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LOGARITHMIC ALGORITHMS FOR FAIR DIVISION PROBLEMS

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Abstract. We study the algorithmic complexity of fair division problems with a focus on minimizing the number of queries needed to find an approximate solution with desired accuracy. We show for several classes of fair division problems that under certain natural conditions on sets of preferences, a logarithmic number of queries with respect to accuracy is sufficient.

Keywords: Computational fair division, envy-free fair division, cake-cutting problem, rental harmony problem, Sperner's lemma, KKM lemma

Mathematics Subject Classification 91B32 68Q17 52C45

1. Introduction

The problem of fair division is old and famous, it has many forms and arises in numerous real-world settings. See, e. g., [12] and references therein for an introduction to the subject. Among the many interesting aspects of this problem, the area of algorithmic issues stands out. The state of art here is well characterized by the following words of [12]: "*The main unresolved issue ... is the general question of finding bounded finite algorithms for envy-free division for any number of players. This is a well-known problem, and any algorithm for envy-free division, even for special small cases ... would be of interest.*"

In this paper we investigate fast algorithms for some particular cases of the rental harmony and cake-cutting problems. To recall the former, suppose a group of friends consider renting a house but they shall first agree on how to allocate its rooms and share the rent. They will rent the house only if they can find a room assignment-rent division which appeals to each of them. We call this problem the *rental harmony* problem, following Su [15]. Our investigation of the rental harmony problem starts with the simplest situation when each of the tenants forms for their own opinion of a fair price for each of the rooms, according to the tenant's own criteria, and is ready to rent a room at this or lower price, regardless (say, not knowing) how the rest of the rent is distributed among other tenants.

In the envy-free *cake-cutting problem*, a "cake" (a heterogeneous divisible resource) has to be divided among d partners with different preferences over parts of the cake [5, 13, 14, 15]. An alternative interpretation of this problem is as several subcontractors distribute parts of some job among themselves. In particular, to emphasize the duality with the rental harmony problem, suppose a group of workers discussing renovating a house but they shall first agree on who is in charge of which room and allocate the reward. The simplest version for this problem is when each of the workers forms their own opinion of a fair price for repairing each of the rooms, according to the worker's own criteria, and is ready to renovate a room for this or greater fee.

A convenient visual-geometric point of view on the fair division problems is provided by the configuration space approach. In case of d agents (friends/ tenants/ partners/ subcontractors/ workers) the configuration space of all possible rent/reward allocations is the standard $(d-1)$-dimensional simplex $A = \Delta^{d-1}$, that is, all representations of a positive number as a sum of d nonnegative ones. The preference sets of agents are then subsets of this simplex, the conditions of existence of a fair division can be formulated as conditions on these sets, and solution existence theorems are related to extensions of the KKM (Knaster–Kuratowski–Mazurkiewicz) lemma.[1] The mentioned above simplest versions are the cases where these subsets are intersections of the simplex with half-spaces whose boundaries are parallel to the faces of the simplex. Another case we consider is of convex preference sets.

In terms of configuration spaces, the fair division problems we study are formulated as follows. There is a simplex and collections of preference sets (satisfying conditions of solution existence). We do not know these sets, but we can ask which of these sets contain any chosen point of the simplex. We are interested in algorithms for finding (approximate) solutions aimed at minimizing the number of queries.

We consider two types of queries. One of these two means that we choose a preference set and a point and check whether this set contains this point. For example, for the rental harmony problem this means that a tenant says if a given room suits them under a given rent distribution. We call this type of query the *binary mode*. Another approach, which we refer to as the *minimal mode*, is when a tenant indicates one of the rooms that suits them (under a given rent distribution).

Another important issue is that we are not looking for a point close to a solution (there is no finite algorithm for finding such a point in general), but a point that is a solution when all preference sets increase by a prescribed ε (in a natural metric on the simplex). Such a point is called an *ε-fair division point.*

Taking the configuration space point of view, we immediately see that an ε-fair division point can be found in less than n^{d-1} queries, where n stands for $\lceil 1/\varepsilon \rceil$. Indeed, it is enough to check all the vertices of some triangulation mesh size less than ε. The main point of the present article is that, under certain natural conditions on preference sets, a logarithmic number of queries with respect to accuracy is sufficient (provided we know in advance that the preference sets satisfy the conditions).

