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# Pseudo-peakons and Cauchy analysis for an integrable fifth-order equation of Camassa-Holm type

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**Abstract**

In this paper, we introduce a hierarchy of integrable higher order equations *of Camassa-Holm (CH) type*, that is, we present infinitely many nonlinear equations depending on inertia operators which generalize the standard momentum operator $A_2 = \partial_{xx} - 1$ appearing in the Camassa-Holm equation $m_t = -m_x u - 2 m u_x$, $m = A_2(u)$. Our higher order CH-type equations are integrable in the sense that they possess an infinite number of local conservation laws, quadratic pseudo-potentials, and zero curvature formulations. We focus mainly on the fifth order CH-type equation and we show that it admits *pseudo-peakons*, this is, bounded solutions with differentiable first derivative and continuous and bounded second derivative, but whose higher order derivatives blow up. Furthermore, we investigate the Cauchy problem of our fifth order equation on the real line and prove local well-posedness for initial conditions $u_0 \in H^s(\mathbb{R})$, $s > 7/2$. In addition, we discuss conditions for global well-posedness in $H^4(\mathbb{R})$ as well as conditions causing local solutions to blow up in a finite time. We finish our paper with some comments on the geometric content of our equations of CH-type.

**Keywords**: Camassa-Holm equation; integrability; local conservation laws; pseudo-peakons; fifth order CH-type equation; well-posedness.

**AMS Classification**: 35G25, 35L05, 35Q53.

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# 1 Introduction

In this paper we explain how to construct a family of integrable equations which generalize the celebrated Camassa-Holm (CH) equation (see [5])

$$m_t = -m_x\, u - 2\, m\, u_x \;, \qquad m = u_{xx} - u \;.$$

Our main motivation for undertaking this project is the observation that the Camassa-Holm equation can be understood as an Euler equation for the inertia operator $A_2 = \partial_{xx} - 1$ on the Lie algebra $Vect(S^1)$ of the (Fréchet) Lie group $Diff(S^1)$, and that, in turn, the latter equation can be interpreted geometrically as determining geodesics for a Riemannian metric on $Diff(S^1)$ defined via $A_2$, see for instance [1, 20], the treatise [18] and also [15]. A similar observation is valid for a group of diffeomorphisms of the line that is slightly more difficult to describe, see [10]. A natural question which arises is whether taking different inertia operators $A_n$ would yield new "Euler equations" which may be of interest for the theory of fluids and which may determine interesting geometry on diffeomorphism groups. In the paper [12] Constantin and Kolev answer this question as follows in the case of $Vect(S^1)$ and the (Fréchet) Lie group $Diff(S^1)$: the only *bi-Hamiltonian* equations (for some natural Poisson structures) arising from inertia operators $A_n = \sum (-1)^k \partial_x^{2k}$ are the inviscid Burgers equation ($k = 0$) and the Camassa-Holm equation ($k = 1$). However, since inertia operators *are* quite interesting, as they equip infinite-dimensional Lie groups with non-trivial (pseudo-)Riemannian geometry (see for instance [20, 10], the classical paper [1] by Arnold, and also [15]), we cannot help but wondering if we could go around the Constantin-Kolev result.

Further motivation for our work comes from the fact that the standard Camassa-Holm hierarchy, see for instance [14, Equation (2.49)] and [21, Theorem 2], is a hierarchy of *nonlocal* equations. It is natural to wonder if there exists a *local* family of equations to which the CH equation belongs. Rasin and Schiff have explicitly computed a sequence of generalized symmetries of the CH, see [22, Section 7], and this work can be considered as providing an answer to the above question, but then we could insist that the members of our hypothetical family ought to be defined via inertia operators, precisely as the CH equation.

We observe in this paper that we can generalize the zero curvature formulation of the Camassa-Holm equation (see [27, 24, 14, 21]) by using some appropriate higher order operators $A_{2n}$ that can be interpreted as higher order inertia operators, thereby obtaining partial differential equations which indeed generalize CH in the sense we seek. In turn, these zero curvature representations allow us to check integrability of the corresponding "higher order CH-type equations", not in the sense of being bi-Hamiltonian, but in the sense of possessing an infinite number of non-trivial local conservation laws. We note that by proceeding in this way we lose the interpretation of our higher order Ch-type equations as standard Euler equations but, as we show in the last section of this work, they do possess interesting geometric content.

Our paper is organized as follows. In Section 2 we introduce the following fifth order

CH-type equation

$$\begin{cases} m_t = -m_x v - 2m v_x, \\ v = u - u_{xx} = (1-\partial^2)u = A_2(u), \\ m = v - v_{xx} = (1-\partial^2)^2 u = -A_4(u), \end{cases} \tag{1}$$

which we derive using the operators $A_2(u) = u - u_{xx}$, and $A_4(u) = -u_{xxxx} + 2u_{xx} - u$. The expansion of (1) reads

$$\begin{aligned} u_t - 2u_{xxt} + u_{xxxxt} &= -3uu_x + 4uu_{xxx} - uu_{xxxxx} + 5u_x u_{xx} - 2u_x u_{xxxx} \\ &\quad -6u_{xx}u_{xxx} + 2u_{xxx}u_{xxxx} + u_{xx}u_{xxxxx}, \end{aligned} \tag{2}$$

We show that (1) admits a Lax pair and an infinite number of non-trivial local conservation laws as well as its Lie algebra of symmetries. Then in Section 3 Equation (1) is shown to possess *pseudo-peakons*, this is, bounded solutions with smooth first derivative and continuous second derivative, but whose higher order derivatives blow up. Pseudo-peakons have been observed before in a "fifth order Camassa-Holm equation", see [19, 30], but we believe that this is the first time a fifth order *integrable* equation possessing this kind of solutions is reported. In Section 4 we discuss local well-posedness of (1) for initial conditions $u_0 \in H^s(\mathbb{R})$, $s > 7/2$, which is to be contrasted with the corresponding result for standard CH equation (for instance, in [26, Theorem 3.1] local well-posedness of the Camassa-Holm equation is proven for initial data in $H^s(\mathbb{R})$, $s > 3/2$). In this section we also prove a theorem on global well-posedness of (1) in $H^4(\mathbb{R})$, and we present conditions causing local solutions to blow up in a finite time. Finally, in Section 5 we make some remarks on the geometric meaning of Equation (1) —in particular, we observe that the operators $A_{2n}$ can indeed be considered as inertia operators— and we introduce our whole family of higher order Camassa-Holm type equations.

# 2 Equations of Camassa–Holm type

Let us begin by recalling the following observation about the important Camassa–Holm equation introduced in [5].

**Theorem 1.** *The compatibility condition of the linear problem*

$$d\psi = (X dx + T dt)\psi \ ,$$

*where* $\psi = (\psi_1, \psi_2)^t$, *and*

$$X = \frac{1}{2}\begin{bmatrix} 0 & \lambda + 2\,m \\ \lambda^{-1} & 0 \end{bmatrix}, \qquad T = \frac{1}{2}\begin{bmatrix} -u_x & -2u\,m + \lambda u - \lambda^2 \\ -1 - u\lambda^{-1} & u_x \end{bmatrix}, \tag{3}$$

*is the Camassa-Holm* $(CH)$ *equation*

$$m_t = -m_x\,u - 2\,m\,u_x \ , \qquad m = u_{xx} - u \ . \tag{4}$$

This theorem appears in [27] and [24]. It is well-known how to obtain quadratic pseudo-potentials and conservation laws from associated $sl(2,\mathbb{R})$-valued linear problems, see [9], Refs. [23, 24] for full details, and Proposition 1 below. We obtain:

**Theorem 2.** *The CH equation* (4) *admits a quadratic pseudo-potential* $\gamma$ *determined by the compatible equations*

$$m = \gamma_x + \frac{1}{2\lambda}\,\gamma^2 - \frac{1}{2}\lambda\,, \qquad \gamma_t = \frac{\gamma^2}{2}\left[1 + \frac{1}{\lambda}u\right] - u_x\gamma - u\,m + \left[\frac{1}{2}u\lambda - \frac{1}{2}\lambda^2\right]\,, \tag{5}$$

*where* $\lambda \neq 0$ *is a parameter. Moreover, Equation* (4) *possesses the parameter-dependent conservation law*

$$\gamma_t = \lambda\left(u_x - \gamma - \frac{1}{\lambda}u\gamma\right)_x\,. \tag{6}$$

We can interpret Equations (5) and (6) as determining a "Miura-like" transformation and a "modified Camassa-Holm" (MOCH) model. This observation is further developed in [16].

We use (5) and (6) to construct conservation laws for the CH equation. Setting $\gamma = \sum_{n=1}^{\infty} \gamma_n \lambda^{n/2}$ and substituting it into (5) we find the conserved densities

$$\gamma_1 = \sqrt{2}\,\sqrt{m}\,, \qquad \gamma_2 = -\frac{1}{2}\,\ln(m)_x\,, \qquad \gamma_3 = \frac{1}{2\sqrt{2}\,\sqrt{m}}\left[1 - \frac{m_x^2}{4\,m^2} + \ln(m)_{xx}\right], \tag{7}$$

$$\gamma_{n+1} = -\frac{1}{\gamma_1}\,\gamma_{n,x} - \frac{1}{2\gamma_1}\sum_{j=2}^{n}\gamma_j\,\gamma_{n+2-j}\,, \qquad n \geq 3\,, \tag{8}$$

while the expansion $\gamma = \lambda + \sum_{n=0}^{\infty}\gamma_n\lambda^{-n}$ implies

$$\gamma_{0,x} + \gamma_0 = m\,, \qquad \gamma_{n,x} + \gamma_n = -(1/2)\sum_{j=0}^{n-1}\gamma_j\,\gamma_{n-1-j}\,, \quad n \geq 1\,. \tag{9}$$

It is shown in [24] that the local conserved densities $\gamma_n$ determined by (7) and (8) correspond to the ones found by Fisher and Schiff in [13] by using an "associated Camassa–Holm equation", while (9) generates the local conserved densities $u$, $u_x^2 + u^2$, and $uu_x^2 + u^3$, and a sequence of nonlocal conservation laws.

Let us now introduce our higher order equations of Camassa-Holm type. Our basic idea is to use $m = A_n(u)$ for higher order operators $A_n$ in the matrices (3), instead of using simply $m = (1 - \partial_{xx})u$: we keep the $\lambda$-pole structure of the matrices $X$ and $T$ appearing in (3) — so that, in particular, we expect our equations to be amenable of analysis via scattering/inverse scattering, see [2] — but we write $m = A_n(u)$ in $X$ and we modify $T$ so that the zero curvature equation $X_t - T_x + [X,T] = 0$ is equivalent to a scalar partial differential equation. Here we work out the case in which $A_n$ is a fourth order operator, and we present a general construction in the final section of this paper.

