \mathbb{C})$ is continuously embedded in $C_0(\mathbb{Q}_p^n, \mathbb{C})$. This is the non-Archimedean analog of the Sobolev embedding theorem, cf. [18, Theorem 10.15].

Lemma 1. *If $s_1 \leq s \leq s_2$, with $s = \theta s_1 + (1-\theta)s_2$, $0 \leq \theta \leq 1$, then $\|f\|_s \leq \|f\|_{s_1}^{\theta} \|f\|_{s_2}^{(1-\theta)}$.*

Proof. Take $f \in \mathcal{H}_s$, then by using the Hölder inequality for the exponents $\frac{1}{q} = \theta, \frac{1}{q'} = 1-\theta$,

$$\begin{aligned}
\|f\|_s^2 &= \int_{\mathbb{Q}_p^n} [\xi]_p^s \left|\widehat{f}(\xi)\right|^2 d^n\xi = \int_{\mathbb{Q}_p^n} [\xi]_p^{\theta s_1 + (1-\theta)s_2} \left|\widehat{f}(\xi)\right|^{2(\theta+(1-\theta))} d^n\xi \\
&= \int_{\mathbb{Q}_p^n} \left([\xi]_p^{s_1} \left|\widehat{f}(\xi)\right|^2\right)^{\theta} \left([\xi]_p^{s_2} \left|\widehat{f}(\xi)\right|^2\right)^{1-\theta} d^n\xi \\
&\leq \left(\int_{\mathbb{Q}_p^n} [\xi]_p^{s_1} \left|\widehat{f}(\xi)\right|^2 d^n\xi\right)^{\theta} \left(\int_{\mathbb{Q}_p^n} [\xi]_p^{s_2} \left|\widehat{f}(\xi)\right|^2 d^n\xi\right)^{1-\theta} d^n\xi.
\end{aligned}$$

□

The following characterization of the spaces $\mathcal{H}_s$ and $\mathcal{H}_\infty$ is useful:

Lemma 2 ([18, Lemma 10.8])**.** *(i) $\mathcal{H}_s = \left\{f \in L^2; \|f\|_s < \infty\right\} = \{T' \in \mathcal{D}; \|T\|_s < \infty\}$, (ii) $\mathcal{H}_\infty = \left\{f \in L^2; \|f\|_s < \infty \text{ for any } s \in \mathbb{R}_+\right\} = \{T' \in \mathcal{D}; \|T\|_s < \infty \text{ for any } s \in \mathbb{R}_+\}$. The equalities in (i)-(ii) are in the sense of vector spaces.*

Proposition 1. *If $s > n/2$, then $\mathcal{H}_s$ is a Banach algebra with respect to the product of functions. That is, if $f, g \in \mathcal{H}_s$, then $fg \in \mathcal{H}_s$ and $\|fg\|_s \leq C(n,s)\|f\|_s \|g\|_s$, where $C(n,s)$ is a positive constant.*

Proof. By the ultrametric property of $\|\cdot\|_p$, $\|\xi\|_p \leq \max\left\{\|\xi-\eta\|_p, \|\eta\|_p\right\}$ for $\xi, \eta \in \mathbb{Q}_p^n$, we have $\max\left\{1, \|\xi\|_p\right\} \leq \max\left\{1, \|\xi-\eta\|_p, \|\eta\|_p\right\}$, which implies that

$$\left[\max\left\{1, \|\xi\|_p\right\}\right]^s \leq \max\left\{1, \|\xi-\eta\|_p^s, \|\eta\|_p^s\right\} = \max\left\{1, \|\xi-\eta\|_p, \|\eta\|_p\right\}^s$$

for $s > 0$. Therefore

$$[\xi]_p^s \leq [\xi-\eta]_p^s + [\eta]_p^s. \tag{3.1}$$

Now, for $f, g \in L^2$, by using (3.1),

$$\begin{aligned}
[\xi]_p^{\frac{s}{2}} \left|\widehat{fg}(\xi)\right| &= \left|[\xi]_p^{\frac{s}{2}} \int_{\mathbb{Q}_p^n} \widehat{f}(\xi-\eta)\widehat{g}(\eta) d^n\eta\right| \\
&\leq \int_{\mathbb{Q}_p^n} [\xi-\eta]_p^{\frac{s}{2}} \left|\widehat{f}(\xi-\eta)\right| |\widehat{g}(\eta)|\, d^n\eta + \int_{\mathbb{Q}_p^n} [\eta]_p^{\frac{s}{2}} |\widehat{g}(\eta)| \left|\widehat{f}(\xi-\eta)\right| d^n\eta \\
&= [\xi]_p^{\frac{s}{2}} \left|\widehat{f}(\xi)\right| * |\widehat{g}(\xi)| + \left|\widehat{f}(\xi)\right| * [\xi]_p^{\frac{s}{2}} |\widehat{g}(\xi)|.
\end{aligned}$$

Then

$$\|fg\|_s \leq \left\| [\xi]_p^{\frac{s}{2}} \left|\widehat{f}(\xi)\right| * |\widehat{g}(\xi)| + \left|\widehat{f}(\xi)\right| * [\xi]_p^{\frac{s}{2}} |\widehat{g}(\xi)| \right\|_2$$
$$\leq \left\| [\xi]_p^{\frac{s}{2}} \left|\widehat{f}(\xi)\right| * |\widehat{g}(\xi)| \right\|_2 + \left\| \left|\widehat{f}(\xi)\right| * [\xi]_p^{\frac{s}{2}} |\widehat{g}(\xi)| \right\|_2 .$$

Since $[\xi]_p^{\frac{s}{2}} \left|\widehat{f}(\xi)\right|$, $[\xi]_p^{\frac{s}{2}} |\widehat{g}(\xi)| \in L^2$, by using the Cauchy-Schwarz inequality with $s > n/2$, we have $\| |\widehat{g}(\xi)| \|_1 \leq A(n,s) \|g\|_s$, $\left\| \left|\widehat{f}(\xi)\right| \right\|_1 \leq A(n,s) \|f\|_s$, i.e. $|\widehat{g}(\xi)|$, $\left|\widehat{f}(\xi)\right| \in L^1$. Now, by the Young inequality, we obtain that

$$\|fg\|_s \leq \|f\|_s \|\widehat{g}\|_1 + \|g\|_s \left\|\widehat{f}\right\|_1 \leq 2A(n,s) \|f\|_s \|g\|_s .$$

