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Zhu, Gui-Lei, et al. "Exceptional Point Generated Robust Asymmetric High-Order Harmonics." arXiv preprint arXiv:2201.07663 (2022).

Exceptional Point Generated Robust Asymmetric High-Order Harmonics

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We propose a metallic-silicon system with a complex optical potential modulated along the length of the waveguide for a robust higher harmonic generation. For right moving fields when the strength of non-Hermiticity becomes equal to the real part of the optical potential, the dynamical equations associated with modal field amplitudes in our proposed system are described by a Jordan form Hamiltonian. This ultimately will allow for a unidirectional higher frequency generation which always has a maximum value for a specific length of the waveguide irrespective of the geometrical imperfections in the design of the waveguide. Furthermore, the phase of the generated higher harmonic mode becomes independent of the coupling between the fundamental frequency and higher harmonic one. Unlike other proposed spatiotemporal modulated systems when the system has a Jordan form Hamiltonian, the fundamental mode remains reciprocal while the harmonic generation is non-reciprocal. Consequently, while the proposed device cannot be used as an optical isolator it can be used for many other devices such as laser cavities, interferometry, and holographic processes.

PACS numbers:

Introduction— Optical metamaterials with complex potentials provide incredible possibilities to alter light-matter interaction, create new functionalities and artificially change optical properties of synthetic structures that are not possible with normal materials[1–5]. Among such fascinating effects in light-matter interactions are unidirectional invisibility[6, 7], topological lasers[8, 9], isolation and circulations[10–15], new lasing schemes[16, 17], low-cost quantum state transfer[18–20] and quantum lattice with multi-point phase transition[21], one and two dimensional wave manipulation[22–24], to name a few. These new functionalities are achieved using different physical mechanisms. For example, in a parity-time symmetric system, the major element is non-Hermiticity where some gain and loss mechanisms break the Hermiticity in a system and results in the bi-orthogonality of states. A consequence of non-Hermiticity is the existence of exceptional points in which the Hilbert space becomes skewed and two or more eigenstates coalesce[12, 25–30]. In general, the exceptional points are not robust and are affected by the other parameters in the system. However, there are reports for the generation of a robust exceptional point where the underlying physics is associated with the nilpotent matrices with an odd number of sites[31].

The other example of metamaterials is the time modulated system and specifically the combination of space and time modulation[32–37]. The space and time modulated systems are known for breaking reciprocity, and generation of optical isolators based on asymmetric harmonic generation. Time modulated systems have been studied in waveguides, ring resonators, and metasurfaces[38–40]. In more recent studies, the spatiotemporal modulation has been extended to complex potentials with gain and loss mechanisms where higher-order knots and asymmetric exceptional points

have been reported[41]. However, the interplay between the exceptional point and harmonic generation has not studied yet. Nonlinear harmonic generation (NHG) is a beneficial procedure for coherently producing laser light in the extreme-ultraviolet region of the spectrum[42–44]. The applications include: investigating surface dynamics[45, 46], holographic imaging[47, 48], probing static molecular structure, or internal molecular dynamics[49, 50].

One of the main challenges in harmonic generation in laser cavities is the susceptibility of the output emission to the fabrication process and low-efficiency of the NHG. Although recently there have been some attempts to overcome the efficiency issue, there are problems in terms of sensitivity of the structure to the fabrication process, losses, and frequency restrictions[51–53].

This paper investigates an alternative method to address the aforementioned difficulty regarding fabrication process and achieve robust higher harmonic generation using time and space modulation of the real and imaginary part of the refractive index. Specifically, we show that an appropriate selection of a spatially and temporally varying modulation of the refractive index causes simultaneous changes in the frequency and momentum of photons traveling in the waveguide allowing a robust NHG. The specific key element in this modulation is the concurrent spatiotemporal modulation of the real and imaginary part of the distributed relative permittivity with equal strength. This specific modulation makes the system operate at the exceptional point associated with the coupled equations governing the dynamics of the modes.