We choose $A_4(u) = -u_{xxxx} + 2\,u_{xx} - u$ and we select

$$X_4 = \begin{bmatrix} 0 & \frac{1}{2}\,\lambda - u_{xxxx} + 2\,u_{xx} - u \\ \frac{1}{2}\,\lambda^{-1} & 0 \end{bmatrix} \tag{10}$$

and

$$T_4 = \frac{1}{2}\begin{bmatrix} -u_x + u_{xxx} & (-2\,u + 2\,u_{xx})\,(-u_{xxxx} + 2\,u_{xx} - u) + \lambda\,(u - u_{xx}) - \lambda^2 \\ -1 - \dfrac{u}{\lambda} + \dfrac{u_{xx}}{\lambda} & u_x - u_{xxx} \end{bmatrix}. \tag{11}$$

A straightforward computation shows that the equation

$$X_{4,t} - T_{4,x} + [X_4, T_4] = 0$$

is equivalent to equation (2).

Equation (2) is integrable in the sense of admitting the parameter-depending Lax pair $\psi_x = X_4\psi, \psi_t = T_4\psi$ and (as we will see momentarily) of possessing an infinite number of non-trivial local conservation laws; we call either (2) or (1) the integrable fifth-order CH-type equation.

We now present conservation laws and symmetries of Equation (2) in an explicit form.

## Conservation laws

After the classical work [28] (and the geometric reinterpretation of [28] appearing in [9, 23]), we compute conservation laws using quadratic pseudo-potentials via the following general proposition:

**Proposition 1.** *Let us assume that a given equation $\Xi(x,t,u,\dots) = 0$ is the integrability condition of an $sl(2,\mathbb{R})$-valued linear problem $\Psi_x = X\Psi$ and $\Psi_t = T\Psi$, in which the matrices $X = (X_{ij})$ and $T = (T_{ij})$ depend on $x, t, u$, finite numbers of derivatives of $u$, and (possibly) a parameter $\lambda$. Then, the following pair of Riccati equations determines a quadratic pseudo-potential for $\Xi = 0$ :*

$$\begin{aligned} -2\,\Gamma_x &= (-X_{12} + X_{21} + 2\,X_{11}) - 2\,\Gamma(X_{12} + X_{21}) + \Gamma^2(-X_{12} + X_{21} - 2\,X_{11}) \\ -2\,\Gamma_t &= (-T_{12} + T_{21} + 2\,T_{11}) - 2\,\Gamma(T_{12} + T_{21}) + \Gamma^2(-T_{12} + T_{21} - 2\,T_{11})\;. \end{aligned}$$

*Moreover, the equation $\Xi = 0$ admits a conservation law with conserved density*

$$X_{12} + X_{21} - \Gamma(-X_{12} + X_{21} - 2\,X_{11})$$

*and flux*

$$T_{12} + T_{21} - \Gamma(-T_{12} + T_{21} - 2\,T_{11})\;.$$

In the case of the linear problem determined by (10) and (11), it is convenient to apply a gauge transformation to the connection $X_4 dx + T_4 dt$ with gauge matrix

$$R = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} . \tag{12}$$

We perform this transformation and then we use Proposition 1. We obtain that Equation (2) admits the quadratic pseudo-potential

$$\frac{\partial \Gamma}{\partial x} = \frac{1}{2}\lambda - u_{xxxx} + 2\,u_{xx} - u - \frac{\Gamma^2}{2\,\lambda} \tag{13}$$

$$\begin{aligned} \frac{\partial \Gamma}{\partial t} &= \frac{1}{2}\,(-2\,u + 2\,u_{xx})\,(-u_{xxxx} + 2\,u_{xx} - u) + \frac{1}{2}\lambda\,(u - u_{xx}) - \frac{1}{2}\lambda^2 \\ &\quad + \Gamma\,(-u_x + u_{xxx}) - \frac{1}{2}\Gamma^2\left(-1 - \frac{u}{\lambda} + \frac{u_{xx}}{\lambda}\right) \end{aligned} \tag{14}$$

and the parameter dependent conservation law

$$-\left(\frac{\Gamma}{\lambda}\right)_t = \left(-u_x + u_{xxx} - \Gamma\left(-1 - \frac{u}{\lambda} + \frac{u_{xx}}{\lambda}\right)\right)_x . \tag{15}$$

Expansion in powers of $\lambda$ as in (7)–(9) yields a sequence of non-trivial local conservation laws. In full detail, we proceed as follows. We adopt the concise form (1) of the CH-type equation (2), this is,

$$m_t + 2v_x m + v m_x = 0 ,$$

where $v = u - u_{xx}$ and $m = v - v_{xx} = u - 2u_{xx} + u_{xxxx}$. Then, setting $\Gamma = \sum_{n=1}^{\infty} \gamma_n \lambda^{n/2}$ and replacing into (13) we obtain

$$\gamma_1 = \sqrt{2}\sqrt{-m} \tag{16}$$

$$\gamma_2 = \frac{-\gamma_{1,x}}{\gamma_1} = \frac{1}{2}\ln(|-m|)_x \tag{17}$$

$$\gamma_3 = \frac{1}{2\sqrt{2}\sqrt{-m}}\left(1 - \frac{1}{4}\frac{m_x^2}{m^2} - \ln(|-m|)_{xx}\right) \tag{18}$$

$$\gamma_n = \frac{-1}{\gamma_1}\gamma_{n,x} - \frac{1}{\gamma_1}\sum \gamma_k \gamma_{n+2-k} , \qquad n \geq 3 . \tag{19}$$

On the other hand, setting now $\Gamma = \lambda + \sum_{n=0}^{\infty} \tilde{\gamma}_n \lambda^{-n}$ and substituting into (13) we find

$$\tilde{\gamma}_{0,x} + \tilde{\gamma}_0 = -m \tag{20}$$

$$\tilde{\gamma}_{n,x} + \tilde{\gamma}_n = -\frac{1}{2}\sum_{k=0}^{n-1} \tilde{\gamma}_k \tilde{\gamma}_{n-1-k} . \tag{21}$$

Equation (20) yields the conserved density $\tilde{\gamma}_0 = -u_{xxx} + u_{xx} + u_x - u$. In order to find further densities we note that Equation (15) means not only that the functions $\tilde{\gamma}_n$ are conserved densities, but also that so are the functions $\tilde{\gamma}_n + \tilde{\gamma}_{n,x}$. From (21) we obtain

$$\int (\tilde{\gamma}_1 + \tilde{\gamma}_{1,x}) dx = -\frac{1}{2}\int \tilde{\gamma}_0^2 dx = -\frac{1}{2}\int (u_{xxx}^2 + 3u_{xx}^2 + 3u_x^2 + u^2) dx , \tag{22}$$

in which we have eliminated total derivatives and also all boundary terms that appear after using integration by parts. Thus,

$$u_{xxx}^2 + 3u_{xx}^2 + 3u_x^2 + u^2 \tag{23}$$

is a local conserved density for (2). This density will be important for our analysis of the global well-posedness of (2), see Theorems 6, 7, and 8 below. Further conserved densities arising from (21) are non-local expressions. For example, taking $n = 2$ in (21) we find

$$\int (\tilde{\gamma}_2 + \tilde{\gamma}_{2,x}) dx = -\int \tilde{\gamma}_0 \tilde{\gamma}_1 dx \ ,$$

this is, after integration by parts,

$$\begin{aligned} \int (\tilde{\gamma}_2 + \tilde{\gamma}_{2,x}) dx &= -\int (u_{xx} - u_x) \tilde{\gamma}_{1,x} dx + \int u(\tilde{\gamma}_{1,x} + \tilde{\gamma}_1) dx \\ &= \int (u_{xxx} - u_{xx}) \tilde{\gamma}_1 dx - \frac{1}{2} \int u(-u_{xxx} + u_{xx} + u_x - u)^2 dx \ . \end{aligned} \tag{24}$$

We can check that all conserved densities $\gamma_k$, $k$ odd, which are included in (16)–(19) are non-trivial. Indeed, it is clear that $\gamma_1$ and $\gamma_3$ determine non-trivial conservation laws. Now, if we expand $\gamma_5$ we obtain the summand $-\gamma_3^2/\gamma_1$. This function contains a term depending only on $-m$, and not on derivatives of $m$. Thus, it is impossible for $\gamma_5$ to be a total derivative. In general, we argue thus: we consider the density $-\gamma_{n+1}$ so that we do not worry about signs. This density will be equal to $\gamma_{n,x}/\gamma_1$ plus a sum of terms of the form $\gamma_k \gamma_{n+2-k}$. If $n + 1$ is odd, this sum decomposes into two pieces: the first summand is of the form $+\gamma_{even\,index}\gamma_{even\,index}$, and the second summand is of the form $+\gamma_{odd\,index}\gamma_{odd\,index}$. This second sum always contains a term depending only on $-m$ and not on derivatives of $m$, and therefore $-\gamma_{n+1}$ must be a non-trivial conserved density.

**Remark 1.** The foregoing computations suggest that Equation (2) has different analytic properties than the standard Camassa-Holm equation, even though it arises as the compatibility condition of a linear problem having the same pole structure as the linear problem (3) associated with the CH equation. In fact, while expansion in powers of $\lambda^{n/2}$ of the density $\Gamma$ of (15) yields conservation laws which are similar to the ones appearing in (7)–(8), we lose one local density if we expand in powers of $\lambda^{-n}$ : In the case of the fifth order CH-type equation (2) we obtain (24) instead of a density similar to the conserved density $u(u_x^2 + u^2)$ arising in the Camassa-Holm case, see (9) and [24].

## Symmetries

Now we compute symmetries for (2). Interestingly, the Lie algebra of point symmetries of this equation is much richer than the CH Lie algebra of point symmetries. We obtain the following result with the help of GeM, see [8], and the MAPLE built-in package PDEtools:

**Proposition 2.**

- *The Lie algebra of point symmetries of Equation* (2) *is generated by the vector fields*

$$V_1 = \frac{\partial}{\partial x}\,, \qquad V_2 = \frac{\partial}{\partial t}\,, \qquad V_{3F} = F(t)e^x\frac{\partial}{\partial u}$$

*and*

$$V_{4G} = G(t)e^{-t}\frac{\partial}{\partial u}\,, \qquad V_5 = t\frac{\partial}{\partial t} - u\frac{\partial}{\partial u}\,,$$

*in which* $F(t)$ *and* $G(t)$ *are arbitrary smooth functions.*

- *The Lie algebra structure of point symmetries of Equation* (2) *is determined by the commutator table*

| | $V_1$ | $V_2$ | $V_{3F}$ | $V_{4G}$ | $V5$ |
|---|---|---|---|---|---|
| $V1$ | 0 | 0 | $V_{3F}$ | $-V_{4G}$ | 0 |
| $V2$ | | 0 | $V_{3F_t}$ | $V_{4G_t}$ | $V_2$ |
| $V3$ | | | 0 | 0 | $-V_{3\,(-tF_t+F)}$ |
| $V4$ | | | | 0 | $-V_{4(tG_t+G)}$ |
| $V5$ | | | | | 0 |

The existence of symmetries $V_1$ and $V_2$ prompts us to look for solutions of the form $u(x,t) = f(x+ct)$. We easily find $u(x,t) = A\,\mathrm{e}^{ct+x} + B\,\mathrm{e}^{-ct-x} - c$, which is not a travelling wave. We consider "weak forms" of travelling waves in the next section.