□

3.1. **The Taibleson operator.** Let $\alpha > 0$, the Taibleson operator is defined as

$$(\boldsymbol{D}^{\alpha}\varphi)(x) = \mathcal{F}^{-1}_{\xi\to x}(\|\xi\|_p^{\alpha} (\mathcal{F}_{x\to\xi}\varphi)),$$

for $\varphi \in \mathcal{D}(\mathbb{Q}_p^n)$. This operator admits the extension

$$(\boldsymbol{D}^{\alpha}f)(x) = \frac{1-p^{\alpha}}{1-p^{-\alpha-n}} \int_{\mathbb{Q}_p^n} \|y\|_p^{-\alpha-n} \{f(x-y) - f(x)\} d^n y$$

to locally constant functions satisfying

$$\int_{\|x\|_p > 1} \|x\|_p^{-\alpha-n} |f(x)| \, d^n x < \infty.$$

The Taibleson operator $\boldsymbol{D}^{\alpha}$ is the $p-$adic analog of the fractional derivative. If $n = 1$, $\boldsymbol{D}^{\alpha}$ agrees with the Vladimirov operator. The operator $\boldsymbol{D}^{\alpha}$ does not satisfy the chain rule neither Leibniz formula. We also use the notation $\boldsymbol{D}_x^{\alpha}$, when the Taibleson operator acts on functions depending on the variables $x \in \mathbb{Q}_p^n$ and $t \geq 0$.

Given $0 = \delta_0 < \delta_1 < \cdots < \delta_{k-1} < \delta_k = \delta$, we define

$$P(\boldsymbol{D}) = \sum_{j=0}^{k} C_j \boldsymbol{D}^{\delta_j}, \text{ where the } C_j \in \mathbb{R}.$$

Lemma 3 ([18, Lemma 10.13 and Theorem 10.15])**.** *For $s \in \mathbb{R}_+$, the mapping $P(\boldsymbol{D}) : \mathcal{H}_{s+2\delta} \longrightarrow \mathcal{H}_s$ is a well-defined continuous mapping between Banach spaces.*

Lemma 4. *Take $s - 2\delta > n/2$ and $f, g \in \mathcal{H}_{s+2\delta}$. Then*

$$\|P(\boldsymbol{D})(fg)\|_s \leq C(n,s,\delta) \|f\|_{s+2\delta} \|g\|_{s+2\delta} ,$$

where $C(n,s,\delta)$ is a positive constant that depends of n, s and δ.

Proof. Since $s > n/2$ and $f, g \in \mathcal{H}_{s+2\delta}$, by Proposition 1, $fg \in \mathcal{H}_{s+2\delta}$, and by Lemma 3, $P(\boldsymbol{D})(fg) \in \mathcal{H}_s$. Now by using Proposition 1,

$$\|P(\boldsymbol{D})(fg)\|_s \leq \sum_{j=0}^{k} |C_j| \left\| \boldsymbol{D}^{\delta_j}(fg) \right\|_s$$

$$= \sum_{j=0}^{k} |C_j| \left(\int_{\mathbb{Q}_p^n} [\xi]_p^s \|\xi\|_p^{2\delta_j} \left| \widehat{fg}(\xi) \right|^2 d^n\xi \right)^{\frac{1}{2}} \leq \sum_{j=0}^{k} |C_j| \left(\int_{\mathbb{Q}_p^n} [\xi]_p^{s+2\delta_j} \left| \widehat{fg}(\xi) \right|^2 d^n\xi \right)^{\frac{1}{2}}$$

$$= \sum_{j=0}^{k} |C_j| \, \|fg\|_{s+2\delta_j} \leq \sum_{j=0}^{k} |C_j| \, C(n, s, \delta_j) \, \|f\|_{s+2\delta_j} \|g\|_{s+2\delta_j}$$

$$\leq \left(\sum_{j=0}^{k} |C_j| \, C(n, s, \delta_j) \right) \|f\|_{s+2\delta} \|g\|_{s+2\delta}.$$

□

4. Local well-posedness of the p−adic Nagumo-type equations

4.1. **Some technical remarks.** Let X, Y Banach spaces, $T_0 \in (0, \infty)$ and let $F : [0, T_0] \times Y \longrightarrow X$ a continuous function. The Cauchy problem

$$\begin{cases} \partial_t u(t) = F(t, u(t)) \\ \\ u(0) = \phi \in Y \end{cases} \tag{4.1}$$

is locally well-posed in Y, if the following conditions are satisfied.

(i) There is $T \in (0, T_0]$ and a function $u \in C([0, T]; Y)$, with $u(0) = \phi$, satisfying the differential equation in the following sense:

$$\lim_{h \to 0} \left\| \frac{u(t+h) - u(t)}{h} - F(t, u(t)) \right\|_X = 0,$$

where the derivatives at $t = 0$ and $t = T$ are calculated from the right and left, respectively.

(ii) The initial value problem (4.1) has at most one solution in $C([0, T]; Y)$.

(iii) The function $\phi \to u$ is continuous. That is, let $\{\phi_n\}$ be a sequence in Y such that $\phi_n \to \phi_\infty$ in Y and let $u_n \in C([0, T_n]; Y)$, resp. $u_\infty \in C([0, T_\infty]; Y)$, be the corresponding solutions. Let $T \in (0, T_\infty)$, then the solutions u_n are defined in $[0, T]$ for all n big enough and

$$\lim_{n \to \infty} \sup_{t \in [0, T]} \|u_n(t) - u_\infty(t)\|_Y = 0.$$

4.2. **Main result.** Consider the following Cauchy problem:

$$\begin{cases} u \in C([0, T], \mathcal{H}_s) \cap C^1([0, T], \mathcal{H}_s); \\ \\ u_t = -\gamma \boldsymbol{D}_x^{\alpha} u - u^3 + (\beta + 1) u^2 - \beta u + P(\boldsymbol{D}_x)(u^m), \quad x \in \mathbb{Q}_p^n, \ t \in [0, T]; \\ \\ u(0) = f_0 \in \mathcal{H}_s, \end{cases} \tag{4.2}$$

where T, γ, α, $\beta>0$, and m is a positive integer. The main result of this work is the following:

Theorem 1. *For $s>n/2+2\delta$, the Cauchy problem (4.2) is locally well-posed in $\mathcal{H}_s$.*

4.3. **Preliminary results.** We denote by $\boldsymbol{V}(t)=e^{-(\gamma\boldsymbol{D}^{\alpha}+\beta\boldsymbol{I})t}$, $t\geq0$, the semigroup in L^2 generated by the operator $\boldsymbol{A}=-\gamma\boldsymbol{D}^{\alpha}-\beta\boldsymbol{I}$, that is,

$$\boldsymbol{V}(t)f\left(x\right)=\mathcal{F}_{\xi\rightarrow x}^{-1}\left(e^{-(\gamma\left\Vert \xi\right\Vert _{p}^{\alpha}+\beta)t}\mathcal{F}_{x\rightarrow\xi}f\right),\text{ for }f\in L^{2},\ t\geq0.$$

Lemma 5. *$\{\boldsymbol{V}(t)\}_{t\geq0}$ is a C^0-semigroup of contractions in $\mathcal{H}_s$, $s\in\mathbb{R}$, satisfying $\left\Vert \boldsymbol{V}(t)\right\Vert_s\leq e^{-\beta t}$ for $t\geq0$. Moreover, $u(x,t)=\boldsymbol{V}(t)f_0(x)$ is the unique solution to the following Cauchy problem:*

$$\left\{ \begin{array}{l} u\in C\left(\left[0,T\right],\mathcal{H}_{s}\right)\cap C^{1}\left(\left[0,T\right],\mathcal{H}_{s}\right);\\ \\ u_{t}=-\gamma\boldsymbol{D}^{\alpha}u-\beta u,\ t\in\left[0,T\right];\\ \\ u(x,0)=f_{0}(x)\in\mathcal{H}_{s},\end{array}\right. \tag{4.3}$$

where T is an arbitrary positive number.

Proof. We just verify the strongly continuity of the semigroup. Since

$$\begin{aligned}\left\Vert \mathcal{F}_{\xi\rightarrow x}^{-1}\left(e^{-(\gamma\left\Vert \xi\right\Vert _{p}^{\alpha}+\beta)t}\mathcal{F}_{x\rightarrow\xi}f\right)-f\left(x\right)\right\Vert _{s}^{2}&\\ =\int\limits_{\mathbb{Q}_{p}^{n}}\left[\xi\right]_{p}^{s}\left\vert \widehat{f}\left(\xi\right)\right\vert ^{2}&\left\{1-e^{-(\gamma\left\Vert \xi\right\Vert _{p}^{\alpha}+\beta)t}\right\}^{2}d^{n}\xi\leq\left\Vert f\right\Vert _{s}^{2},\end{aligned}$$

it follows from the dominated convergence theorem that

$$\lim_{t\rightarrow0+}\left\Vert \boldsymbol{V}(t)f-f\right\Vert _{s}=0.$$

The existence and uniqueness of a solution for the Cauchy problem (4.3) follows from a well-known result, see e.g. [20, Theorem 4.3.1]. □

Lemma 6. *Let $f_0\in\mathcal{H}_s$, $s\in\mathbb{R}$, $\lambda\geq0$. Then, there exists a positive constant $C(\lambda,\alpha)$ that depends of λ and α such that*

$$\left\Vert \boldsymbol{V}(t)f_{0}\right\Vert _{s+\lambda}\leq e^{-\beta t}\left(1+C(\lambda,\alpha)\left(\frac{\lambda}{2\alpha\gamma t}\right)^{\frac{\lambda}{2\alpha}}\right)\left\Vert f_{0}\right\Vert _{s}\quad\text{for }t>0. \tag{4.4}$$

Proof. We first notice that

$$\left\Vert \boldsymbol{V}\left(t\right)f_{0}\right\Vert _{s+\lambda}^{2}=\int\limits_{\mathbb{Q}_{p}^{n}}\left[\xi\right]_{p}^{s+\lambda}e^{-2(\gamma\left\Vert \xi\right\Vert _{p}^{\alpha}+\beta)t}\left\vert f_{0}\left(\xi\right)\right\vert ^{2}d^{n}\xi$$

$$\leq e^{-2\beta t}\left(\sup_{\xi\in\mathbb{Q}_{p}^{n}}\left[\xi\right]_{p}^{\lambda}e^{-2\gamma\left\Vert \xi\right\Vert _{p}^{\alpha}t}\right)\left\Vert f_{0}\right\Vert _{s}^{2}\leq e^{-2\beta t}\left(1+\sup_{\xi\in\mathbb{Q}_{p}^{n}\smallsetminus\mathbb{Z}_{p}^{n}}\left\Vert \xi\right\Vert _{p}^{\lambda}e^{-2\gamma\left\Vert \xi\right\Vert _{p}^{\alpha}t}\right)\left\Vert f_{0}\right\Vert _{s}^{2}$$

$$\leq e^{-2\beta t}\left(1+\sup_{\xi\in\mathbb{Q}_{p}^{n}}\left\Vert \xi\right\Vert _{p}^{\lambda}e^{-2\gamma\left\Vert \xi\right\Vert _{p}^{\alpha}t}\right)\left\Vert f_{0}\right\Vert _{s}^{2}.$$

We now set $y = \|\xi\|_p$ and $h(y) = y^{\lambda} e^{-2\gamma y^{\alpha} t}$. By using the fact that $h(y)$ reaches its maximum at $y_{\max} = \left(\frac{\lambda}{2\alpha\gamma t}\right)^{\frac{1}{\alpha}}$, we conclude that

$$\sup_{\xi\in\mathbb{Q}_p^n} \|\xi\|_p^{\lambda} e^{-2\gamma\|\xi\|_p^{\alpha} t} \leq \left(\frac{\lambda}{2\alpha\gamma t}\right)^{\frac{\lambda}{\alpha}} e^{-\frac{\lambda}{\alpha}} \leq \mathrm{C}\left(\lambda,\alpha\right)\left(\frac{\lambda}{2\alpha\gamma t}\right)^{\frac{\lambda}{\alpha}}.$$