In particular, in the case when all preference sets are intersections of the simplex with half-spaces of a certain kind (motivated by the cake division problem) Theorem 4.1 says that $(d-1)^2\lceil \log_2(n\cdot(d-1))\rceil$ queries suffice in binary mode, and in the case when all preference sets are intersections with half-spaces of another specific kind (motivated by the rent division problem) Theorem 4.3 states that $(d-1)\lceil \log_{\frac{d}{d-1}} n\rceil$ queries suffice in minimal mode.

If the preference sets are convex and $d = 3$, Theorem 5.2 says that $O(\log^2(n))$ queries in binary mode are enough for the rental harmony problem. A similar result concerning the cake-cutting problem is Theorem 6 in [4].

2. Stating the problem

Let d be a positive integer[2], let A be a $(d-1)$-dimensional regular simplex with edges of length 1 in $\mathbb{R}^{d-1}$, and let $v_1, \dots, v_d$ be the vertices of A. For $j \in \{1, \dots, d\}$, we denote the facet $\operatorname{Conv}\left(\{v_k\}_{k\neq j}\right)$ of A by F_j. For the rental harmony problem, A corresponds to all representations of total price as a sum of d nonnegative numbers; and F_j is precisely the set of price distributions with zero price for the jth room.

[1]The existence of a solution to envy-free division problems was discussed in [1, 3, 6, 7, 11, 15]. Gale [7] proved a colorful (rainbow) version of the KKM lemma that can be applied for the existence theorem proved by Su [15].

[2]In those situations where the case $d = 1$ is degenerate, we will assume by default that $d \geq 2$.

Assume that for each $i \in \{1, \dots, d\}$ a covering $\{A_{i1}, \dots, A_{id}\}$ of A consisting of d closed sets is fixed (these coverings are preferences of our agents):

$$\bigcup_{j \in \{1,\dots,d\}} A_{ij} = A,$$

but the elements of these coverings are **not known** to us.

We want to find a way to split the total price into d prices of rooms such that every person is "satisfied" with it. We will denote by ε the maximum "error" we allow. For $\varepsilon \geq 0$, we denote the ε-neighbourhoods of the preference sets as

$$A_{ij}^{\varepsilon} = \{x \in \mathbb{R}^{d-1} : \operatorname{dist}(x, A_{ij}) \leq \varepsilon\}$$

and the intersections of these ε-neighbourhoods as

$$I_i^{\varepsilon} = A_{i1}^{\varepsilon} \cap \ldots \cap A_{id}^{\varepsilon}.$$

We introduce the following definition.

Definition 2.1. We say that x in A is an *ε-fair division point* if there exists a permutation σ of $(1, \dots, d)$ such that $x \in A_{j\sigma(j)}^{\varepsilon}$ for all $j \in \{1, \dots, d\}$. We say that x a *fair division point* if x is an ε-fair division point for $\varepsilon = 0$.

General theory provides some reasonable sufficient conditions when a fair division point exists. For example, the rainbow KKM lemma tells it exists whenever

$$\operatorname{Conv}\left(\{v_j\}_{j \in J}\right) \subset \bigcup_{j \in J} A_{ij} \quad \text{for all } i \in \{1, \dots, d\}, J \subset \{1, \dots, d\},$$

and a rainbow generalization of Sperner's lemma implies that it exists whenever

$$\operatorname{Conv}\left(\{v_j\}_{j \in J}\right) \cap \bigcup_{j \notin J} A_{ij} = \emptyset \quad \text{for all } i \in \{1, \dots, d\}, J \subset \{1, \dots, d\}.$$

Our goal is to find an ε-fair division point using the smallest possible number of queries.

Now we give a formal definition for the two types of queries. One of these two is as follows: we choose an index i and a point $x \in A$ and receive an index j such that $x \in A_{ij}$ as an answer; we call this type of query the *minimal mode.* Another approach is: we choose indices i and j and a point $x \in A$ and receive "yes" if $x \in A_{ij}$ and "no" otherwise; this type of query is called the *binary mode.*

3. Preference sets with the inclusion property

In this section we prove two auxiliary constructive statements showing that if the collection of preference sets has a specific property (given in Definition 3.2), then one can find a solution point (i) in a set of points each of which is found separately for each of the agents according to their preferences, and (ii) without information about (the preferences of) one of the agents.[3] [4] Lemma 3.1 is a purely combinatorial statement, and Theorem 3.3 applies this combinatorics to our problem.