**Remark 2.** It is proven in [24] that the Camassa-Holm equation admits the nonlocal symmetry $V = \gamma\exp(\delta/\lambda)\,\partial/\partial u$, in which $\gamma$ satisfies (5) and $\delta$ is a potential of the conservation law (6). Also, it is observed in [25] that a nonlocal symmetry of the same form as $V$ allows one to *classify* all integrable equations belonging to a one-parameter family of equations admitting quadratic pseudopotentials and conservation laws (see [25, Theorem 6.6]). Thus, we wonder if Equation (2) — being integrable and generalizing the Camassa-Holm equation — admits a symmetry similar to $V$. In fact, this is not so: computations carried out with the help of the MAPLE packages DifferentialGeometry and JetCalculus tell us that it is not possible to choose $L \in \mathbb{R}$ so that $V = \gamma\exp(L\,\delta)\,\partial/\partial u$ — in which $\gamma$ solves (13), (15) and $\delta$ is a potential of the conservation law (15) — be a symmetry of (2). More generally we can prove:

**Proposition 3.** *The fifth order CH-type equation* (2) *does not admit a non-trivial nonlocal symmetry of the form* $V = f(\gamma,\delta)\,\partial/\partial u$*, in which* $\gamma$ *is a solution to* (13) *and* (15)*, and* $\delta$ *is a potential of the conservation law* (15)*.*

*Proof.* The method of proof is standard, and so we only sketch its main points. We use the MAPLE packages DifferentialGeometry and JetCalculus in order to carry out our computations.

Let $\Delta$ be the left hand side of Equation (2). We consider a vector $V$ as in the enunciate of the proposition and we compute the Lie derivative

$$L_{pr(V)}\Delta \ ,$$

in which $pr(V)$ is the fifth prolongation of $V$. This derivative depends on higher derivatives of $\gamma$ and $\delta$. We get rid of these derivatives using the four compatible equations (13), (15), $\delta_x = \gamma$, $\delta_t = \lambda(u_x - u_{xxx}) - \lambda\gamma - (u - u_{xx})\gamma$, and their differential consequences. We obtain a long expression which depends only of $x$-derivatives of $u$; in fact, the highest $x$-derivative that appears in this expression is $u_{xxxxxxx}$. We will call this expression (and the ones obtained from it as explained below) $E$, simply. Differentiating $E$ with respect to $u_{xxxxxxx}$ and then differentiating the resulting expression with respect to $u$, we obtain the necessary condition

$$f_{\gamma\gamma} = 0$$

for $V$ to be a symmetry, this is, $f(\gamma, \delta) = f_1(\delta)\gamma + f_2(\delta)$. Replacing into $E$, differentiating with respect to $u_{xxxxxxx}$, and then differentiating the resulting expression with respect to $u_x$, yield the new necessary condition $f_1(\delta) = C1 \exp(3\delta/\lambda)$. Replacing this constraint into $E$ and differentiating with respect to $u_{xxxxxxx}$ once again, we find that our next necessary condition is $C1 = 0$. We replace this new constraint into $E$ and differentiate the resulting expression with respect to $u_{xxxxxx}$ and to $u_x$. We obtain the new necessary condition $f_2(\delta) = C3 + C4 \exp(2\delta/\lambda)$. Replacing one last time into $E$ and differentiating with respect to $u_{xxxxxx}$, we obtain the conditions $C3 = C4 = 0$, so that if a vector field $V = f(\gamma, \delta)\, \partial/\partial u$ were a symmetry of (2), then the function $f$ had to vanish identically. □

# 3 Pseudo-peakons

In this section we study solutions to the fifth order CH-type equation (1). As pointed out after Proposition 2, we can find explicit solutions rather easily. Besides the elementary solution already reported therein, we can check, for instance, that

$$u(x,t) = \left(c_1\, e^{-2\,x} + c_2\, e^{-2\,x} x + c_3 + c_4\, x\right) e^{x - d(t)} \ ,$$

in which $c_1, \ldots, c_4$ are constant numbers and $d(t)$ is an arbitrary function of $t$, solves (1). This function $u(x,t)$ is not a solution to the standard Camassa-Holm equation. The main goal of this section is to show that, much more interestingly, the integrable Equation (1) admits pseudo-peakon and multi-pseudo-peakon solutions, as anticipated in Section 1.

Casting the regular travelling wave setting $\xi = x - ct$ in the CH-type equation (1), through a lengthy computation we obtain the following single *pseudo-peakon* solution:

$$u = \frac{c}{2} e^{-|\xi|}(1 + |\xi|) \ , \quad \xi = x - ct \ , \tag{25}$$

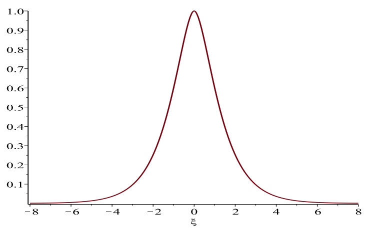


Figure 1: The single pseudo-peakon solution (25)

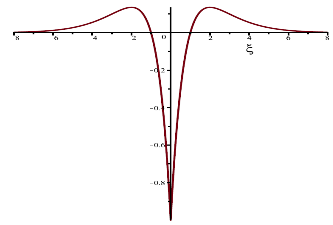

Figure 2: The peaked second derivative (26)

which looks like a peakon since there are absolute-value functions involved. But, this function in spirit has continuous derivatives up to the second order,

$$u' = -\frac{c}{2}e^{-|\xi|}\xi \ , \qquad u'' = \frac{c}{2}e^{-|\xi|}(|\xi| - 1) \ , \tag{26}$$

which show us that the solution $u$ is differentiable, with continuous and bounded second order derivative, but whose third order derivative blows up (see Figures 1 and 2).

We can also compute multi-pseudo-peakon solutions. They are of the form

$$u = \sum_{j=1}^{N} \frac{p_j(t)}{2} e^{-|x-q_j(t)|}(1 + |x - q_j(t)|) \ , \tag{27}$$

where $p_j(t), \ q_j(t)$ satisfy the following canonical Hamiltonian dynamical system

$$\dot{q}_j = \frac{\partial H}{\partial p_j}, \tag{28}$$

$$\dot{p}_j = -\frac{\partial H}{\partial q_j}, \tag{29}$$

with Hamiltonian function:

$$H = \frac{1}{2}\sum_{i,j=1}^{N} p_i p_j e^{-|q_i - q_j|} \ . \tag{30}$$

Now we make the crucial observation that (28)–(30) coincides exactly with the finite-dimensional peakon dynamical system of the CH equation, see [5]. This fact allows us to have a full picture of multi-pseudo-peakon solutions, as we show momentarily.

First, let us calculate explicitly 2-pseudo-peakons. When $N = 2$, we have the 2−pseudo-peakon equations below:

$$\begin{cases} p_{1,t} = p_1 p_2 \,\mathrm{sgn}(q_1 - q_2)\, e^{-|q_1-q_2|} \ , \\ p_{2,t} = p_1 p_2 \,\mathrm{sgn}(q_2 - q_1)\, e^{-|q_1-q_2|} \ , \\ q_{1,t} = p_1 + p_2 \, e^{-|q_1-q_2|} \ , \\ q_{2,t} = p_2 + p_1 \, e^{-|q_1-q_2|} \ , \end{cases} \tag{31}$$

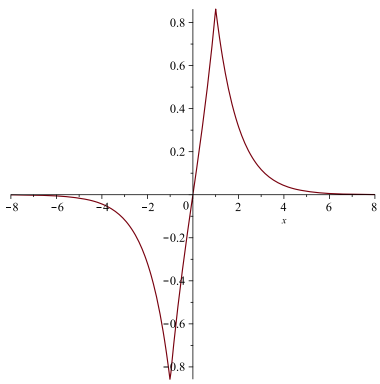

Figure 3: The two pseudo-peakon interaction plotted in 3D for negative time $t$.

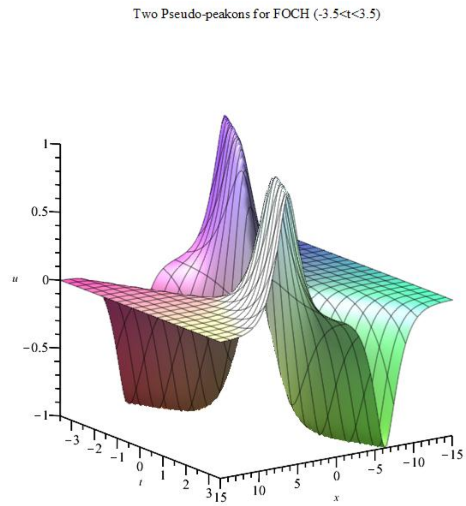


Figure 4: The two pseudo-peakon interaction plotted in 3D for all times.

which can be solved with the following explicit solutions:

$$\begin{cases} p_1(t) = -p_2(t) = A\coth(At), \\ q_1(t) = -q_2(t) = \ln\cosh(At), \end{cases} \tag{32}$$

where $A$ is an arbitrary constant. Thus, the 2-pseudo-peakon solution of the fifth order CH-type equation (1) is given by

$$\begin{aligned} u(x,t) &= \frac{p_1(t)}{2}e^{-|x-q_1(t)|}(1+|x-q_1(t)|) + \frac{p_2(t)}{2}e^{-|x-q_2(t)|}(1+|x-q_2(t)|) \\ &= \frac{A}{2}\coth(At)\left[e^{-|x-\ln\cosh(At)|}(1+|x-\ln\cosh(At)|) - e^{-|x+\ln\cosh(At)|}(1+|x+\ln\cosh(At)|)\right], \end{aligned} \tag{33}$$

where $\cosh(At) = \frac{e^{At}+e^{-At}}{2}$, and $\coth(At) = \frac{e^{At}+e^{-At}}{e^{At}-e^{-At}}$. If we fix time $t = \frac{\cosh^{-1}e}{A}$ and we select $A = \frac{2}{\coth(\cosh^{-1}e)}$, the above 2-pseudo-peakon solution reads as the following simplest form

$$u(x,t) = e^{-|x-1|}(1+|x-1|) - e^{-|x+1|}(1+|x+1|),$$

which we may plot in a 2D picture for the two pseudo-peakon interaction (see Figure 3).

**Remark 3.** The plots below show 2-pseudo-peakons in 3D. Figure 4 shows a 3D interactional dynamics of the 2-pseudo-peakon solution for all times. Figure 5 and Figure 6 show a 3D interactional dynamics of the two-pseudo-peakon solution for the negative times and the positive times, respectively. During the interaction of two-pseudo-peakons, it follows from the explicit solution (33) that the solution $u$ suddenly crashes to zero when the time

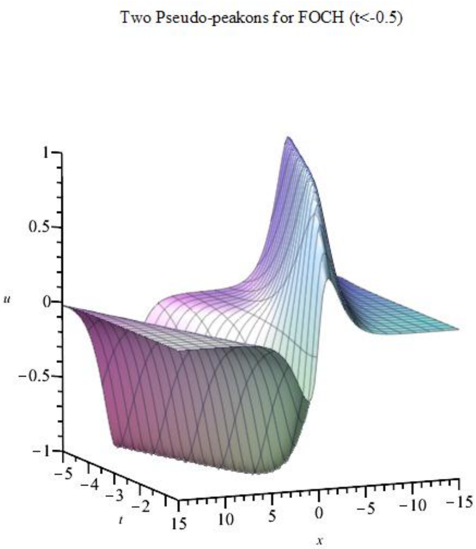


Figure 5: The two pseudo-peakon interaction plotted in 3D for negative time $t$.