Exceptional points are particular singularities that arise in non-Hermitian complex systems, in which two or more eigenvalues and eigenstates coalesce. As a result of the coalescence of the eigenvectors at the exceptional
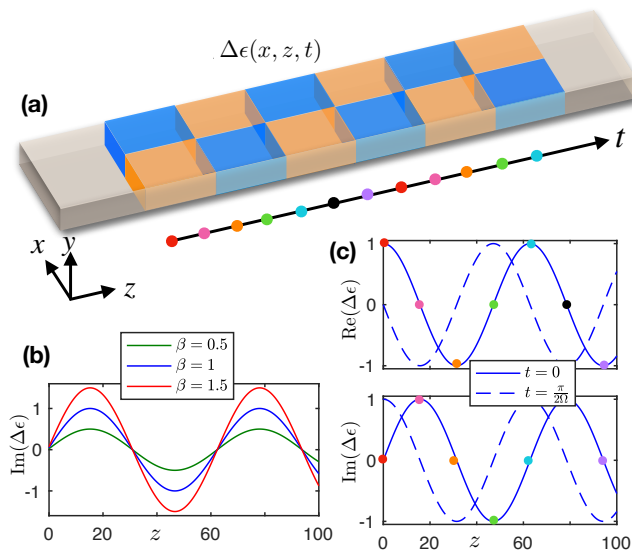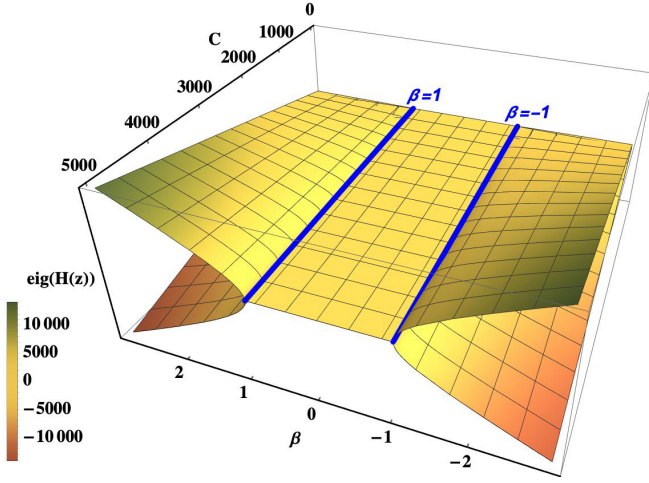
FIG. 2: The real part of the Hamiltonian's eigenvalue in Eq.(3) is depicted as a function of C and β. We can observe that when $\beta = 1, -1$ eigenvalues coalesce; thus, we have exceptional point. Interestingly, the value of C doesn't change the location of exceptional points if $\beta = 1, -1$. In other words, exceptional poiint is robust against geometrical imperfections.

the corresponding normalized intensities associated with each mode. Substituting $E(x, z, t)$ and Eq. (1) into Maxwell equations and applying the slowly varying envelope approximation we obtain the coupled-mode equations for the modal field amplitudes $\vec{\Psi}(z) = (a_1(z) \quad a_2(z))^T$

$$i\frac{d}{dz}\vec{\Psi} = H(z)\vec{\Psi} \qquad (3)$$

in which

$$H(z) = \begin{pmatrix} 0 & C(1-\beta)e^{-i\Delta kz} \\ C(1+\beta)e^{i\Delta kz} & 0 \end{pmatrix}, \qquad (4)$$

where $C = \frac{\epsilon_0}{8}\int_{-\infty}^{\infty} E_1(x)\delta(x)E_2(x)dx$ is the coupling strength. It is clear that the integral has nonzero values inside the waveguide and the width of the waveguide directly changes the interval of the integral; thus affects the coupling C. However, we observe that when $\beta = \beta_r = 1$ the matrix $H(z)$ becomes a Jordan form and a nilpotent matrix of the second order since $H^2 = 0$. Therefore, as it is shown in Fig. 2, β_r can be considered as the second order exceptional point of the $H(z)$, irrespective of the value of C or z. Specifically, for $\beta_r = 1$, the intensity of symmetric mode, namely a_1 remains constant while a_2 would change through the evolution along the z direction.

