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On characterization of the exponential distribution via hypoexponential distributions

George Yanev
The University of Texas Rio Grande Valley, george.yanev@utrgv.edu

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George Yanev[1,2*]

[1*]School of Mathematical and Statistical Sciences, University of Texas Rio Grande Valley, 1201 W. University Dr., Edinburg 78539, Texas, USA.

[2*]Institute of Mathematics and Informatics, Bulgarian Academy of Sciences, 8 Acad. G. Bontchev St., Sofia 1113, Bulgaria.

Corresponding author(s). E-mail(s): george.yanev@utrgv.edu;

Abstract

The sum of independent, but not necessary identically distributed, exponential random variables follows hypoexponential distribution. We study a situation when the rate parameters of the exponential variables are not all different from each other. We obtain a representation for the Laplace transform of the hypoexponential distribution in the case of two repeated parameter values. Applying this decomposition, we prove a characterization of the exponential distribution.

Keywords: characterizations, exponential distribution, hypoexponential distribution

MSC Classification: 62G30 , 62E10.

1 Introduction and main results

Sums of exponentially distributed random variables play a central role in many stochastic models of real-world phenomena. The *hypoexponential distribution* arises as a convolution of n independent exponential distributions each with their own rate λ_i, the rate of the i^{th} exponential distribution. Many processes can be divided into sequential phases. If the time periods spent in different phases are independent but not necessary identically distributed exponential variables, then the overall time is hypoexponential. For example, the absorption time for a finite-state Markov chain follows this distribution. We will write $X_i \sim \text{Exp}(\lambda_i)$, $\lambda_i > 0$, if X_i has density

$$f_i(x) = \lambda_i \mathrm{e}^{-\lambda_i x}, \quad x \geq 0 \quad \textit{(exponential distribution).}$$

The distribution of the sum $Y_n = X_1 + X_2 + \ldots + X_n$, where λ_i for $i = 1, \ldots, n$ are not all identical, is called (general) *hypoexponential distribution* (e.g., [1] and [2]). Assume that all λ_i's are distinct, i.e., $\lambda_i \neq \lambda_j$ when $i \neq j$. It is well-known that under this condition, the density of $Y_n \sim \text{HypoE}(\lambda_1, \lambda_2, \ldots, \lambda_n)$ is given by (see [3], p.309 and [4], p.40, Problem 12)

$$f_{Y_n}(x) = \sum_{j=1}^{n} \ell_j f_j(x), \quad x \geq 0.$$

Here the weight ℓ_j is defined as $\ell_j = \prod_{i=1, i \neq j}^{n} \lambda_i (\lambda_i - \lambda_j)^{-1}$. Thus, the density of the sum of independent exponential variables with distinct parameters is linear combination of the individual densities. For example, the density of Y_2 is

$$f_{Y_2}(x) = \frac{\lambda_2}{\lambda_2 - \lambda_1} f_1(x) + \frac{\lambda_1}{\lambda_1 - \lambda_2} f_2(x).$$

It is called *hypoexponential* distribution because its coefficient of variation is less than one, in contrast to the *hyperexponential* distribution which has coefficient of variation greater than one and the *exponential* distribution which has coefficient of variation equals one. An interesting connection with the Hirschman-Widder densities is discussed in [5].

Let X_1 and X_2 be two independent copies of a non-negative random variable X and $\mathsf{E}[X] < \infty$. If $X \sim \text{Exp}(\lambda)$, then $X_1 + X_2/2 \sim \text{HypoE}(\lambda, 2\lambda)$. It was proved in [6] that this property of the exponential distribution is not shared by any other continuous distribution, i.e., for $\lambda > 0$

$$X_1 + \frac{1}{2} X_2 \sim \text{HypoE}(\lambda, 2\lambda) \quad \text{iff} \quad X \sim \text{Exp}(\lambda).$$

The key argument in the proof is that the exponential distribution's LT

$$\Phi(t) = \frac{\lambda}{\lambda + t}, \qquad t \geq 0 \tag{1}$$

is the unique LT solution of the equation

$$\Phi(t)\Phi\left(\frac{t}{2}\right) = 2\Phi(t) - \Phi\left(\frac{t}{2}\right), \qquad t \geq 0.$$