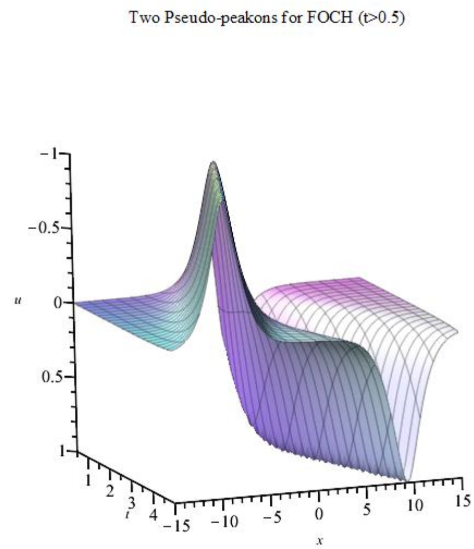


Figure 6: The two pseudo-peakon interaction plotted in 3D for positive time $t$.

$t$ passes from negative to positive via $t = 0$. After the time $t = 0$, the two-pseudo-peakon solution continues travelling from left to right, but the amplitudes already flipped along with the time (see Figure 5 and Figure 6 for details).

Now we consider multi-pseudo-peakons in full generality. It is known that the Hamiltonian system (28)–(30) —that in our context will be called the multi-pseudo-peakon system— is completely integrable in the Liouville sense. This fact is discussed in [5] and fully studied by Calogero and Françoise in [4]. Since the Hamiltonian (30) is not continuously differentiable, we cannot conclude that the trajectories of the system are given by quadratures via the Arnold-Liouville theorem. *However*, in the papers [2, 3] Beals, Sattinger and Szmigielski are able to solve the Hamiltonian system (28)–(30) via inverse spectral methods and continued fractions. More precisely we have, after the summary appearing in [6, Theorem 2.1]:

**Theorem 3.** *The solutions of the Hamiltonian system*

$$\frac{dx_j}{d\tau} = \frac{\partial H}{\partial m_j} \, , \qquad \frac{dm_j}{d\tau} = -\frac{\partial H}{\partial x_j} \, , \qquad H(x_1, \cdots, x_N, m_1, \cdots, m_N) = \frac{1}{4} \sum_{j,k=1}^{N} m_j m_k e^{-2|x_j - x_k|} \tag{34}$$

*are given by*

$$x_j = \frac{1}{2} \log \left( \frac{1 + y_j}{1 - y_j} \right) \, , \qquad m_j = g_j (1 - y_j^2) \, , \tag{35}$$

*in which*

$$y_j = 1 - \frac{\Delta^2_{N-j}}{\Delta^0_{N-j+1}} \, , \qquad g_j = \frac{(\Delta^0_{N-j+1})^2}{\Delta^1_{N-j+1} \, \Delta^1_{N-j}} \, .$$

*The functions* $\Delta_k^l(\tau)$ *are given by*

$$\Delta_k^l = \det(A_{i+j+l}(\tau))_{i,j=1}^{k-1}\,, \qquad A_k(\tau) = \sum_{j=0}^{N}(-\lambda_j)^k\, a_j(\tau)\,, \quad a_0(\tau) = \frac{1}{2}\,, \quad \lambda_0 = 0\,,$$

*with* $\frac{d}{d\tau}a_j(\tau) = -\frac{2a_j(\tau)}{\lambda_j}$ *and* $\lambda_j \neq 0$ *for* $j \geq 1$.

In this theorem we assume that the numbers $\lambda_j$ are all distinct and have the same sign, and that the initial conditions $a_j(0)$ are positive, see [6, p. 158].

Now, in order to obtain solutions to the system (28)–(30), we apply the symplectic transformation $x_j \mapsto (1/2)\, q_j$, $m_j \mapsto 2p_j$. Then, the Hamiltonian appearing in (34) becomes (30), and trajectories of (34) map onto trajectories of (28)–(30). We adjust the time evolution by setting $t = 2\tau$ and we are ready: replacing $q_j(t)$ and $p_j(t)$ into our formula (27) we obtain explicit expressions for $N$-pseudo-peakons.

# 4 The Cauchy problem for the CH-type equation

In this section we consider the Cauchy problem for the fifth order CH-type equation (1). We rewrite (1) as

$$\begin{cases} m_t + 2\,(u - u_{xx})_x\, m + (u - u_{xx})\, m_x = 0, \\ m = -A_4(u) = (1-\partial_x^2)^2 u = u - 2u_{xx} + u_{xxxx}\,. \end{cases} \tag{36}$$

The operator $(1-\partial_x^2)^{-2}$ can be expressed by

$$(1-\partial_x^2)^{-2} f = G * f = \int_{\mathbb{R}} G(x-y) f(y) dy$$

for any $f \in L^2(\mathbb{R})$ with $G = \frac{1}{4}(1+|x|)e^{-|x|}$. It follows

$$\begin{aligned} u(x,t) &= G * m = \frac{1}{4}\int_{\mathbb{R}}(1+|x-\xi|)e^{-|x-\xi|}m(\xi,t)d\xi \\ &= \frac{1}{4}e^{-x}\int_{-\infty}^{x}(1+x-\xi)e^{\xi}m(\xi,t)d\xi + \frac{1}{4}e^{x}\int_{x}^{+\infty}(1-x+\xi)e^{-\xi}m(\xi,t)d\xi. \end{aligned} \tag{37}$$

Then,

$$\begin{aligned} u_x(x,t) &= -\frac{1}{4}e^{-x}\int_{-\infty}^{x}(1+x-\xi)e^{\xi}m(\xi,t)d\xi + \frac{1}{4}e^{x}\int_{x}^{+\infty}(1-x+\xi)e^{-\xi}m(\xi,t)d\xi \\ &\quad + \frac{1}{4}e^{-x}\int_{-\infty}^{x}e^{\xi}m(\xi,t)d\xi - \frac{1}{4}e^{x}\int_{x}^{+\infty}e^{-\xi}m(\xi,t)d\xi \end{aligned}$$

and

$$\begin{aligned} u_{xx}(x,t) &= \frac{1}{4}e^{-x}\int_{-\infty}^{x}(1+x-\xi)e^{\xi}m(\xi,t)d\xi+\frac{1}{4}e^{x}\int_{x}^{+\infty}(1-x+\xi)e^{-\xi}m(\xi,t)d\xi \\ &\quad -\frac{1}{2}e^{-x}\int_{-\infty}^{x}e^{\xi}m(\xi,t)d\xi-\frac{1}{2}e^{x}\int_{x}^{+\infty}e^{-\xi}m(\xi,t)d\xi. \end{aligned} \tag{38}$$

(37) minus (38), we have

$$(u-u_{xx})(x,t)=\frac{1}{2}e^{-x}\int_{-\infty}^{x}e^{\xi}m(\xi,t)d\xi+\frac{1}{2}e^{x}\int_{x}^{+\infty}e^{-\xi}m(\xi,t)d\xi. \tag{39}$$

Differentiating $u-u_{xx}$ with respect to $x$, we have

$$(u_x-u_{xxx})(x,t)=-\frac{1}{2}e^{-x}\int_{-\infty}^{x}e^{\xi}m(\xi,t)d\xi+\frac{1}{2}e^{x}\int_{x}^{+\infty}e^{-\xi}m(\xi,t)d\xi. \tag{40}$$

## 4.1 Local well-posedness and blow-up scenario

Firstly, we present the local well-posedness theorem for the CH type equation (36).

**Theorem 4.** *Let $u_0\in H^s(\mathbb{R})$ with $s>\frac{7}{2}$. Then there exist a $T>0$ depending on $\|u_0\|_{H^s}$, such that the CH type equation* (36) *has a unique solution*

$$u\in C([0,T);H^s(\mathbb{R}))\cap C^1([0,T);H^{s-1}(\mathbb{R})).$$

*Morever, the map $u_0\in H^s\to u\in C([0,T);H^s(\mathbb{R}))\cap C^1([0,T);H^{s-1}(\mathbb{R}))$ is continuous.*

The proof (via Kato's theory) is similar to the ones appearing in [11] and [26, Section 3]. To make the paper concise, we would like to omit the details.

The maximum value of $T$ in Theorem 4 is called the lifespan of the solution. If $T<\infty$, that is,

$$\lim_{t\to T^-}\|u\|_{H^s}=\infty\ ,$$

we say the solution blows up in finite time. Let us present the precise blow-up scenario.

**Theorem 5.** *Assume that $u_0\in H^4(\mathbb{R})$ and let $T$ be the maximal existence time of the solution $u(x,t)$ to the CH type equation* (36) *with the initial data $u_0(x)$, then the corresponding solution of the CH type equation* (36) *blows up in finite time if and only if*

$$\lim_{t\to T}\inf_{x\in\mathbb{R}}\{(u_x-u_{xxx})(x,t)\}=-\infty.$$

*Proof.* By direct calculation, we have $\|u\|_{H^4}^2\le\|m\|_{L^2}^2\le 3\|u\|_{H^4}^2$. Multiplying (36) by $m$, direct calculation we have

$$\begin{aligned} \frac{d}{dt}\int_{\mathbb{R}}m^2dx &= -2\int_{\mathbb{R}}2\left(-u_{xx}+u\right)_x m^2+\left(-u_{xx}+u\right)m_x\,mdx \\ &\le -3\inf_{x\in\mathbb{R}}\{u_x-u_{xxx}\}\int_{\mathbb{R}}m^2dx. \end{aligned}$$

If

$$\inf_{x\in\mathbb{R}}\{u_x - u_{xxx}\} \geq -M,$$

then

$$\frac{d}{dt}\int_{\mathbb{R}} m^2 dx \leq 3M \int_{\mathbb{R}} m^2 dx.$$

By using the Gronwall inequality,

$$\|m\|_{L^2}^2 \leq e^{3M}\|m_0\|_{L^2}^2.$$

Therefore the $H^4$-norm of the solution is bounded on $[0, T)$. On the other hand, by the Sobolev's embedding $\|u_x - u_{xxx}\|_{L^\infty} \leq \|u\|_{H^4}$. This inequality tells us that if $H^4$-norm of the solution is bounded, then the $L^\infty$-norm of $u_x - u_{xxx}$ is bounded. We have completed the proof of Theorem 5. □