[3] Of course, to move from a solution point to the corresponding solution itself (for example, to a specific resource allocation between the agents), information about the excluded agent's preferences may also be needed.

[4] This is directly related to the results of [1], see also [6], where it is proved that there exists an envy-free rent division that remains so for any change in the preferences of one of the tenants.

Lemma 3.1. *Let $\{\leq_1, \ldots, \leq_d\}$ be d linear orderings of $D = \{1, \ldots, d-1\}$. Then D contains an element i_0 such that for each $j_0 \in \{1, \ldots, d\}$ there exists a bijective map $\pi\colon D \to \{1, \ldots d\} \setminus \{j_0\}$ with the property that for all $i \in D$ we have*

$$i_0 \leq_{\pi(i)} i.$$

Proof. We will prove the lemma by induction on d. If $d = 1$, there is nothing to prove, so we may assume D is not empty. Since there are d orderings and $d-1$ elements, it follows that there exists an element $k \in D$ that is maximal with respect to two distinct orderings l_1 and l_2. Now we apply induction hypothesis to the set $D \setminus \{k\}$ and all of the orderings except l_1, and get some $i_0 \in D \setminus \{k\} \subset D$. We claim that i_0 has the desired property for the initial problem as well.

Suppose we choose (in $\{1, \ldots, d\}$) some $j_0 \neq l_1$. By the choice of i_0 there exists a map

$$\pi'\colon D \setminus \{k\} \to \{1, \ldots, d\} \setminus \{l_1, j_0\}$$

with the desired property. Since k is maximal with respect to l_1, we can define π as

$$\pi(i) = \begin{cases} \pi'(i) & \text{if } i \neq k, \\ l_1 & \text{if } i = k. \end{cases}$$

In the remaining case with $j_0 = l_1$, by the choice of i_0 there exists a map

$$\pi'\colon D \setminus \{k\} \to \{1, \ldots, d\} \setminus \{l_1, l_2\}$$

with the desired property. Since k is maximal with respect to l_2, it follows that we can define π as

$$\pi(i) = \begin{cases} \pi'(i) & \text{if } i \neq k, \\ l_2 & \text{if } i = k. \end{cases}$$

$\square$

Definition 3.2. We say that our collection of preference sets A_{ij} satisfies the *inclusion property* if for each triplet (i_1, i_2, j) with $i_1, i_2 \in \{1, \ldots, d-1\}$ and $j \in \{1, \ldots, d\}$ we have either $A_{i_1 j} \subset A_{i_2 j}$ or $A_{i_2 j} \subset A_{i_1 j}$.

Theorem 3.3. *Suppose the collection of sets A_{ij} satisfies the inclusion property. Then each sequence of points $x_1, \ldots, x_{d-1}$ with $x_i \in I_i = A_{i1} \cap \ldots \cap A_{id}$ for each $i \in \{1, \ldots, d-1\}$ contains a fair division point. Moreover, there exists an algorithm that, given such a sequence and the related data about inclusions, finds a fair division point in the sequence.*

Proof. Inclusion orderings on A_{ij} determine orderings on $\{1, \ldots, d-1\}$ by the rule

$$i_1 \leq_j i_2 \;\Leftrightarrow\; A_{i_1 j} \subset A_{i_2 j}.$$

We apply Lemma 3.1 to these orderings and obtain some index $i_0 \in \{1, \ldots, d-1\}$ having the property described in Lemma 3.1. We claim that x_{i_0} is a fair division point.