□

Proposition 2. *Let $s > n/2 + 2\delta$ and $F(u) = (\beta+1)u^2 - u^3 + P(\boldsymbol{D})(u^m)$. Then $F : \mathcal{H}_s \longrightarrow \mathcal{H}_{s-2\delta}$ is a continuous function satisfying*

$$\|F(u) - F(w)\|_{s-2\delta} \leq L(\|u\|_s, \|w\|_s)\|u - w\|_s, \tag{4.5}$$

for $u, w \in \mathcal{H}_s$, here $L(\cdot,\cdot)$ is a continuous function, which is not decreasing with respect to each of their arguments. In particular,

$$\|F(u)\|_{s-2\delta} \leq L(\|u\|_s, 0)\|u\|_s. \tag{4.6}$$

Proof. We first notice that

$$F(u) - F(w) = (\beta+1)(u^2 - w^2) - (u^3 - w^3) + P(\boldsymbol{D})(u^m - w^m)$$
$$= (\beta+1)(u-w)(u+w) - (u-w)(u^2+uw+w^2) + P(\boldsymbol{D})((u-w)q(u,w)),$$

where $q(u,w) = \sum_{k=0}^{m-1} u^k w^{m-1-k}$. By using Proposition 1 and Lemma 4, the condition $s > n/2$ implies that if $u, w \in \mathcal{H}_s$, then any polynomial function in u, w belongs to $\mathcal{H}_s$, and

$$\|F(u) - F(w)\|_{s-2\delta} \leq C\left\{(\beta+1)\|u-w\|_{s-2\delta}\|u+w\|_{s-2\delta} + \right.$$
$$\left. \|u-w\|_{s-2\delta}\|u^2+uw+w^2\|_{s-2\delta} + \|u-w\|_s \left\|q(u,w)\right\|_s\right\},$$

where $C = C(n,s,\delta)$. Then

$$\|F(u) - F(w)\|_{s-2\delta} \leq A(\|u\|_s, \|w\|_s)\|u - w\|_s,$$

where

$$A(\|u\|_s, \|w\|_s) = C\left\{(\beta+1)\|u+w\|_s + \|u^2+uw+w^2\|_s + \left\|q(u,w)\right\|_s\right\}$$
$$\leq C\left\{(\beta+1)\|u\|_s + (\beta+1)\|w\|_s + \|u^2\|_s + \|uw\|_s + \|w^2\|_s + \sum_{k=0}^{m-1}\left\|u^k w^{m-1-k}\right\|_s\right\}$$
$$\leq C(\beta+1)\|u\|_s + C(\beta+1)\|w\|_s + C^2\|u\|_s^2 + C^2\|u\|_s\|w\|_s + C^2\|w\|_s^2 +$$
$$C^{m+1}\sum_{k=0}^{m-1}\|u\|_s^k\,\|w\|_s^{m-1-k} =: L(\|u\|_s, \|w\|_s).$$

□

For $M, T > 0$ and $f_0 \in \mathcal{H}_s$, we set

$$\mathcal{X}(M,T,f_0) := \left\{u(t) \in C\left([0,T];\mathcal{H}_s\right);\ \sup_{t\in[0,T]}\|u(t) - V(t)f_0\|_s \leq M\right\}.$$

We endow $\mathcal{X}(M,T,f_0)$ with the metric $d(u(t),v(t)) = \sup_{t\in[0,T]}\|u(t)-v(t)\|_s$. The resulting metric space is complete.

Proposition 3. *Take $f_0 \in \mathcal{H}_s$ with $s > n/2 + 2\delta$, $\delta > 0$. Then, there exists $T = T(\|f_0\|_s, M) > 0$ and a unique function $u \in C([0,T]; \mathcal{H}_s)$ satisfying the integral equation*

$$u(t) = \boldsymbol{V}(t)f_0 + \int_0^t \boldsymbol{V}(t-\tau)F(u(\tau))d\tau, \tag{4.7}$$

such that $u(0) = f_0$. Here $F(u) = (\beta+1)u^2 - u^3 + P(\boldsymbol{D})(u^m)$ as before.

Remark 1. *Since $F(u)$ is not a locally Lipschitz function because inequality (4.6) involves two different norms, the existence of mild solutions of type (4.7) does not follow directly from standard results in semigroup theory, see e.g.* [20, Theorem 5.2.2].

Proof. Given $u \in \mathcal{X}(M,T,f_0)$, we set

$$\boldsymbol{N}u(t) = \boldsymbol{V}(t)f_0 + \int_0^t \boldsymbol{V}(t-\tau)F(u(\tau))d\tau.$$

Claim 1. $\boldsymbol{N} : \mathcal{X}(M,T,f_0) \longrightarrow C([0,T]; \mathcal{H}_s)$.

Take $u \in \mathcal{X}(M,T,f_0)$, then

$$\begin{aligned} \|\boldsymbol{N}u(t_1) - \boldsymbol{N}u(t_2)\|_s &\leq \|(\boldsymbol{V}(t_1) - \boldsymbol{V}(t_2))f_0\|_s \\ &+ \left\| \int_0^{t_1} \boldsymbol{V}(t_1-\tau)F(u(\tau))d\tau - \int_0^{t_2} \boldsymbol{V}(t_2-\tau)F(u(\tau))d\tau \right\|_s . \end{aligned} \tag{4.8}$$

Since $\{\boldsymbol{V}(t)\}_{t\geq 0}$ is a C_0-semigroup in $\mathcal{H}_s$, cf. Lemma 5, the first term on the right-hand side of the inequality (4.8) tends to zero when $t_2 \to t_1$. To study the second term, we assume without loss of generality that $0 < t_1 < t_2 < T$. Then

$$\begin{aligned} &\left\| \int_0^{t_1} \boldsymbol{V}(t_1-\tau)F(u(\tau))d\tau - \int_0^{t_2} \boldsymbol{V}(t_2-\tau)F(u(\tau))d\tau \right\|_s \\ &\leq \int_0^{t_1} \|\{\boldsymbol{V}(t_1-\tau) - \boldsymbol{V}(t_2-\tau)\}F(u(\tau))\|_s d\tau + \int_{t_1}^{t_2} \|\boldsymbol{V}(t_2-\tau)F(u(\tau))\|_s d\tau. \end{aligned}$$