## 4.2 Global existence and blow up phenomena

In this subsection, firstly, we establish a sufficient condition that guarantees the global existence of the solution to CH type equation (36). We give the particle trajectory as

$$\begin{cases} q_t(x,t) = (u - u_{xx})(q(x,t),t), \qquad 0 < t < T, x \in \mathbb{R}, \\ q(x,0) = x, \qquad x \in \mathbb{R}, \end{cases} \tag{41}$$

where $T$ is the lifespan of the solution. Taking derivative (41) with respect to $x$, we obtain

$$\frac{dq_t(x,t)}{dx} = q_{tx} = ((u_x - u_{xxx})q_x)(q(x,t),t), \qquad t \in (0,T).$$

Therefore

$$\begin{cases} q_x = exp\{\int_0^t (u_x - u_{xxx})(q,s)ds\}, \quad 0 < t < T, \quad x \in \mathbb{R}, \\ q_x(x,0) = 1, \qquad x \in \mathbb{R}, \end{cases}$$

which is always positive before the blow-up time. Therefore, the function $q(x,t)$ is an increasing diffeomorphism of the line before blow-up. In fact, direct calculation yields

$$\frac{d}{dt}(m(q(x,t),t)q_x^b) = [m_t(q) + (u_x - u_{xxx})(q,t)m_x(q) + 2(u_x - u_{xxx})(q,t)m(q)]q_x^2 = 0.$$

Hence, the following identity can be proved:

$$m(q(x,t),t)q_x^2 = m_0(x), \qquad 0 < t < T, x \in \mathbb{R}. \tag{42}$$

From (42), we know that if the initial data $m_0(x,t) \geq 0$, then $m(q(x_0,t),t) \geq 0$. Before going to our main results, we recall the useful conservation law which was found in Section 2

$$\int_{\mathbb{R}} (u^2 + 3u_x^2 + 3u_{xx}^2 + u_{xxx}^2)dx = \int_{\mathbb{R}} (u_0^2 + 3u_{0x}^2 + 3u_{0xx}^2 + u_{0xxx}^2)dx := E_0^2. \tag{43}$$

**Theorem 6.** *Suppose that* $u_0 \in H^4(\mathbb{R})$, *and* $m_0 = (1-\partial_x^2)^2 u_0$ *does not change sign. Then the corresponding solution to the CH-type equation* (36) *exists globally.*

*Proof.* From (42), we known that $m(x,t)$ also does not change sign. By Theorem 5, we only need to bounded $(u_x - u_{xxx})(x,t)$. For $m_0 \geq 0$, by (39) and (40), we obtain

$$(u_x - u_{xxx})(x,t) + (u - u_{xx})(x,t) = e^x \int_x^{+\infty} e^{-\xi} m(\xi,t) d\xi \geq 0.$$

It follows

$$\begin{aligned}
&(u_x - u_{xxx})(x,t) \geq -(u - u_{xx})(x,t) \\
&\geq -\int_{\mathbb{R}} u^2 + 3u_x^2 + 3u_{xx}^2 + u_{xxx}^2 dx = -E_0^2.
\end{aligned}$$

Similarly, for $m_0 \leq 0$, we have

$$(u_x - u_{xxx})(x,t) \geq (u - u_{xx})(x,t) \geq -E_0^2.$$

The proof of Theorem 6 is completed. □

**Theorem 7.** *Suppose that* $u_0 \in H^4(\mathbb{R})$, *and there exists* $x_0 \in \mathbb{R}$ *such that* $m_0(x) \leq 0$ *on* $(-\infty, x_0]$ *and* $m_0(x) \geq 0$ *on* $[x_0, \infty)$. *Then the corresponding solution to the CH-type equation* (36) *exists globally.*

*Proof.* From (42), we known that $m(x,t) \leq 0$ on $(-\infty, q(x_0,t)]$ and $m(x,t) \geq 0$ on $[q(x_0,t), \infty)$. For the points $x \in (-\infty, q(x_0,t)]$, we have

$$(u_x - u_{xxx})(x,t) = (u - u_{xx})(x,t) - e^{-x} \int_{-\infty}^x e^{\xi} m(\xi,t) d\xi \geq (u - u_{xx})(x,t).$$

For the points $x \in [q(x_0,t), \infty)$, we have

$$(u_x - u_{xxx})(x,t) = -(u - u_{xx})(x,t) + e^x \int_x^{+\infty} e^{-\xi} m(\xi,t) d\xi \geq -(u - u_{xx})(x,t).$$

It means for any $x \in \mathbb{R}$, we have

$$(u_x - u_{xxx})(x,t) \geq -\|u - u_{xx}\|_{L^\infty} \geq -E_0^2,$$

where we have used the conservation law (43). Then, we complete the proof of Theorem 7. □

**Theorem 8.** *Assume that* $u_0 \in H^4(\mathbb{R})$ *and there exists* $x_0 \in \mathbb{R}$ *such that*

$$(u_{0x} - u_{0xxx})(x_0) < -\frac{E_0}{\sqrt{2}}, \tag{44}$$

*where* $E_0$ *is defined in* (43), *then the corresponding solution* $u(x,t)$ *to CH type equation* (36) *blows up at a finite time* $T$ *bounded by*

$$T \leq \frac{1}{-\frac{1}{2}(u_{0x} - u_{0xxx})(x_0) + \frac{E_0^2}{2(u_{0x} - u_{0xxx})(x_0)}}.$$

**Remark 4.** Due to $u_{0x}$ being bounded by the conservation law $E_0$ with $\|u_{0x}\|_{L^\infty} \leq \frac{\|u_{0x}\|_{H^1}}{\sqrt{2}} \leq \frac{E_0}{\sqrt{2}}$, the condition (44) holds true only if $u_{0xxx}(x_0) > \sqrt{2}E_0$.

*Proof.* Let

$$I(t) = \frac{1}{2}e^{-q(x_0,t)} \int_{-\infty}^{q(x_0,t)} e^x m(x,t)dx$$

and

$$II(t) = \frac{1}{2}e^{q(x_0,t)} \int_{q(x_0,t)}^{+\infty} e^{-x} m(x,t)dx.$$

From (39) and (40), we have

$$(u - u_{xx})(q(x_0,t),t) = I(t) + II(t)$$

and

$$(u_x - u_{xxx})(q(x_0,t),t) = -I(t) + II(t).$$

Differential $(u_x - u_{xxx})(q(x_0,t),t)$ with respect to $t$,

$$\frac{d}{dt}(u_x - u_{xxx})(q(x_0,t),t) = -\frac{d}{dt}I(t) + \frac{d}{dt}II(t). \tag{45}$$

Then, we estimate $\frac{d}{dt}I(t)$.

$$\begin{aligned}\frac{d}{dt}I(t) &= \frac{1}{2}(u-u_{xx})m(q(x_0,t),t) - \frac{1}{2}(u-u_{xx})e^{-q(x_0,t)}\int_{-\infty}^{q(x_0,t)} e^x m(x,t)dx \\ &\quad +\frac{1}{2}e^{-q(x_0,t)}\int_{-\infty}^{q(x_0,t)} e^x m_t(x,t)dx.\end{aligned}$$

The third term in the right hand side can been estimated as

$$\begin{aligned}
&\frac{1}{2}e^{-q(x_0,t)}\int_{-\infty}^{q(x_0,t)} e^x m_t(x,t)dx \\
=\ & -\frac{1}{2}e^{-q(x_0,t)}\int_{-\infty}^{q(x_0,t)} e^x((u-u_{xx})m_x + 2(u-u_{xx})_x m)(x,t)dx \\
=\ & -\frac{1}{2}e^{-q(x_0,t)}\int_{-\infty}^{q(x_0,t)} e^x(((u-u_{xx})m)_x + (u-u_{xx})_x m)(x,t)dx \\
=\ & -\frac{1}{2}(u-u_{xx})m(q(x_0,t),t) + \frac{1}{2}e^{-q(x_0,t)}\int_{-\infty}^{q(x_0,t)} e^x(((u-u_{xx})m) - (u-u_{xx})_x m)(x,t)dx \\
=\ & -\frac{1}{2}(u-u_{xx})m(q(x_0,t),t) + \frac{1}{2}e^{-q(x_0,t)}\int_{-\infty}^{q(x_0,t)} e^x\left((u-u_{xx})^2 - (u-u_{xx})(u-u_{xx})_{xx}\right. \\
& \left. -(u-u_{xx})_x(u-u_{xx}) + (u-u_{xx})_x(u-u_{xx})_{xx}\right)dx \\
=\ & -\frac{1}{2}(u-u_{xx})m(q(x_0,t),t) - \frac{1}{2}(u-u_{xx})(u-u_{xx})_x + \frac{1}{4}(u-u_{xx})_x^2 \\
& +\frac{1}{2}e^{-q(x_0,t)}\int_{-\infty}^{q(x_0,t)} e^x\left((u-u_{xx})^2 + \frac{1}{2}(u-u_{xx})_x^2\right)dx.
\end{aligned}$$

It follows that

$$\begin{aligned}\frac{d}{dt}I(t) &= -\frac{1}{2}(u-u_{xx})e^{-q(x_0,t)}\int_{-\infty}^{q(x_0,t)} e^x m(x,t)dx - \frac{1}{2}(u-u_{xx})(u-u_{xx})_x + \frac{1}{4}(u-u_{xx})_x^2 \\ &\quad +\frac{1}{2}e^{-q(x_0,t)}\int_{-\infty}^{q(x_0,t)} e^x\left((u-u_{xx})^2+\frac{1}{2}(u-u_{xx})_x^2\right)dx \\ &= -\frac{1}{2}(u-u_{xx})^2+\frac{1}{4}(u-u_{xx})_x^2+\frac{1}{2}e^{-q(x_0,t)}\int_{-\infty}^{q(x_0,t)} e^x\left((u-u_{xx})^2+\frac{1}{2}(u-u_{xx})_x^2\right)dx.\end{aligned}$$

Note that

$$\begin{aligned}&\int_{-\infty}^{q(x_0,t)} e^x\left((u-u_{xx})^2+(u-u_{xx})_x^2\right)dx \\ \geq\ & 2\int_{-\infty}^{q(x_0,t)} e^x(u-u_{xx})(u-u_{xx})_x dx \\ \geq\ & e^{q(x_0,t)}(u-u_{xx})^2(q(x_0,t),t) - \int_{-\infty}^{q(x_0,t)} e^x(u-u_{xx})^2dx,\end{aligned}$$

which yields that

$$\int_{-\infty}^{q(x_0,t)} e^x\left((u-u_{xx})^2+\frac{1}{2}(u-u_{xx})_x^2\right)dx \geq \frac{1}{2}e^{q(x_0,t)}(u-u_{xx})^2(q(x_0,t),t).$$

Therefore,

$$\frac{d}{dt}I(t) \geq -\frac{1}{4}(u-u_{xx})^2(q(x_0,t),t)+\frac{1}{4}(u-u_{xx})_x^2(q(x_0,t),t). \tag{46}$$