Indeed, x_{i_0} is in A_{dj_0} for some j_0 since $\{A_{d1}, \ldots, A_{dd}\}$ is a covering of A. Then by the choice of i_0 there exists a map

$$\pi\colon \{1, \ldots, d-1\} \to \{1, \ldots, d\} \setminus \{j_0\}$$

such that $i_0 \leq_{\pi(i)} i$. Now we define the desired permutation σ as

$$\sigma(i) = \begin{cases} \pi(i) & \text{if } i \in \{1, \ldots, d-1\}, \\ j_0 & \text{if } i = d. \end{cases}$$

The condition $i_0 \leq_{\pi(i)} i$ means that $A_{i_0 \pi(i)} \subset A_{i \pi(i)}$. Then for each $i \in \{1, \ldots, d-1\}$ we have

$$x_{i_0} \in A_{i_0 \pi(i)} \subset A_{i \pi(i)} = A_{i \sigma(i)},$$

and $x_{i_0} \in A_{dj_0} = A_{d\sigma(d)}$ straight from the definition of j_0. $\square$

4. Logarithmic algorithms for the "linear" preference sets

We use the barycentric coordinate system on our regular simplex A in $\mathbb{R}^{d-1}$. Namely, recall that $v_1, \ldots, v_d$ are the vertices of A so that each $x \in A$ can be uniquely expressed in the form

$$x = \alpha_1 v_1 + \ldots + \alpha_d v_d$$

with all $\alpha_i \geq 0$ and $\alpha_1 + \ldots + \alpha_d = 1$. We will use the notation $[\alpha_1, \ldots, \alpha_d]$ for x.

Let $\{A_{ij}\},\ i = 1, \ldots, d,\ j = 1, \ldots, d$, as above, be a collection of preference sets. In this section we consider two kinds of *linear preference sets* (*LPS*):

(i) each of A_{ij} has the form $\{[\alpha_1, \ldots, \alpha_d] \in A : \alpha_j \geq a_{ij}\}$;
(ii) each of A_{ij} has the form $\{[\alpha_1, \ldots, \alpha_d] \in A : \alpha_j \leq a_{ij}\}$,

where $\{a_{ij}\}$ is a set of real numbers.

Case (i) is a particular case of the cake-cutting problem. The condition that $\{A_{i1}, \ldots, A_{id}\}$ is a covering of A implies that $\sum_{j=1}^{d} a_{ij} \leq 1$ for all i. In addition, the standard cake-cutting preferences condition that "players prefer any piece with mass to an empty piece" means here that all a_{ij} are in $(0, 1)$. In (ii) we have a particular case of the rental harmony problem with $\sum_{j=1}^{d} a_{ij} \geq 1$ for all i. Note that in both cases A_{ij} is the intersection of A and a half-space that is bounded by the hyperplane $\alpha_j = a_{ij}$.

The following theorem is our "logarithmic complexity" result for the cake-cutting problem.

Theorem 4.1. *Suppose that each of the sets A_{ij}, being a proper subset of the simplex A, is the intersection of A and a closed half-space with boundary parallel to F_j such that $v_j \in A_{ij}$. Then we can find an ε-fair division point in binary mode using at most*

$$(d-1)^2 \lceil \log_2(n \cdot (d-1)) \rceil$$

queries, where $n = \lceil 1/\varepsilon \rceil$.

Proof. By assumptions we have LPS case (i). Fix an index $i \in \{1, \ldots, d-1\}$ and denote $\delta = \frac{\varepsilon}{d-1}$. Since A_{ij} is a proper subset of A, it follows that $0 < a_{ij} < 1$. For every $j \in \{1, \ldots, d-1\}$ we can use binary search to find approximations c_{ij} such that

$$\max(a_{ij} - \delta, 0) \leq c_{ij} < a_{ij}$$

with $\lceil \log_2 \frac{1}{\delta} \rceil$ queries. We claim that I_i^ε contains the point

$$x_i = [c_{i1}, c_{i2}, \ldots, c_{i(d-1)}, 1 - c_{i1} - \ldots - c_{i(d-1)}].$$

Indeed, x_i lies in A since

$$0 \leq \sum_{j=1}^{d-1} c_{ij} < \sum_{j=1}^{d-1} a_{ij} \leq 1.$$

Then for each $j \in \{1, \ldots, d-1\}$ we have $x_i \notin A_{ij}$ straight from the definition, which means that $x_i \in A_{id}$. Furthermore, we have

$$y_i = [a_{i1}, a_{i2}, \ldots, a_{i(d-1)}, 1 - a_{i1} - \ldots - a_{i(d-1)}] \in A_{i1} \cap \ldots \cap A_{i(d-1)},$$

and

$$|x_i - y_i| = \left| \sum_{j=1}^{d-1} (c_{ij} - a_{ij})(v_j - v_d) \right| \leq \sum_{j=1}^{d-1} \left| c_{ij} - a_{ij} \right| \leq (d-1)\delta = \varepsilon.$$