We can analytically solve coupled differential equations in Eq.(3) for any values of β. For example, for the initial excitation of symmetric mode from the left side of the waveguide namely $\vec{\Psi}(0) = (a_1(0) \quad a_2(0))^T = (1 \quad 0)^T$
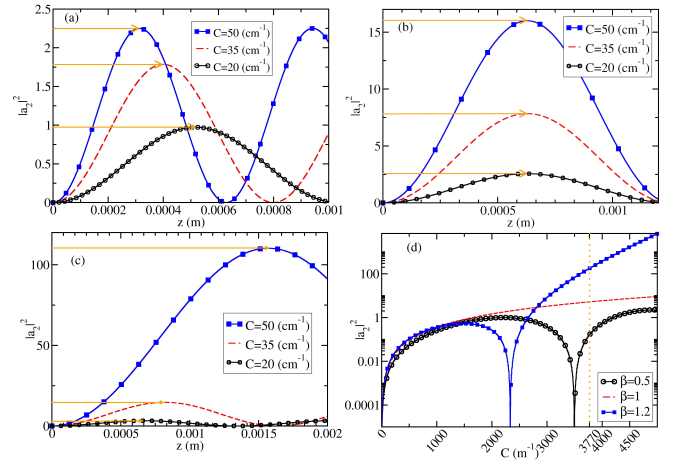
FIG. 3: (a),(b),(c) Intensities of NHG are depicted for $\beta = 0.5(a), 1(b), 1.1(c)$ as a function of waveguide's length for three different value of C. Arrows are showing the length which NHG's intensity is maximum on that. Clearly, this length, is associated with the value of C when $\beta = 0.5, 1.1$. while for $\beta = 1$ it is independent from C.(d) Log-linear plot of intensities of NHG are depicted for $\beta = 0.5, 1, 1.2$ as a Function of C at the end of a waveguide with specific length. We also have identified the value of C for which the exceptional point occurs with a vertical dash line for $\beta = 1.2$. Even in the exact phase for $\beta \neq 1$ the changes in C result in significant changes in a_2. However at $\beta = 1$ it is linearly proportional to C^2.

we would have

$$\begin{pmatrix} a_1(z) \\ a_2(z) \end{pmatrix} = \begin{pmatrix} e^{-\frac{iz\Delta k}{2}}\left[\cosh\frac{z\zeta}{2} + \frac{i\Delta k \sinh\frac{z\zeta}{2}}{\zeta}\right] \\ -2iCe^{\frac{iz\Delta k}{2}}(1+\beta)\frac{\sinh\frac{z\zeta}{2}}{\zeta} \end{pmatrix}, \quad (5)$$

in which $\zeta = \sqrt{4C^2(\beta^2-1) - \Delta k^2}$. From Eq.(5) it is clear that for $\beta = \beta_r$, $\vec{\Psi}(z)|_{\beta=\beta_r}$ will be given by

$$\vec{\Psi}(z)|_{\beta=\beta_r} = \begin{pmatrix} a_1(z) \\ a_2(z) \end{pmatrix} = \begin{pmatrix} 1 \\ -\frac{4iC}{\Delta k}\sin(\frac{\Delta kz}{2})e^{\frac{iz\Delta k}{2}} \end{pmatrix}. \qquad (6)$$

Before we make further discussion on the generated harmonic amplitude a_2 we would like to bring to the attention of the reader that a_1 remains constant and is not affected by the time dependent parity-time symmetric modulation. This effect reminds us about unidirectional invisibility effect when $\Omega = 0$[6].

From relation (6) it is obvious that for β_r the maximum of the amplitude of the generated higher harmonic term $|a_2(z)|^2$ always occurs at length $z = \frac{(2m+1)\pi}{\Delta k}$ where m is an integer number. However, if $\beta \neq \beta_r = 1$ then the maximum of the intensity of generated higher harmonic $|a_2|^2$ depends on the value of C. In practical applications, the ideal length for designing a waveguide would be a length that the intensity of the generated harmonic finds its maximum. However, in other harmonic generation methods this length is affected by the coupling C. Thus, it is an extremely advantageous aspect of our proposal

where this length becomes independent of the value of C. This conclusion is depicted in Fig. 3 where we plotted the $|a_2|^2$ as a function of z for $\beta = 0.5, 1, 1.1$ in panel (a-c), respectively. In each panel we considered $\Delta k = 50$ cm^{-1} and $C = 20, 35, 50$ cm^{-1}. Notice that for $\beta > \sqrt{1 + (\frac{\Delta k}{2C})^2}$ the system undergoes a phase transition to the broken phase, and it becomes unstable. The $\beta = \beta_{\text{EP}} = \sqrt{1 + (\frac{\Delta k}{2C})^2}$ is the exceptional point defined by the quasi-energies $\pm\mathcal{E}$ in the system[41]. This can be seen by applying Floquet theory on Eq. (2), where we find the quasi-energy of the system as

$$\mathcal{E}_{\pm} = \frac{\Delta k}{2} \pm \sqrt{(\frac{\Delta k}{2})^2 - C^2(\beta^2 - 1))}. \qquad (7)$$