Similarity,

$$\begin{aligned}\frac{d}{dt}II(t) &= -\frac{1}{2}(u-u_{xx})m(q(x_0,t),t)+\frac{1}{2}(u-u_{xx})e^{q(x_0,t)}\int_{q(x_0,t)}^{+\infty} e^{-x}m(x,t)dx \\ &\quad +\frac{1}{2}e^{q(x_0,t)}\int_{q(x_0,t)}^{+\infty} e^{-x}m_t(x,t)dx.\end{aligned}$$

The third term in the right hand side can been estimated as

$$\begin{aligned}&\frac{1}{2}e^{q(x_0,t)}\int_{q(x_0,t)}^{+\infty} e^{-x}m_t(x,t)dx \\ =\ & \frac{1}{2}(u-u_{xx})m(q(x_0,t),t)-\frac{1}{2}(u-u_{xx})(u-u_{xx})_x-\frac{1}{4}(u-u_{xx})_x^2 \\ & -\frac{1}{2}e^{q(x_0,t)}\int_{q(x_0,t)}^{\infty} e^{-x}\left((u-u_{xx})^2+\frac{1}{2}(u-u_{xx})_x^2\right)dx.\end{aligned}$$

It follows that

$$\begin{aligned}\frac{d}{dt}II(t) &= \frac{1}{2}(u-u_{xx})e^{q(x_0,t)}\int_{q(x_0,t)}^{+\infty} e^{-x}m(x,t)dx \\ &\quad -\frac{1}{2}(u-u_{xx})(u-u_{xx})_x-\frac{1}{4}(u-u_{xx})_x^2 \\ &\quad -\frac{1}{2}e^{q(x_0,t)}\int_{q(x_0,t)}^{\infty} e^{-x}\left((u-u_{xx})^2+\frac{1}{2}(u-u_{xx})_x^2\right)dx.\end{aligned}$$

By same argument, we have

$$\int_{q(x_0,t)}^{\infty} e^{-x}\left((u-u_{xx})^2+\frac{1}{2}(u-u_{xx})_x^2\right)dx \geq \frac{1}{2}e^{-q(x_0,t)}(u-u_{xx})^2(q(x_0,t),t).$$

Therefore

$$\frac{d}{dt}II(t) \leq \frac{1}{4}(u-u_{xx})^2(q(x_0,t),t)-\frac{1}{4}(u-u_{xx})_x^2(q(x_0,t),t). \tag{47}$$

Inserting (46) and (47) into (45), we have

$$\begin{aligned}&\frac{d}{dt}(u_x-u_{xxx})(q(x_0,t),t) \\ =\ & (u-u_{xx})^2(q(x_0,t),t)-\frac{1}{2}(u-u_{xx})_x^2(q(x_0,t),t) \\ & -\frac{1}{2}e^{-q(x_0,t)}\int_{-\infty}^{q(x_0,t)} e^{x}\left((u-u_{xx})^2+\frac{1}{2}(u-u_{xx})_x^2\right)dx \\ & -\frac{1}{2}e^{q(x_0,t)}\int_{q(x_0,t)}^{\infty} e^{-x}\left((u-u_{xx})^2+\frac{1}{2}(u-u_{xx})_x^2\right)dx.\end{aligned} \tag{48}$$

Combining the above estimates into (48), we obtain

$$\frac{d}{dt}(u_x-u_{xxx})(q(x_0,t),t) \leq \frac{1}{2}(u-u_{xx})^2(q(x_0,t),t)-\frac{1}{2}(uu_{xx})_x^2(q(x_0,t),t). \tag{49}$$

By the fact $\|f\|_{L^\infty}^2 < \frac{1}{2}\|f\|_{H^1}^2$, we have

$$\|u-u_{xx}\|_{L^\infty}^2 < \frac{1}{2}\|u-u_{xx}\|_{H^1}^2 = \frac{1}{2}E_0^2.$$

Let $\varphi(t) = (u_x-u_{xxx})(q(x_0,t),t)$, we can rewrite (49) as

$$\varphi'(t) \leq -\frac{1}{2}\varphi^2(t)+\frac{1}{4}E_0^2.$$

We complete the proof of Theorem 8 by using the hypothesis of the theorem and a standard argument on the Riccati type equations. □

**Theorem 9.** *Assume that* $u_0 \in H^4(\mathbb{R})$ *and there exists* $x_0 \in \mathbb{R}$ *such that* $m_0(x_0) = (1-\partial_x^2)^2 u_0(x_0) = 0$,

$$\int_{-\infty}^{x_0} e^x m_0(x,t)dx > 0 \quad \text{and} \quad \int_{x_0}^{\infty} e^{-x} m_0(x,t)dx < 0, \tag{50}$$

*then the corresponding solution* $u(x,t)$ *to CH type equation* (36) *blows up in finite time.*

*Proof.* Thanks to (42), we obtain $m(q(x_0,t),t) = 0$ for all $t$ in its lifespan. The inequality (49) is also correct in this proof. The initial condition (50) means that $(u_0 - u_{0xx})_x(x_0,t) < 0$ and $(u_0 - u_{0xx})_x^2(x_0,t) > (u_0 - u_{0xx})^2(x_0,t)$. We claim that $(u_0 - u_{0xx})_x(q(x_0,t),t)$ is decreasing, $(u-u_{xx})_x^2(q(x_0,t),t) > (u-u_{xx})^2(q(x_0,t),t)$ for all $t \geq 0$. Suppose not, i.e. there exists a $t_0$ such that $(u-u_{xx})_x^2(q(x_0,t),t) > (u-u_{xx})^2(q(x_0,t),t)$ on $[0,t)$ and $(u-u_{xx})_x^2(q(x_0,t_0),t_0) \leq (u-u_{xx})^2(q(x_0,t_0),t_0)$. Recall (46) and (47), we get on $[0,t_0)$

$$\frac{d}{dt}I(t) \geq -\frac{1}{4}(u-u_{xx})^2(q(x_0,t),t) + \frac{1}{4}(u-u_{xx})_x^2(q(x_0,t),t) \geq 0$$

and

$$\frac{d}{dt}II(t) \leq \frac{1}{4}(u-u_{xx})^2(q(x_0,t),t) - \frac{1}{4}(u-u_{xx})_x^2(q(x_0,t),t) \leq 0.$$

It follows from the continuity property of ODEs that

$$(u-u_{xx})^2(q(x_0,t),t) - (u-u_{xx})_x^2(q(x_0,t),t) = 4I(t)II(t) < 4I(0)II(0) < 0,$$

for all $t > 0$, this implies that $t_0$ can be extended to the infinity. This is a contradiction. So the claim is true. By using (46) and (47) again, we get

$$\begin{aligned}
&\frac{d}{dt}[(u-u_{xx})_x^2 - (u-u_{xx})^2](q(x_0,t),t) \\
= \;& -\frac{d}{dt}\left\{\int_{-\infty}^{q(x_0,t)} e^x m(x,t)dx \times \int_{q(x_0,t)}^{+\infty} e^{-x} m(x,t)dx\right\} \\
= \;& 4\frac{d}{dt}[I(t)II(t)] \\
= \;& 4\frac{d}{dt}I(t) \times II(t) + 4\frac{d}{dt}II(t) \times I(t) \\
\geq \;& -[(u-u_{xx})_x^2 - (u-u_{xx})^2](q(x_0,t),t)II(t) + [(u-u_{xx})_x^2 - (u-u_{xx})^2](q(x_0,t),t)I(t) \\
= \;& -(u-u_{xx})_x(q(x_0,t),t)[(u-u_{xx})_x^2 - (u-u_{xx})^2](q(x_0,t),t),
\end{aligned} \tag{51}$$

where we have used $(u-u_{xx})_x(q(x_0,t),t) = -I(t) + II(t)$. Recall (49), it follows

$$\begin{aligned}
&(u_x - u_{xxx})(q(x_0,t),t) \\
\leq \;& \int_0^t \frac{1}{2}[(u-u_{xx})^2(q(x_0,s),s) - (u-u_{xx})_x^2(q(x_0,s),s)]ds - (u_x - u_{xxx})(x_0,0).
\end{aligned} \tag{52}$$

Substituting (52) into (51), it yields

$$\frac{d}{dt}[(u-u_{xx})_x^2-(u-u_{xx})^2](q(x_0,t),t)\geq\frac{1}{2}[(u-u_{xx})_x^2-(u-u_{xx})^2](q(x_0,t),t) \times\left\{\int_0^t[(u-u_{xx})_x^2-(u-u_{xx})^2](q(x_0,s),s)ds+2(u_x-u_{xxx})(x_0,0)\right\}. \tag{53}$$

Before completing the proof, we need the following technical lemma.

**Lemma 1.** [29] *Suppose that $\Psi(t)$ is twice continuously differentiable satisfying*

$$\begin{cases}\Psi''(t)\geq C_0\Psi'(t)\Psi(t), & t>0, \quad C_0>0,\\ \Psi(t)>0, \quad \Psi'(t)>0.\end{cases} \tag{54}$$

*Then $\Psi(t)$ blows up in finite time. Moreover, the blow up time can be estimated in terms of the initial datum as*

$$T\leq\max\left\{\frac{2}{C_0\Phi(0),\,\frac{\Phi(0)}{\Phi'(0)}}\right\}.$$

Let $\Psi(t)=\int_0^t[(u-u_{xx})_x^2-(u-u_{xx})^2](q(x_0,s),s)ds+2(u_x-u_{xxx})(x_0,0)$, then (53) is an equation of type (54) with $C_0=\frac{1}{2}$. The proof is completed by applying Lemma 1. □

# 5 Final Remarks

In this final section we collect three different remarks: first, we introduce the $(2n+1)th$ order CH-type equations, $n\geq 1$; second, we discuss the relation of these equations with the geometry of the diffeomorphism group $Diff(S^1)$; third, we connect our class of equations with the geometry of pseudo-spherical surfaces.

## 5.1 Higher order Camassa-Holm type equations

In this subsection we consider differential operators $A_{2n}$ of order $2n$ and define $m=A_{2n}(u)$. Specifically, we choose the operators

$$\begin{aligned} A_{2n} &= (-1)^{n+1}\partial_x^{2n}+2\sum_{k=1}^{n-1}(-1)^{n+1-k}\,\partial_x^{2(n-k)}-1\,, \qquad (55)\\ B_{2n} &= \sum_{k=0}^{n-1}(-1)^{n-k}\,\partial_x^{2(n-k)-1}\,,\\ C_{2n} &= \sum_{k=0}^{n-1}(-1)^{n-k}\,\partial_x^{2(n-k-1)}\,. \end{aligned}$$

We consider the matrices

$$X_{2n} = \begin{bmatrix} 0 & \frac{1}{2}\,\lambda + A_{2n}(u) \\ \frac{1}{2}\,\lambda^{-1} & 0 \end{bmatrix} \tag{56}$$

and

$$T_{2n} = \begin{bmatrix} \frac{1}{2}\,B_{2n}(u) & C_{2n}(u)\,A_{2n}(u) - \frac{1}{2}\,\lambda\,C_{2n}(u) - \frac{1}{2}\,\lambda^2 \\ -\dfrac{1}{2} + \dfrac{1}{2\lambda}\,C_{2n}(u) & -\frac{1}{2}\,B_{2n}(u) \end{bmatrix} . \tag{57}$$

A straightforward computation allows us to check that the equation

$$X_{2n,t} - T_{2n,x} + [X_{2n}, T_{2n}] = 0$$

is equivalent to the $(2n+1)$-*order equation of Camassa-Holm type*

$$A_{2n,t}(u) - 2\,C_{2n}(u)_x\,A_{2n}(u) - C_{2n}(u)\,A_{2n}(u)_x = 0 \; . \tag{58}$$

Proposition 1 allows us to compute quadratic pseudo-potentials and conservation laws. Indeed, after applying a gauge transformation to $X_{2n}dx + T_{2n}dt$ with gauge matrix (12) and using Prop. 1, we obtain the Riccati equation

$$\frac{\partial \Gamma}{\partial x} = \frac{1}{2}\,\lambda + A_{2n}(u) - \frac{\Gamma^2}{2\,\lambda} \; , \tag{59}$$

a corresponding equation for $\Gamma_t$, and the conserved density $-\Gamma/\lambda$. It follows by expanding $\Gamma$ as in (7)–(9) that Equation (58) is integrable. We will consider its pseudo-peakon solutions and Cauchy problem elsewhere.