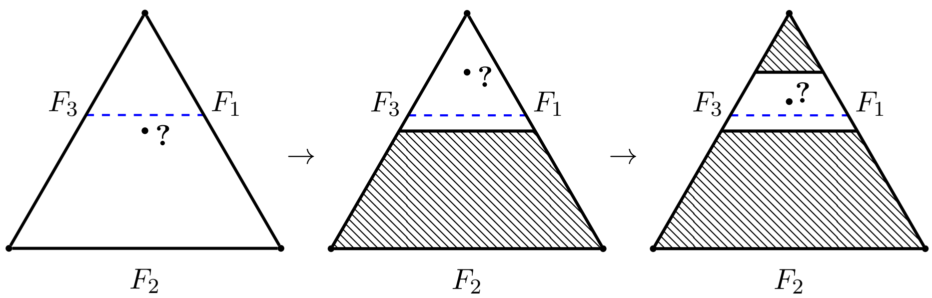
We spent $(d-1)\lceil \log_2(n \cdot (d-1)) \rceil$ queries on the fixed index i. Repeating this for all $i \in \{1, \ldots, d-1\}$ we obtain $d-1$ points $x_1, \ldots, x_{d-1}$ lying in the corresponding intersections.

For $i \in \{1, \dots, d-1\}$ and $j \in \{1, \dots, d\}$ we set

$$A'_{ij} = \{[\alpha_1, \dots, \alpha_d] \in A : \alpha_j \geq c_{ij}\}.$$

These sets satisfy the inclusion property and are known to us, so by Theorem 3.3 we can find a fair division point z for them. But $x_i \in I_i^\varepsilon$ implies that $A'_{ij} \subset A^\varepsilon_{ij}$, so z is also an ε-fair division point for the initial problem. □

Example 4.2. The following figure illustrates the first two steps of binary search for finding (an approximation for) a_{12}. The blue dashed line shows the boundary of the actual set A_{12}. The point with the question mark is the point about which we ask whether it lies in A_{12}, and then cut off a piece of A according to the answer.



Theorem 4.3. *Suppose that each of the sets A_{ij} is the intersection of the simplex A and a closed half-space with boundary parallel to F_j such that $F_j \subset A_{ij}$. Then we can find an ε-fair division point in minimal mode using at most $(d-1)\lceil \log_{\frac{d}{d-1}} n \rceil$ queries, where $n = \lceil 1/\varepsilon \rceil$.*

Proof. We have LPS case (ii). We fix an index $i_0 \in \{1, \dots, d-1\}$ and ask which of $A_{i_0 1}, \dots, A_{i_0 d}$ contains the center c_0 of A. Some $A_{i_0 j_0}$ contains c_0, so we draw the hyperplane H_{j_0} parallel to F_{j_0} through c_0, and $H_{j_0} \cap A$ also lies in $A_{i_0 j_0}$. Clearly, this reduces the problem to the smaller regular simplex that H_{j_0} cuts off from A. This new simplex is similar to the initial one with coefficient $\frac{d-1}{d}$. So, after $\lceil \log_{\frac{d}{d-1}} n \rceil$ queries the resulting regular simplex $A^\dagger$ has edges of length at most ε. Then $I^\varepsilon_{i_0}$ contains $A^\dagger$, and we take any point of $A^\dagger$ as x_{i_0}.