By using Lemma 6 with $\lambda = \alpha$ and Proposition 2,

$$\begin{aligned} &\|(\boldsymbol{V}(t_1-\tau) - \boldsymbol{V}(t_2-\tau))F(u(\tau))\|_s \\ &\leq \|\boldsymbol{V}(t_1-\tau)F(u(\tau))\|_s + \|\boldsymbol{V}(t_2-\tau)F(u(\tau))\|_s \\ &\leq \left\{ 2 + C_0\left(\frac{1}{2\gamma(t_1-\tau)}\right)^{\frac{1}{2}} + C_0\left(\frac{1}{2\gamma(t_2-\tau)}\right)^{\frac{1}{2}} \right\} \|F(u(\tau))\|_{s-\alpha} \\ &\leq 2\left\{1 + C_0\left(\frac{1}{2\gamma(t_1-\tau)}\right)^{\frac{1}{2}}\right\} \sup_{\tau\in[0,T]} \|F(u(\tau))\|_{s-\alpha} \\ &= A(T,s,\alpha)\left\{1 + C_0\left(\frac{1}{2\gamma(t_1-\tau)}\right)^{\frac{1}{2}}\right\} \in L^1([0,t_1]). \end{aligned}$$

Now, by applying the dominated convergence theorem,

$$\lim_{t_2\to t_1} \int_0^{t_1} \|(\boldsymbol{V}(t_1-\tau) - \boldsymbol{V}(t_2-\tau))F(u(\tau))\|_s d\tau = 0.$$

By a similar argument, one shows that

$$\left\| \boldsymbol{V}\left(t_{2}-\tau\right) F(u(\tau))\right\|_{s-2\delta} \leq 1+C_{0}\left(\frac{1}{2\gamma(t_{2}-\tau)}\right)^{\frac{1}{2}} L(\|u\left(\tau\right)\|_{s},0)\|u\left(\tau\right)\|_{s},$$

and since

$$\|u(\tau)\|_{s} \leq\|u(\tau)-\boldsymbol{V}(\tau) f_{0}\|_{s}+\|\boldsymbol{V}(\tau) f_{0}\|_{s} \leq M+\|f_{0}\|_{s}, \text{ for all } \tau \in[0, T], \tag{4.9}$$

we have

$$\begin{aligned} & \int_{t_1}^{t_2} \left\| \boldsymbol{V}(t_2-\tau) F(u(\tau))\right\|_s d\tau \tag{4.10} \\ & \leq L(M+\|f_0\|_s,0)(M+\|f_0\|_s)\left(\int_{t_1}^{t_2}\left(1+C_0\left(\frac{1}{2\gamma(t_2-\tau)}\right)^{\frac{1}{2}}\right)d\tau\right) \\ & = L\left(M+\|f_0\|_s,0\right)\left(M+\|f_0(\cdot)\|_s\right)\left((t_2-t_1)+C_0\left(\sqrt{\frac{2(t_2-t_1)}{\gamma}}\right)\right), \end{aligned}$$

and consequently, by applying the dominated convergence theorem,

$$\lim_{t_2\to t_1}\int_{t_1}^{t_2}\|\boldsymbol{V}(t_2-\tau)F(u(\tau))\|_s d\tau=0.$$

Claim 2. There exists T_0 such that $\boldsymbol{N}(\mathcal{X}(M,T_0,f_0))\subseteq\mathcal{X}(M,T_0,f_0)$.

By using a reasoning similar to the one used to established inequality (4.10), one gets

$$\begin{aligned} \|(\boldsymbol{N}u)(t)-\boldsymbol{V}(t)f_0\|_s & \leq \int_0^t \|\boldsymbol{V}(t-\tau)F(u(\tau))\|_s d\tau \\ \leq L\left(M+\|f_0\|_s,0\right)\left(M+\|f_0\|_s\right) & \left(\int_0^t\left(1+C_0\left(\frac{1}{2\gamma(t-\tau)}\right)^{\frac{1}{2}}\right)d\tau\right) \\ \leq L\left(M+\|f_0\|_s,0\right)\left(M+\|f_0\|_s\right) & \left(T+C_0\left(\sqrt{\frac{2T}{\gamma}}\right)\right). \end{aligned}$$

Now taking T_0 such that

$$L\left(M+\|f_0\|_s,0\right)\left(M+\|f_0\|_s\right)\left(T_0+C_0\left(\sqrt{\frac{2T_0}{\gamma}}\right)\right)\leq M, \tag{4.11}$$

we conclude that $\boldsymbol{N}u\in\mathcal{X}(M,T_0,f_0)$, for all $u(t)\in\mathcal{X}(M,T_0,f_0)$.

Claim 3. There exists T_0' such that $\boldsymbol{N}$ is a contraction on $\mathcal{X}(M,T_0',f_0)$.

Given $u(t)$, $v(t)\in\mathcal{X}(M,T_0,f_0)$, by using Proposition 2, with

$$C_0'=L\left(M+\|f_0\|_s,M+\|f_0\|_s\right),$$

see (4.9), we have

$$\begin{aligned}\|\boldsymbol{N}u(t)-\boldsymbol{N}v(t)\|_s &\leq \int_0^t \|\boldsymbol{V}(t-\tau)[F(u(\tau))-F(v(\tau))]\|_s d\tau\\ &\leq \int_0^t \left(1+C_0\left(\frac{1}{2\gamma(t-\tau)}\right)^{\frac{1}{2}}\right)\|F(u(\tau))-F(v(\tau))\|_{s-\alpha}\ d\tau\\ &\leq C_0'\int_0^t \left(1+C_0\left(\frac{1}{2\gamma(t-\tau)}\right)^{\frac{1}{2}}\right)\|u(\tau)-v(\tau)\|_s d\tau\\ &\leq C_0'\left(\sup_{\tau\in[0,T_0]}\|u(\tau)-v(\tau)\|_s\right)\int_0^t \left(1+C_0\left(\frac{1}{2\gamma(t-\tau)}\right)^{\frac{1}{2}}\right)d\tau\\ &\leq C_0'\left(T_0+C_0\left(\sqrt{\frac{2T_0}{\gamma}}\right)\right)d(u(t),v(t)).\end{aligned}$$