## 5.2 Camassa-Holm type equations and Diff($\mathbf{S^1}$)

In this subsection we connect the (periodic case of the) Camassa-Holm type equations with the geometry of $Diff(S^1)$, the Fréchet Lie group of diffeomorphisms of the circle. We recall some basic facts on the geometry of this group (some of them already mentioned in Section 1) following the exposition appearing in [15]:

We set $G = Diff(S^1)$ and we write its Lie algebra as $\mathfrak{g} = Vect(S^1)$, see [18]. We also denote by $\mathfrak{g}'$ the dual of $\mathfrak{g}$, and by $\langle\ ,\ \rangle : \mathfrak{g}\times\mathfrak{g}' \to \mathbb{R}$ the pairing that induces such a duality. Given a linear map $A : \mathfrak{g} \to \mathfrak{g}'$ we define the $\mathbb{R}$-bilinear mapping $(\ \cdot \mid \cdot\ )_A : \mathfrak{g}\times\mathfrak{g} \to \mathbb{R}$ as $(\ X \mid Y\ )_A = \langle\ X, AY\ \rangle$ whenever $X$ and $Y$ are in $\mathfrak{g}$. If such a bilinear map is symmetric and non-degenerate, we say that $A$ is an *inertia operator*. In this case, we define an adjoint representation with respect to $A$ by

$$(\ \mathrm{ad}(X)Y \mid Z\ )_A = -(\ Y \mid \mathrm{ad}_A(X)Z\ )_A \tag{60}$$

for all $X, Y, Z$ in $\mathfrak{g}$.

Let us fix an inertia operator $A : \mathfrak{g} \to \mathfrak{g}'$. This operator induces a (pseudo-)Riemannian metric on $G$: we let $r_\gamma$ be the right translation by $\gamma \in G$, and we denote by $r_{\gamma *} : T_\sigma G \to$

$T_{\sigma(\gamma)}G$ the induced map on the tangent bundle. We define the (pseudo-)Riemannian metric induced by $A$ as

$$( \xi \mid \tau )_A(\gamma) = ( r_{\gamma^{-1}*}\xi \mid r_{\gamma^{-1}*}\tau )_A \tag{61}$$

for all $\tau, \xi \in T_\gamma G$. Now let us consider a smooth curve $\{ \gamma(t) \mid t \in T \}$ in $G$, where $T$ is an open interval in $\mathbb{R}$, and let $\dot\gamma(t) \in T_{\gamma(t)}G$, $t \in T$, be its velocity vector. Then, $r_{\gamma(t)^{-1}*}\dot\gamma(t) = X(t)$ is in $\mathfrak{g}$, and we get a curve $\{ X(t) \mid t \in T \}$ in $\mathfrak{g}$. The *Euler equation* for $X(t)$ is

$$\frac{d}{dt}X(t) = -\mathrm{ad}_A(X(t))X(t) . \tag{62}$$

Euler's equation determines geodesics on $G$ with respect to the (pseudo-)Riemannian metric (61), see [1] and also [10, 20]: if $X(t)$ solves (62), then the curve $\gamma(t)$ determined by $r_{\gamma(t)^{-1}*}\dot\gamma(t) = X(t)$ is a geodesic on $G$.

Let $x$ in $[0, 2\pi[$ be the standard coordinate in $S^1$. Every smooth vector field on $S^1$ can be written as $X(x)\partial_x$, where $X : S^1 \to \mathbb{R}$ is a smooth function. The Lie bracket between $X = X(x)\partial_x$ and $Y = Y(x)\partial_x$ is given by $[X, Y] = (X\, Y_x - X_x\, Y)(x)\, \partial_x$. We complexify $\mathfrak{g} = \mathrm{Vect}(S^1)$, that is we set

$$\mathrm{Vect}(S^1)_{\mathbb{C}} = \mathfrak{g} \oplus i\, \mathfrak{g} = \mathrm{Vect}(S^1) \oplus i\, \mathrm{Vect}(S^1) ,$$

where $i = \sqrt{-1}$. Thus if $z = e^{i\,x}$, then $\{ l_n = z^n\partial_x \mid n \in \mathbb{Z} \}$ is a basis for $\mathrm{Vect}(S^1)_{\mathbb{C}}$, i.e. for every *complex* vector field of the form $X(x)\partial_x$ we have a Fourier decomposition

$$X(x) = \sum_{n\in\mathbb{Z}} X_n z^n , X_n \in \mathbb{C} .$$

We note that if we set $L_n = il_n$, the collection $\{ L_n \mid n \in \mathbb{Z} \}$ is also a basis for $\mathrm{Vect}(S^1)_{\mathbb{C}}$ and that if we extend the Lie bracket linearly to $\mathrm{Vect}(S^1)_{\mathbb{C}}$ we have the relations $[L_m, L_n] = (m - n)L_{m+n}$, $m, n \in \mathbb{Z}$.

There is a non-degenerate, positive-definite, $L_2-$ inner product on $\mathfrak{g}$: if $X = X(x)\partial_x$ and $Y = Y(x)\partial_x$, then

$$\langle X , Y\rangle = \int_{S^1} X(x)Y(x)d\,x .$$

We use this product to single-out a convenient dual space $\mathfrak{g}'$ as in the beginning of this subsection. Extending such a product complex-linearly to $\mathrm{Vect}(S^1)_{\mathbb{C}}$ we have

$$\langle l_m , l_n\rangle = 2\pi\delta_{m,-n} = -\langle L_m , L_n\rangle , \qquad (m, n) \in \mathbb{Z}^2 .$$

We fix a finite sequence of real numbers $\mathfrak{c} = \{c_k\}_{k=0}^N$ for some $N \in \mathbb{N}$ and we set

$$A_{\mathfrak{c}} = \sum_{k=0}^{N} (-1)^k c_k\, \partial_x^{2k} . \tag{63}$$

We observe that

$$( X \mid Y )_{A_\mathfrak{c}} = \langle X \, , \, A_\mathfrak{c} Y \rangle = \sum_{k=0}^{N} c_k \, \langle \partial_x^k X \, , \, \partial_x^k Y \rangle = \langle A_\mathfrak{c} X \, , \, Y \rangle$$

for every $X$ and $Y$, and therefore in terms of the basis for $\mathrm{Vect}(S^1)_\mathbb{C}$ we have

$$( l_m \mid l_n )_{A_\mathfrak{c}} = \sum_{k=0}^{N} (-1)^k c_k \; m^k n^k \langle l_m \, , \, l_n \rangle = 2\pi \delta_{m,-n} \sum_{k=0}^{N} c_k \; m^{2k} \; ,$$

for $m, n \neq 0$, while $( l_0 \mid l_n )_{A_\mathfrak{c}} = c_0 \langle \partial_x , z^n \partial_x \rangle = 2\pi c_0 \delta_{n,0}$. It follows easily that *a symmetric operator* $A_\mathfrak{c}$ *is an inertia operator if and only if* $( l_m \mid l_{-m} )_{A_\mathfrak{c}}$ *is different from zero for every* $m$ *in* $\mathbb{Z}$.

We are ready to study the geometric interpretation of our equations (58) . First of all, we note that our operators $A_{2n}$ introduced in (55), are indeed inertia operators. We write $A_{2n}$ as

$$A_{2n} = (-1)^n (-1) \partial_x^{2n} + \sum_{j=1}^{n-1} (-1)^j (-2) \partial_x^{2j} - 1 \; ,$$

so that, using the notation introduced in (63) we have $c_0 = -1$, $c_j = -2$ for $j = 1, \cdots, n-1$, $c_n = -1$. Then, $( l_0 \mid l_0 )_{A_\mathfrak{c}} \neq 0$ , $( l_{\pm 1} \mid l_{\pm 1} )_{A_\mathfrak{c}} \neq 0$, and for $m \neq 0, \pm 1$,

$$\sum_{k=0}^{n} c_k \; m^{2k} = -1 - m^{2n} - 2 \left( \frac{m^{2n} - 1}{m^2 - 1} - 1 \right) = 1 - m^{2n} - 2 \, \frac{m^{2n} - 1}{m^2 - 1}$$

which is clearly not zero as well. Now we compute $ad_{A_{2n}}$ using (60); we use " $'$ " to indicate derivative with respect to $x$:

$$\begin{aligned} \langle [X, Y], A_{2n}(Z) \rangle &= \int_{S^1} (XY' - X'Y) A_{2n}(Z) dx \; = \; - \int_{S^1} Y \left[ (X A_{2n}(Z))' + X' A_{2n}(Z) \right] dx \\ &= -\langle Y, (X A_{2n}(Z))' + X' A_{2n}(Z) \rangle \; = \; -\langle Y, A_{2n} A_{2n}^{-1} \{ (X A_{2n}(Z))' + X' A_{2n}(Z) \} \rangle \; , \end{aligned}$$

so that (60) yields

$$ad_{A_{2n}}(X) Z = A_{2n}^{-1} \left\{ (X A_{2n}(Z))' + X' A_{2n}(Z) \right\} = A_{2n}^{-1} \left\{ 2 X' A_{2n}(Z) + X A_{2n}(Z)' \right\} \; . \quad (64)$$

This formula implies that Equation (62) in the case $n = 1$ is precisely the Camassa-Holm equation, while the case $n = 2$ gives the equation

$$(-\partial_x^4 + 2\partial_x^2 - 1) X_t - 2 X_x X_{xxxx} + 4 X_x X_{xx} - 3 X X_x - X X_{xxxxx} + 2 X X_{xxx} = 0 \; ,$$

which *is not* our fifth order CH-type equation (2). In order to find (2) in the present framework we write it as in (36), this is,

$$2 \left( -X_{xx} + X \right)_x m + m_t + \left( -X_{xx} + X \right) m_x = 0 \; , \qquad m = A_4(X) \; . \quad (65)$$