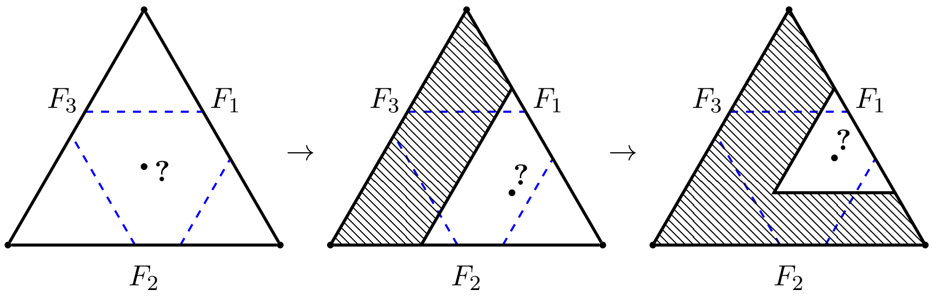
Repeating the described procedure for all indices $i \in \{1, \dots, d-1\}$ and each time using exactly the specified number of queries, we obtain $d-1$ points lying in the corresponding intersections I_i^ε.

Analogously to the proof of 4.1, for $i \in \{1, \dots, d-1\}$ and $j \in \{1, \dots, d\}$ we set

$$A'_{ij} = \{[\alpha_1, \dots, \alpha_d] \in A : \alpha_j \leq x_{ij}\}, \quad \text{where } x_{ij} \text{ is defined by } x_i = [x_{i1}, \dots, x_{id}].$$

These sets satisfy the inclusion property and are known to us, so by Theorem 3.3 we can find a fair division point z for them. But $x_i \in I_i^\varepsilon$ implies that $A'_{ij} \subset A^\varepsilon_{ij}$, so z is also an ε-fair division point for the initial problem. □

Example 4.4. The following figure illustrates the first two steps of binary search for finding $x_1 \in I_1^\varepsilon$. The blue dashed lines show the boundaries of actual sets A_{11}, A_{12}, and A_{13}. The point with the question mark is the point about which we ask which of A_{11}, A_{12}, and A_{13} contains it. At the first step we find out that the point lies in A_{13}, and then cut off a piece of A according to the answer. Similarly, at the second step we get A_{12}.



5. Case of convex preference sets

In this section we deal with the case where the preference sets A_{ij} in the rental harmony problem are convex. If $d = 2$ then convex sets containing a fixed vertex of the simplex are intersections of the simplex with half-spaces containing this vertex, which provides a trivial solution with $\lceil \log_2 n \rceil$ queries (this may be viewed as a trivial case of Theorem 4.3). Here, we focus on the situation with $d = 3$.

First, we prove an analogue of Theorem 3.3 for the case where our sets are convex (while do not necessarily have the inclusion property) and we know all d points in intersections (instead of $d - 1$).

Lemma 5.1. *Suppose all preference sets A_{ij} are convex and $F_j \subset A_{ij}$ for all i and j. Then there exists an algorithm that, given $\varepsilon > 0$ and a collection $x_1, \dots, x_d$ with $x_i \in I_i^\varepsilon$ for each i, finds an ε-fair division point $x = x(x_1, \dots, x_d)$.*

Proof. First of all, observe that A_{ij}^ε are clearly also convex. Put $A'_{ij} = \operatorname{Conv}\big(F_j, x_i\big)$. Observe that each A'_{ij} depends only on x_i and j, but not on A_{ij}. It can be easily seen that any fixed-point free reindexing for vertex indices in $\{v_1, \dots, v_d\}$ turns $\{A'_{i1}, \dots, A'_{id}\}$ into a KKM-covering for each i (see Sec. 2).

Then the rainbow KKM lemma implies that there exists a point z and a permutation σ of $(1, \dots, d)$ such that $z \in A'_{i\sigma(i)}$ for all i. Moreover, it is clear that there exists an algorithm finding z: for example, by checking all permutations of $(1, \dots, d)$ we can find a permutation σ such that

$$\bigcap_{1 \le i \le d} A'_{i\sigma(i)} \neq \emptyset$$

and take any point in this intersection. But $F_j \subset A_{ij}^\varepsilon$ and $x_i \in I_i^\varepsilon \subset A_{ij}^\varepsilon$ imply that $A'_{ij} \subset A_{ij}^\varepsilon$. Thus z is also an ε-fair division point for the initial problem. $\square$

Theorem 5.2. *Suppose that $d = 3$ and each of the sets A_{ij} is a convex set containing F_j. Then we can find an ε-fair division point in binary mode with at most $6(\lceil \log_2 n \rceil^2 + \lceil \log_2 n \rceil) = O(\log^2 n)$ queries, where $n = \lceil 1/\varepsilon \rceil$.*

Proof. By Lemma 5.1 it is sufficient to find a point in each of I_i^ε, so we want to do this for an arbitrary fixed i with at most $2(\lceil \log_2 n \rceil^2 + \lceil \log_2 n \rceil)$ queries.