We can easily show that at the exceptional point β_{EP} associated with quasi energies the $|a_2|^2$ increases in power low manner namely it becomes proportional to z^2. Thus, $\beta = \beta_r$ is always a stable point for the system to operate in it.

In Fig. 3(d), we plotted the intensity of a_2 for a specific length $z = 1.6$ mm as a function of C for three values of β. We have identified the value of C for which the exceptional point occurs with a vertical dash line for $\beta = 1.2$. Note that the exceptional point as a function of C is given by $\sqrt{\frac{\Delta k^2}{4(\beta^2 - 1)}}$ and thus only for $\beta > 1$ we can find a C that makes the system to operate at the exceptional point. It is clear that even when we are in the exact phase for $\beta \neq \beta_r$ the changes in C result in significant changes in a_2 while at $\beta = \beta_r$ it is linearly proportional to C^2.

In a dynamical system, intensity is not the only important element. Another important quantity is the phase. Specifically, in different applications such as designing a laser cavity based on an NHG, in the interferometry processes, and in Holographic imaging the phase of the output field is crucial in the device performance. According to Eq. (5) any mechanism that affects the phase of a_2 results in changing the phase of NHG in output. From Eq. (6) it is clear that the phase of output amplitude a_2 is independent of C when $\beta = \beta_r$. However, for other values of β as long as we are in the exact phase, the phase of the a_2 changes by $\pm\pi$ depending on the value of C. To demonstrate this, in figure 4 we plotted the changes in the phase of the output amplitude a_2 for three values of β for a specific length of a waveguide, namely $z = 2$ mm as a function of C.

All the above discussions are valid when we excite the system from the left side and the direction of propagation is toward the right. On the other hand, we know that usually, the spatiotemporal modulations result in optical isolation. For the right incident excitation and at $\beta = \beta_r = 1$ it is easy to show that the Hamiltonian associated with Eq.(3) reduces to a 2×2 zero matrix and thus $(a_1(z) \quad a_2(z)) = (a_1(0) \quad a_2(0))$ and thus not only is not
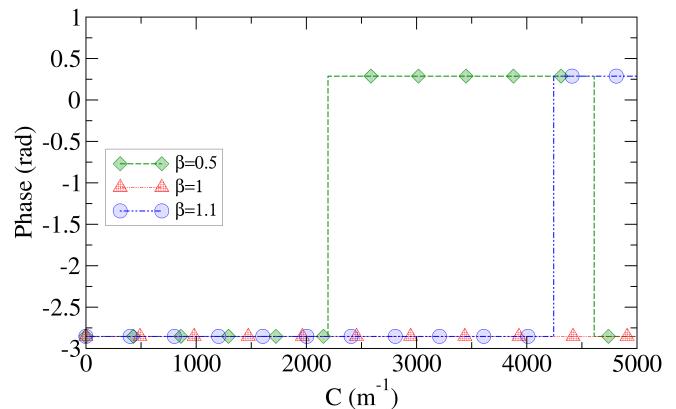
FIG. 4: Phase of a_2 is represented as a function of C for three different values of β at the end of a waveguide with specific length $z = 2$ mm. This phase changes the electric field phase in Eq.(2). clearly, for $\beta \neq 1$ output field will experience sudden changes in its phase respected to C. In contrast when $\beta = 1$ only the trivial phase due to the phase-mismatching condition will be added to the output field.

any harmonic generated by right excitation but also the system is reciprocal even we have asymmetric transport. This means that at $\beta = \beta_r$ we do not have any optical isolation.

Conclusion— We have proposed a new application for exceptional points in complex spatiotemporal modulated systems where they allow for a robust and efficient harmonic generation. While the non-reciprocity becomes suppressed at the exceptional point of the dynamical system, the robust harmonic generation becomes valuable specifically in applications such as interferometry, Harmonic generation-based lasers, and holography applications.

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