Comparing (65) and (64), we see that our fifth order CH-type equation can be written, in geometrical terms, as

$$\frac{d}{dt} X = -ad_{A_4}(A_2(X)) \cdot X \ .$$

This equation is a Hamiltonian equation with respect to the standard Poisson bracket on $\mathfrak{g}'$ with Hamiltonian function

$$H(X) = -\frac{1}{2} \int_{S^1} (X^2 + X_x^2) dx \ .$$

More generally, Equation (64) implies that our $(2n+1)$th order CH-type equation (58) can be written as

$$\frac{d}{dt} X = ad_{A_{2n}}(C_{2n}(X)) \cdot X \ .$$

This equation is also Hamiltonian; the corresponding Hamiltonian function is

$$H(X) = \frac{1}{2} \int_{S^1} (X^2 + (\partial_x X)^2 + \cdots + (\partial_x^{n-1} X)^2 \,) dx \ .$$

## 5.3 Camassa-Holm type equations and classical theory of surfaces

The very construction of Equation (65) —and more generally of (58)— as the integrability condition of an $sl(2,\mathbb{R})$-valued over-determined linear problem implies that (65) (resp. (58)) describes surfaces of constant Gaussian curvature $K = -1$ in the following sense, see Chern and Tenenblat's [9] or the later review [25]:
If $\omega^1 = (X_{2n\,21} + X_{2n\,12})dx + (T_{2n\,21} + T_{2n\,12})dt$, $\omega^2 = 2X_{2n\,11}dx + 2T_{2n\,11}dt$, and $\omega^3 = (X_{2n\,21} - X_{2n\,12})dx + (T_{2n\,21} - T_{2n\,12})dt$, in which $X_{2n}$ and $T_{2n}$ are given by (56) and (57) and $X_{2n\,ij}$, $T_{2n\,ij}$ denote the $(i,j)$ entry of $X_{2n}$ and $T_{2n}$ respectively, then

$$d\omega^1 = \omega^3 \wedge \omega^2, \qquad d\omega^2 = \omega^1 \wedge \omega^3, \qquad \text{and} \qquad d\omega^3 = \omega^1 \wedge \omega^2 \tag{66}$$

whenever $u(x,t)$ solves (58). If $\omega^1(u(x,t)) \wedge \omega^2(u(x,t)) \neq 0$, these structure equations say that the domain of $u(x,t)$ has the structure of a surface of constant Gaussian curvature equal to $-1$, with metric $(\omega^1)^2 + (\omega^2)^2$ and connection one–form $\omega_{12} = \omega^3$.

The importance of this observation is that the methods used in Section 2 for obtaining conservation laws and pseudo-potentials originate within the geometry of pseudo-spherical surfaces, as noted by Chern and Tenenblat in [9]. Later papers on these topics are [23, 25], and [24] for the particular case of the Camassa-Holm equation.

A recent endeavour within the theory of equations describing pseudo-spherical surfaces is to investigate local isometric immersions into $E^3$ of the pseudo-spherical surfaces described intrinsically by solutions of equations such as (58). It is a classical result that every pseudo-spherical surface can be locally immersed in $E^3$, and that the existence of such immersion is due to the fact that one can find further one-forms

$$\omega_{13} = a\,\omega_1 + b\,\omega_2\ , \qquad \omega_{23} = b\,\omega_1 + c\,\omega_2\ ,$$

satisfying the equations

$$d\omega_{13} = \omega_{12} \wedge \omega_{23}\ , \qquad d\omega_{23} = -\omega_{12} \wedge \omega_{13}\ , \qquad a\,c - b^2 = -1\ . \tag{67}$$

Generally speaking, the functions $a, b, c$ are determined by solving differential equations, and therefore it is quite surprising that (for one-forms associated to differential equations describing pseudo-spherical surfaces) in some cases they can be expressed in closed form as functions of the independent variables and of a *finite number* of derivatives of the dependent variable $u$, see for instance [7, p. 1650021-4]. Even more, it has been noticed that in some important instances (*e.g.* the Degasperis-Procesi equation), the functions $a, b, c$ *depend only on the independent variables* $x, t$, see [7, Theorem 1.1], while (up to a technical condition) it is not possible to find functions $a, b, c$ depending on $x, t$ and at most a finite number of derivatives of $u$ in the case of the Camassa-Holm equation. Thus, it is very natural to ask whether or not we can find functions $a, b, c$ depending on finite order jets for our equations (58). Our first result concerns the fifth order CH-type equation (2).

**Proposition 4.** *1. The fifth order CH-type equation* (2) *describes pseudo-spherical surfaces with associated one-forms*

$$\omega^1 = \left(\frac{\lambda}{2} - m + \frac{1}{2\lambda}\right) dx + \left(vm + \frac{\lambda v}{2} - \frac{\lambda^2}{2} - \frac{1}{2} - \frac{v}{2\lambda}\right) dt\ , \qquad \omega^2 = -v_x dt\ , \tag{68}$$

$$\omega^3 = \left(-\frac{\lambda}{2} + m + \frac{1}{2\lambda}\right) dx + \left(-vm - \frac{\lambda v}{2} + \frac{\lambda^2}{2} - \frac{1}{2} - \frac{v}{2\lambda}\right) dt \tag{69}$$

*in which*

$$v = u - u_{xx}\ , \qquad m = v - v_{xx}\ .$$

2. *There are no functions $a, b, c$ depending only on independent variables $x, t$ such that the one forms $\omega^1, \omega^2, \omega_{12} = \omega^3$ given by* (68), (69) *and*

$$\omega_{13} = a\,\omega_1 + b\,\omega_2\ , \qquad \omega_{23} = b\,\omega_1 + c\,\omega_2$$

*satisfy the structure equations* (66) *and* (67).

*Proof.* Item 1 is a straightforward computation: we simply note that Equation (2) can be written as in (1), that is, as the system

$$-\,2\,m\,v_x - m_x\,v = m_t\ , \qquad v = u - u_{xx}\ , \qquad m = v - v_{xx}\ , \tag{70}$$

and we check that the one-forms appearing in (68) and (69) satisfy the structure equations (66). Item 2 is proven by a strategy similar in spirit to the one used in the proof of

Proposition 3: let us assume that there exist functions $a, b, c$ depending only on $x, t$ such that Equations (67) hold. We write down (67) and we obtain two equations that have to be satisfied identically whenever $u(x,t)$ solves (70), the first one corresponding to $d\omega_{13} = \omega_{12} \wedge \omega_{23}$ and the second one corresponding to $d\omega_{23} = -\omega_{12} \wedge \omega_{13}$. We will simply call them Equations M and N respectively. The sketch that follows has been checked using MAPLE 2021:

Taking derivatives of M with respect to $v_x$ and then with respect to $m$ we get $a(x,t) = c(x,t)$. Replacing into M and taking derivative with respect to $v_x$ we find $b_x = 0$, while replacing $a(x,t) = c(x,t)$ into N and then differentiating with respect to $v_x$ and $m$ yields $b = 0$. Replacing into N once again gives us $c_x = 0$, and then replacing into M we find $c_t = 0$. Now we use the Gauss equation $ac - b^2 = -1$ and we conclude that $a, b, c$ cannot exist. □

Now we consider Equations (58) in full generality. In order to study the local immersion problem we proceed differently than in the interesting paper [7]. Instead of trying to find one-forms $\omega_{13}$ and $\omega_{23}$ satisfying (67) directly, we simply construct a rather explicit local immersion, taking advantage of the following observation appearing in [17]:

**Lemma 2.** *There exists a local diffeomorphism $\Psi : V \subseteq \mathbb{R}^2 \to \overline{W}$, where $\overline{W}$ is a subset of the Poincaré upper half plane, and a smooth function $\mu$ such that*

$$\begin{aligned} \Psi^*\theta^1 &= \cos\mu\,\omega^1 + \sin\mu\,\omega^2 \\ \Psi^*\theta^2 &= -\sin\mu\,\omega^1 + \cos\mu\,\omega^2 \\ \Psi^*\theta^3 &= \omega^3 + d\mu \end{aligned}$$

*where*

$$\theta^1 = \frac{d\overline{x}}{\overline{t}}\,, \qquad \theta^2 = \frac{d\overline{t}}{\overline{t}}\,, \qquad \theta^3 = \frac{d\overline{x}}{\overline{t}}\,.$$

The one-forms $\theta^1, \theta^2$ give the standard metric on the Poincaré upper half plane (hereafter denoted by $\mathbb{H}$) $ds^2 = \theta^1 \otimes \theta^1 + \theta^2 \otimes \theta^2$, and $\theta^3$ is the corresponding connection one-form. The proof of this lemma is in [17, pp. 90-91]. Now we note that $\mathbb{H}$ can be immersed into $E^3$ explicitly. A well known immersion is given by the function $F : U \subseteq \mathbb{H} \to E^3$, $U = \{(\overline{x}, \overline{t}) \in \mathbb{H} : \overline{t} > 1\}$, with

$$F(\overline{x}, \overline{t}) = (f(\overline{t})\cos\overline{x}\,,\; f(\overline{t})\sin\overline{x}\,,\; g(\overline{t}))\,,$$

where

$$f(\overline{t}) = \frac{1}{\overline{t}} \qquad g(\overline{t}) = \ln\left(\sqrt{\overline{t}^2 - 1} + \overline{t}\right) - \frac{\sqrt{\overline{t}^2 - 1}}{\overline{t}}\,.$$

Thus, *a local isometric immersion from the pseudo-spherical structure on $V$ induced by our Ch-type equation* (58) *into $E^3$ is given by the composition $\Phi = F \circ \Psi$.* Certainly, this immersion is in principle highly “nonlocal”, since it depends on the diffeomorphism $\Psi$ that is found by means of the Frobenius theorem, see [17, p. 91]. However, we believe that this nonlocality is interesting in its own right:

The first and third equations appearing in Lemma 2 imply that we can obtain the function $\mu$ via the Pfaffian system

$$\cos\mu\,\omega^1 + \sin\mu\,\omega^2 = \omega^3 + d\mu\ .$$

The change of variables $\Gamma = \tan(\mu/2)$ transforms this equation into the Riccati system

$$2d\Gamma = (\omega^1 - \omega^3) + 2\Gamma\omega^2 - \Gamma^2(\omega^1 + \omega^3)\ ,$$

and using the explicit expressions for the one-forms $\omega^i$, $i = 1, 2, 3$, appearing at the beginning of this subsection we obtain that this system is equivalent to

$$2\Gamma_x = \lambda + 2A_{2n}(u) - \frac{1}{\lambda}\Gamma^2$$

and an equation for $\Gamma_t$ which we will not write down. This equation for $\Gamma_x$ is precisely the quadratic pseudo-potential equation (59) determining local conservation laws for the CH-type equation (58)! Also, we can check that if we write $\Psi(x,t) = (\phi(x,t), \psi(x,t))$, then $\ln(\psi)$ is a potential for the local conservation laws of (58), while $\phi$ is a further potential depending on $\psi$.

Thus, local isometric immersions of our CH-type equations are, essentially, constructed via their local conservation laws.

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