Based on the barycentric coordinates notation introduced in Sec. 4, for the triples of numbers (a, b, c) with nonzero sum $a + b + c$ we use the notation

$$[a, b, c]_* = [a/(a + b + c), b/(a + b + c), c/(a + b + c)]$$

and introduce the grid

$$\{[a, b, c]_* : a, b, c \in \mathbb{N}_0, a + b + c = n\}.$$

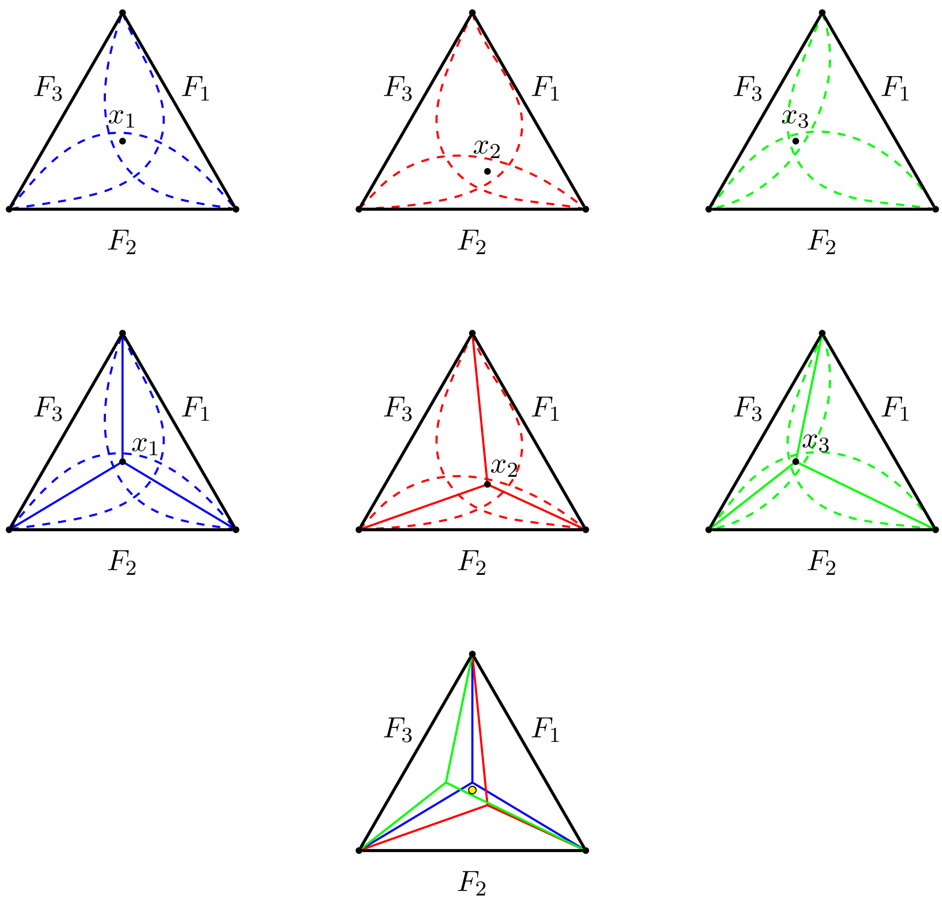


FIGURE 1. An example of new covering for Lemma 5.1

Our queries will only be about grid points. We define

$$k_2(a) = \max\{b \in \mathbb{N}_0 : 0 \le b \le n-a, [a, b, n-a-b]_* \in A_{i2}\},$$
$$k_3(a) = \min\{b \in \mathbb{N}_0 : 0 \le b \le n-a, [a, b, n-a-b]_* \in A_{i3}\}.$$