Thus, taking T_0' such that

$$C:=C_0'\left(T_0'+C_0\left(\sqrt{\frac{2T_0'}{\gamma}}\right)\right)<1, \tag{4.12}$$

we obtain that $d(\boldsymbol{N}u(t),\boldsymbol{N}v(t))\leq Cd(u(t),v(t))$, that is, $\boldsymbol{N}$ is a strict contraction in $\mathcal{X}(M,T_0',f_0)$. We pick T such that the inequalities (4.11) and (4.11) hold true, and apply the Banach Fixed Point Theorem to get $u(t)\in\mathcal{X}(M,T,f_0)$ a unique fixed point of $\boldsymbol{N}$, which satisfies the integral equation (4.7), where $T=T(\|f_0\|_s,M)>0$. □

Remark 2. *Let $\mathcal{X}$ be a Banach space and let $\boldsymbol{A}:Dom(\boldsymbol{A})\to\mathcal{X}$ be an operator with dense domain such that $\boldsymbol{A}$ is the infinitesimal generator of a contraction semigroup $(\boldsymbol{S}_t)_{t\geq 0}$. Fix $T>0$ and let $f:[0,T]\to\mathcal{X}$ be a continuous function. Consider the Cauchy problem:*

$$\begin{cases} u\in C\left([0,T],Dom(\boldsymbol{A})\right)\cap C^1\left([0,T],\mathcal{X}\right);\\ u_t=\boldsymbol{A}u+f(t),\quad t\in[0,T];\\ u(0)=u_0\in\mathcal{X}.\end{cases} \tag{4.13}$$

Then

$$u(t)=\boldsymbol{S}(t)u_0+\int_0^t \boldsymbol{S}(t-\tau)f(\tau))d\tau, \tag{4.14}$$

for $t\in[0,T]$, see e.g. [3, Lemma 4.1.1]. *Conversely, if $u_0\in Dom(\boldsymbol{A})$, $f\in C\left([0,T],\mathcal{X}\right)$,*

$$\int_{(0,T)}\|f(\tau)\|_{\mathcal{X}}\ d\tau<\infty,$$

then a solution of (4.14) is a solution of the Cauchy problem (4.13), see e.g. [3, Proposition 4.1.6].

Proposition 4. *The problem (4.2) is equivalent to the integral equation (4.7). More precisely, if $s>n/2+2\delta$, and $u(t)\in C([0,T];\mathcal{H}_s)\cap C^1((0,T];\mathcal{H}_{s-2\delta})$ is a solution of (4.2), then $u(t)$ satisfies the integral equation (4.7). Conversely,* *if* $s > n/2 + 2\delta$*, and* $u(t) \in C([0,T];\mathcal{H}_s)$ *is a solution of (4.7), then* $u(t) \in C^1([0,T];\mathcal{H}_{s-2\delta})$ *and it satisfies (4.2).*

Proof. It follows from Remark 2, Propositions 3, 2, by taking $\boldsymbol{A} = -\gamma \boldsymbol{D}_x^{\alpha} - \beta \boldsymbol{I}$, $Dom(\boldsymbol{A}) = \mathcal{H}_s$, $\mathcal{X} = \mathcal{H}_{s-2\delta}$, $f(t) = F(u(t))$. We first recall that $\mathcal{D} \hookrightarrow \mathcal{H}_s \hookrightarrow \mathcal{H}_{s-2\delta}$, where $\hookrightarrow$ means continuous embedding, an that $\mathcal{D}$ is dense in $\mathcal{H}_{s-2\delta}$. If $u(t)$ is a solution of (4.2), then, since $F(u(t)) \in C([0,T];\mathcal{H}_{s-2\delta})$, by Proposition 2, $u(t)$ is a solution of (4.7). Conversely, if $u(t)$ is a solution of (4.7), since

$$\int_{(0,T)} \|F(u(\tau))\|_{s-2\delta} \, d\tau < \infty,$$

by Proposition 2, $u(t)$ is a solution of (4.2). □

Lemma 7 ([20, Theorem 5.1.1])**.** *If* $h \in L^1(0,T)$*, with* $T > 0$*, is real-valued function such that. If*

$$h(t) \leq a + b \int_0^t h(s)ds,$$

for $t \in (0,T)$ *a.e., where* $a \in \mathbb{R}$ *and* $b \in [0,\infty)$ *then* $h(t) \leq ae^{bt}$ *for almost all* t *in* $(0,T)$*.*

Proposition 5. *Let* $f_0,\ f_1 \in \mathcal{H}_s$ *and* $u(t), v(t) \in C[0,T];\mathcal{H}_s)$ *be the corresponding solutions of equation (4.7) with initial conditions* $u(0) = f_0$ *and* $v(0) = f_1$*, respectively. If* $s > n/2 + 2\delta$*, then*

$$\|u(t) - v(t)\|_s \leq e^{L(W,W)} \|f_0 - f_1\|_s,$$

where L *is given in Proposition 1 and*

$$W := \max \left\{ \sup_{t\in[0,T]} \|u(t)\|_s, \sup_{t\in[0,T]} \|v(t)\|_s \right\}.$$

Proof. By using (4.7), we have

$$u(t) - v(t) = \boldsymbol{V}(t)(f_0 - f_1) + \int_0^t \boldsymbol{V}(t-\tau)\{F(u(\tau)) - F(v(\tau))\}d\tau.$$

By using Proposition 1, we get

$$\begin{aligned} \|u(t) - v(t)\|_s &\leq \|f_0 - f_1\|_s + \int_0^t \|\boldsymbol{V}(t-\tau)\{F(u(\tau)) - F(v(\tau))\}\|_s d\tau \\ &\leq \|f_0 - f_1\|_s + \int_0^t \|F(u(\tau)) - F(v(\tau))\|_{s-\alpha} \, d\tau \\ &\leq \|f_0 - f_1\|_s + L(W,W) \int_0^t \|u(\tau) - v(\tau)\|_s d\tau. \end{aligned}$$

Now the result follow from Lemma 7, by taking $h(t) = \|u(t) - v(t)\|_s$, $a = \|f_0 - f_1\|_s$, $b = L(W,W)$. □

Proposition 6. *Let* $s > n/2 + 2\delta$ *and* $\delta \geq 0$*. Then, the map* $f_0 \mapsto u(t)$ *is continuous in the following sense: if* $f_0^{(n)} \to f_0$ *in* $\mathcal{H}_s$ *and* $u_n(t) \in C([0,T_n];\mathcal{H}_s)$*, with* $T_n = T\left(\left\|f_0^{(n)}\right\|_s, M\right) > 0$*, are the corresponding solutions to the Cauchy* *problem (4.2) with* $u_n(0) = f_0^{(n)}$*. Then, there exist* $T > 0$ *and a positive integer* $N = N(\gamma, f_0, T)$ *such that* $T_n \geq T$ *for all* $n \geq N$ *and*

$$\lim_{n\to\infty} \sup_{t\in[0,T]} \|u_n(t) - u(t)\|_s = 0. \tag{4.15}$$

Proof. By Proposition 3, the $T_n = T\left(\left\|f_0^{(n)}\right\|_s, M\right) > 0$ are continuous functions of $\left\|f_0^{(n)}\right\|_s$, then, given $T^* > 0$ there exists $N \in \mathbb{N}$ such that $T^* \leq T_n$ for all $n \geq N$. We set $T := \min\{T^*, T_1, T_2, \ldots, T_{N-1}\}$. Therefore, all the $u_n(t)$ are defined on $[0, T]$, furthermore, $u \in \mathcal{X}\left(M, T, f_0^{(n)}\right)$ for all n, and

$$\|u_n(t)\|_s \leq \left\|f_0^{(n)}\right\|_s + M \leq \delta + M,$$

where $\delta = \sup_{n\in\mathbb{N}} \left\|f_0^{(n)}\right\|_s$. Now

$$\sup_{t\in[0,T]} \|u_n(t)\|_s \leq \delta + M \text{ for all } n, \text{ and } \sup_{t\in[0,T]} \|u(t)\|_s \leq \delta + M.$$

On the other hand, by reasoning as in the proof of Proposition 5, we have

$$\|u_n(t) - u(t)\|_s \leq \left\|f_0^{(n)} - f_0\right\|_s + L(\delta + M, \delta + M) \int_0^t \|u_n(\tau) - u(\tau)\|_s d\tau,$$

and by applying Lemma 7

$$\|u_n(t) - u(t)\|_s \leq e^{TL(\delta+M,\delta+M)} \left\|f_0^{(n)} - f_0\right\|_s,$$

which in turns implies (4.15). □

4.4. **Proof of the Main result.** The local well-posedness of the Cauchy problem (4.2) in $\mathcal{H}_s$, $s > n/2 + 2\delta$, follows from Propositions 3, 5, 6.

5. The Blow-up phenomenon

In this section, we study the blow-up phenomenon for the solution of the equation

$$\begin{cases} u_t = -\gamma \boldsymbol{D}_x^{\alpha} u + F(u) + \boldsymbol{D}_x^{\alpha_1} u^3, & x \in \mathbb{Q}_p^n,\ t \in [0, T]; \\ \\ u(0) = f_0 \in \mathcal{H}_\infty, \end{cases} \tag{5.1}$$

where $F(u) = -u^3 + (\beta + 1)\, u^2 - \beta u$. We will say that a non-negative solution $u(x, t) \geq 0$ of (5.1) blow-up in a finite time $T > 0$, if $\lim_{t\to T^-} \sup_{x\in\mathbb{Q}_p^n} u(x, t) = +\infty$. This limit makes sense since $\mathcal{H}_\infty(\mathbb{Q}_p^n, \mathbb{C})$ is continuously embedded in $C_0(\mathbb{Q}_p^n, \mathbb{C})$, [18, Theorem 10.15].

5.1. $p-$**adic wavelets and pseudo-differential operators.** We denote by $C(\mathbb{Q}_p, \mathbb{C})$ the $\mathbb{C}-$vector space of continuous $\mathbb{C}-$valued functions defined on $\mathbb{Q}_p$.

We fix a function $\mathfrak{a} : \mathbb{R}_+ \to \mathbb{R}_+$ and define the pseudo-differential operator

$$\begin{array}{ccc} \mathcal{D} & \to & C(\mathbb{Q}_p, \mathbb{C}) \cap L^2 \\ \\ \varphi & \to & \boldsymbol{A}\varphi, \end{array}$$

where $(\boldsymbol{A}\varphi)(x) = \mathcal{F}^{-1}_{\xi\to x} \left\{\mathfrak{a}\left(|\xi|_p\right) \mathcal{F}_{x\to\xi}\varphi\right\}$.

The set of functions $\{\Psi_{rnj}\}$ defined as

$$\Psi_{rnj}(x) = p^{\frac{-r}{2}} \chi_p \left(p^{-1} j \left(p^r x - n\right)\right) \Omega\left(\left|p^r x - n\right|_p\right), \tag{5.2}$$

where $r \in \mathbb{Z}$, $j \in \{1, \cdots, p-1\}$, and n runs through a fixed set of representatives of $\mathbb{Q}_p/\mathbb{Z}_p$, is an orthonormal basis of $L^2(\mathbb{Q}_p)$ consisting of eigenvectors of operator $\boldsymbol{A}$:

$$\boldsymbol{A}\Psi_{rnj} = \mathfrak{a}(p^{1-r})\Psi_{rnj} \text{ for any } r,\ n,\ j, \tag{5.3}$$

see e.g. [18, Theorem 3.29], [1, Theorem 9.4.2]. Notice that

$$\widehat{\Psi}_{rnj}(\xi) = p^{\frac{r}{2}} \chi_p \left(p^{-r} n \xi\right) \Omega\left(\left|p^{-r}\xi + p^{-1} j\right|_p\right),$$

and then

$$\mathfrak{a}\left(|\xi|_p\right) \widehat{\Psi}_{rnj}(\xi) = \mathfrak{a}(p^{1-r}) \widehat{\Psi}_{rnj}(\xi).$$

In particular, $\boldsymbol{D}_x^{\alpha} \Psi_{rnj} = p^{(1-r)\alpha} \Psi_{rnj}$, for any r, n, j and $\alpha > 0$, and since $p^{(1-r)\alpha}$,

$$\boldsymbol{D}_x^{\alpha} \operatorname{Re}(\Psi_{rnj}) = p^{(1-r)\alpha} \operatorname{Re}(\Psi_{rnj}),\ \boldsymbol{D}_x^{\alpha} \operatorname{Im}(\Psi_{rnj}) = p^{(1-r)\alpha} \operatorname{Im}(\Psi_{rnj}).$$