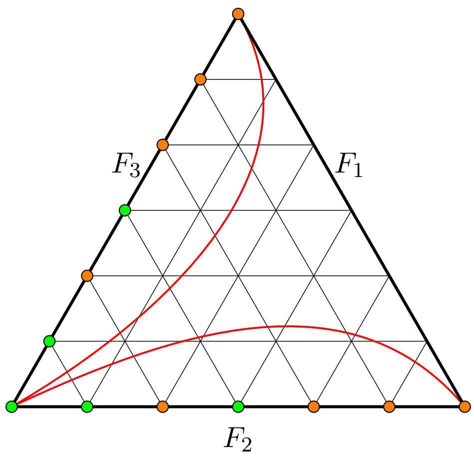


FIGURE 2. Green points correspond to covered values and orange points correspond to uncovered ones

Observe that $[a, 0, n-a]_* \in A_{i2}$ and $[a, n-a, 0]_* \in A_{i3}$. Then convexity of A_{i2} and A_{i3} implies that the values of $k_2(a)$ and $k_3(a)$ for any fixed a can be found with binary search using $2\lceil \log_2 n \rceil$ queries.

We say that a coordinate a is *covered* if $k_2(a)+1 \geq k_3(a)$ and *uncovered* otherwise. In other words, a is covered if the union $A_{i2} \cup A_{i3}$ contains all points of the grid with the first coordinate a. Clearly, $a = n$ is covered. Then we spend $2\lceil \log_2 n \rceil$ queries to calculate $k_2(0)$ and $k_3(0)$ and check whether $a = 0$ is covered. If it is then $[0, k_2(a), n-k_2(a)]_*$ is the desired point. Otherwise we can run a binary search through a to obtain a coordinate a_0 such that it is uncovered while a_0+1 is covered (note that this is possible despite the set of covered values has "gaps" in general): each iteration of binary search requires at most $2\lceil \log_2 n \rceil$ queries, so the total search takes at most $2\lceil \log_2 n \rceil^2$ ones.

After finding a_0, we turn to the following sequence Z of $k = k_3(a_0) - k_2(a_0) + 2$ points (see Fig. 3):

$$
\begin{array}{rlll}
z_1 := & [a_0, & k_2(a_0), & n - a_0 - k_2(a_0)]_*, \\
z_2 := & [a_0+1, & k_2(a_0), & n - a_0 - k_2(a_0) - 1]_*, \\
z_3 := & [a_0+1, & k_2(a_0)+1, & n - a_0 - k_2(a_0) - 2]_*, \\
 & \ldots & & \\
z_{k-2} := & [a_0+1, & k_3(a_0)-2, & n - a_0 - k_3(a_0) + 1]_*, \\
z_{k-1} := & [a_0+1, & k_3(a_0)-1, & n - a_0 - k_3(a_0)]_*, \\
z_k := & [a_0, & k_3(a_0), & n - a_0 - k_3(a_0)]_*.
\end{array}
$$

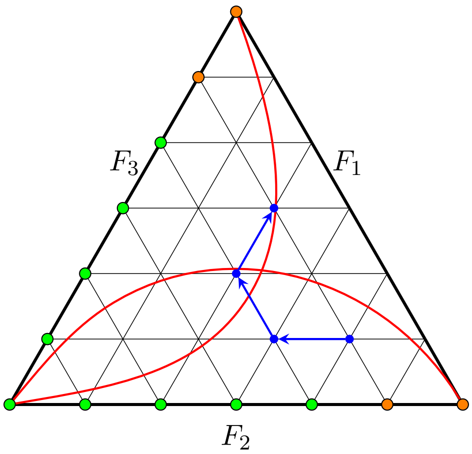


FIGURE 3. The sequence Z is highlighted in blue

Observe that by construction we have $z_1 \in A_{i2}$ and $z_k \in A_{i3}$ while the union $A_{i2} \cup A_{i3}$ contains Z. This implies that for some m we have $z_m \in A_{i2}$ and $z_{m+1} \in A_{i3}$. Besides, observe that each pair of consecutive points in Z is a pair of vertices of a triangle of our grid with its third vertex lying at the level a_0 between z_1 and z_k. Since all of the points of the segment between z_1 and z_k are in A_{i1}, it follows that z_m and z_{m+1} are in I_i^ε. It remains to notice that we can find z_m since we know $k_2(a_0+1)$ and $k_3(a_0+1)$. □

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