And,

$$\begin{aligned} \{\Psi_{rn1}(x)\}^2 &= p^{-r} \chi_p \left(2p^{-1}\left(p^r x - n\right)\right) \Omega\left(\left|p^r x - n\right|_p\right) \\ &= p^r \{\Psi_{rn1}(x)\}^2 = p^{\frac{r}{2}} \Psi_{rn2}(x), \end{aligned}$$

then

$$\boldsymbol{D}_x^{\alpha} \operatorname{Re}\left(\{\Psi_{rn1}(x)\}^2\right) = p^{\frac{r}{2}} p^{(1-r)\alpha} \operatorname{Re}(\Psi_{rn2}(x)) = p^{(1-r)\alpha} \operatorname{Re}\left(\{\Psi_{rn1}(x)\}^2\right).$$

5.2. **The blow-up.** In this section, we assume that $u(x,t)$ is real-valued non-negative solution of the Cauchy problem (4.2) in $\mathcal{H}_\infty$. We set $w(x) := \operatorname{Re}\left(\{\Psi_{rn1}(x)\}^2\right)$, so $\boldsymbol{D}_x^{\alpha} w(x) = p^{(1-r)\alpha} w(x)$. Thus $w(x)dx$ defines a (positive) measure. We also set $G(t) := \int_{\mathbb{Q}_p} u(x,t) w(x) dx$, where $u(x,t)$ is a positive solution of (5.1), then

$$\begin{aligned} G'(t) &= \int_{\mathbb{Q}_p} u_t(x,t) w(x) dx = -\gamma \int_{\mathbb{Q}_p} (\boldsymbol{D}_x^{\alpha} u)(x,t) w(x) dx \\ &+ \int_{\mathbb{Q}_p} F(u(x,t)) w(x) dx + \int_{\mathbb{Q}_p} (\boldsymbol{D}_x^{\alpha_1} u^3)(x,t) w(x) dx. \end{aligned} \tag{5.4}$$

Now, by using that $\boldsymbol{D}_x^{\alpha} u(\cdot,t)$, $w \in L^2$, and $F(u(\cdot,t))$, $\boldsymbol{D}_x^{\alpha_1} u^3(\cdot,t) \in L^2$ since for $s > n/2$, $\mathcal{H}_s$ is a Banach algebra contained in L^2 cf. Proposition 1, and applying the Parseval-Steklov theorem, we get (5.4) can be rewritten as

$$G'(t) = \int_{\mathbb{Q}_p} \left(-\gamma p^{(1-r)\alpha} u(x,t) + F(u(x,t)) + p^{(1-r)\alpha_1} u^3(x,t)\right) w(x) dx.$$

Since the function $H(y) = -\gamma p^{(1-r)\alpha} y + F(y) + p^{(1-r)\alpha_1} y^3$ is convex because

$$H''(y) = -6y + 2(\beta+1) + p^{(1-r)\alpha_1} 6y = 6y\left(p^{(1-r)\alpha_1} - 1\right) + 2(\beta+1) \geq 0,$$

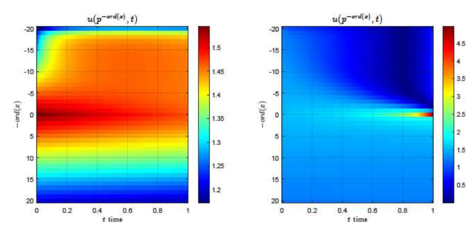
for $y \geq 0$, and $r \leq 0$, we can use the Jensen's inequality to get $G'(t) \geq H(G(t))$, then the function $G(t)$ can not remain finite for every $t \in [0, \infty)$. Then there exists

$T \in (0, \infty)$ such that $\lim_{t \to T^-} G(t) = +\infty$, hence $u(x,t)$ blow ups at the time T. Then we have established the following result:

Theorem 2. *Let $u(x,t)$ be a positive solution of (5.1). Then there $T \in (0, +\infty)$ depending on the initial datum such that* $\lim_{t \to T^-} \sup_{x \in \mathbb{Q}_p^n} u(x,t) = +\infty$.

6. Numerical Simulations

In this section, we present two numerical simulations for the solution of problem (5.1) (in dimension one) for a suitable initial datum. We solve and visualize (using a heat map) the radial profiles of the solution of (5.1). We consider equation (5.1) for radial functions $u(x, \cdot)$. In [15], Kochubei obtained a formula for $\boldsymbol{D}_x^{\alpha} u\,(x,t)$ as an absolutely convergent real series, we truncate this series and then we apply the classic Euler Forward Method (see e.g. [23]) to find the values of $u(p^{-ord(x)}, t)$, when $-20 \leq ord(x) \leq 20$ (vertical axis) and when $t = \{t_k : \ t_k = 1/k, k = 1, \ldots, 300\}$ (horizontal axis). In Figure 1, on the left, the heat map of the numerical solution of the homogeneous equation $u_t(x,t) = -\boldsymbol{D}_x^{\alpha} u(x,t)$ with initial data $u(x,0) = 4e^{-p^{|ord(x)|}/100}$ (Gaussian bell type), and parameters $p = 3$, $\alpha = 0.2$, $\gamma = 1$. On the right side, we have the numerical solution of the equation $u_t(x,t) = -\boldsymbol{D}_x^{\alpha} u(x,t) - u^3(x,t) + (\beta+1)u^2(x,t) - \beta u(x,t) + \boldsymbol{D}_x^{\alpha_1} u^3(x,t)$, with $p = 3$, $\alpha = 0.2$, $\alpha_1 = 0.1$, and $\beta = 0.7$.



On the left side of the Figure 1, we observe that the solution u is uniformly decreasing with respect to the variable t. This behavior is typical for solutions of diffusion equations. These equations have been extensively studied, see e.g. [18], [35] and the references therein.

On the right side of Figure 1, we see that the evolution of $u(x,t)$ is controlled by the diffusion term $-\boldsymbol{D}_x^{\alpha} u(x,t)$, up to a time T (blow-up time), this behavior is similar to that described above. When $t > T$, the reactive term $-u^3(x,t) + (\beta+1)u^2(x,t) - \beta u(x,t) + \boldsymbol{D}_x^{\alpha_1} u^3(x,t)$ takes over and $u(x,t)$ grows rapidly towards infinity.

The method converges quite fast, but still lacks a mathematical formalism to support it, for this reason we refer to it as a numerical simulation of the solution.

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