Motivated by this result, in [7] we extended it in two directions: (i) for any number $n \geq 2$ of independent copies $X_1, X_2, \ldots, X_n$ of X, and (ii) for the linear combination $\mu_1 X_1 + \mu_2 X_2 + \ldots + \mu_n X_n$ with arbitrary positive and distinct coefficients $\mu_1, \mu_2, \ldots, \mu_n$. Namely, it was proved in [7], under some additional assumptions, that for $\lambda > 0$

$$\mu_1 X_1 + \ldots + \mu_n X_n \sim \text{HypoE}\left(\frac{\lambda}{\mu_1}, \ldots, \frac{\lambda}{\mu_n}\right) \quad \text{iff} \quad X \sim \text{Exp}(\lambda). \tag{2}$$

This characterization was obtained by showing that (1) is the unique solution of the LT equation

$$\Phi(\mu_1 t)\Phi(\mu_2 t) \cdots \Phi(\mu_n t) = \sum_{j=1}^{n} \bar{\ell}_j \Phi(\mu_j t), \qquad t \geq 0, \tag{3}$$

where $\bar{\ell}_j = \prod_{i=1, i \neq j}^{n} \mu_j (\mu_j - \mu_i)^{-1}$. Thus, the case of rate parameters λ_i's being all different from each other was settled down. The other extreme case of equal λ_i's, which leads to Erlang distribution of the sum, is trivial. Recently the case of both positive and negative μ_i's was considered in [8].

Does a similar characterization hold when the rate parameters λ_i's of $\text{HypoE}(\lambda_1, \lambda_2, \ldots, \lambda_n)$ *are not all different?* It is our goal in this paper to show that, at least in one particular case, the answer to this question is positive.

Without the condition that all parameters λ_i's are different, the hypoexponential density has a quite complex form (see [9]). This makes the analysis of this case difficult. Here, we turn to one intermediate situation, allowing two repeated values (ties) among λ_i's. More precisely, let $X_1, X_2, \ldots, X_{r+n}$ be independent copies of X with LT Φ. Consider the sum

$$Y_{r,n} := \sum_{k=1}^{r} w X_k + \sum_{k=r+1}^{r+n} X_k, \quad w > 0 \quad \text{and} \quad w \neq 1.$$

Due to the independence assumption, the LT of $Y_{r,n}$ equals $\Phi^r(wt)\Phi^n(t)$. If Φ is given by (1), then $\Phi^r(wt)\Phi^n(t)$ is a product of certain linear fractions. Therefore, we can decompose it into two sums involving the LT's of wX and $X \sim \text{Exp}(\lambda)$. To formulate the theorem below, we need to introduce the following sums for any integers $n \geq 1$ and $m \geq 0$

$$S_{n,m+1} = \sum_{k=1}^{n} S_{k,m}, \qquad S_{n,0} \equiv 1.$$

In particular, $S_{n,1} = n$ and $S_{n,2} = n(n+1)/2$.

Theorem 1 *If $X \sim \text{Exp}(\lambda)$, then for any positive integers r and n, and positive real $w \neq 1$, the random variable $Y_{r,n}$ has a hypoexponential distribution with LT $\Phi_{r,n}(t)$, $t \geq 0$, which satisfies*

$$\frac{(w-1)^{r+n}}{w^n}\Phi_{r,n}(t) = (w-1)^r\Phi^r(wt)\left(\frac{w-1}{w}\right)^n\Phi^n(t) \tag{4}$$
$$= \sum_{i=1}^{r} S_{n,r-i}(-1)^{r-i}(w-1)^i\Phi^i(wt) + (-1)^r\sum_{j=1}^{n} S_{n-j+1,r-1}\left(\frac{w-1}{w}\right)^j\Phi^j(t).$$

Theorem 1 shows that a necessary condition for $X \sim \text{Exp}(\lambda)$ is that its LT Φ is a solution of equation (4). In particular, setting $r = 1$ in (4), we have

$$(w-1)\Phi(wt)\left(\frac{w-1}{w}\right)^n\Phi^n(t) = (w-1)\Phi(wt) - \sum_{i=1}^{n}\left(\frac{w-1}{w}\right)^i\Phi^i(t). \tag{5}$$

The next theorem shows, under some additional assumptions, that (5) is both a necessary and sufficient condition for $X \sim \text{Exp}(\lambda)$.

Theorem 2 *Suppose that $X_1, X_2, \ldots, X_{n+1}$, $n \geq 1$, are independent copies of a non-negative and absolutely continuous random variable X. Assume further that X satisfies Cramér's condition: there is a number $t_0 > 0$ such that $\mathsf{E}[e^{tX}] < \infty$ for all $t \in (-t_0, t_0)$. For fixed positive integer n, fixed positive real $w \neq 1$, and $\lambda > 0$*

$$wX_1 + \sum_{k=2}^{n+1} X_k \sim \text{HypoE}\left(\frac{\lambda}{w}, \lambda, \ldots, \lambda\right) \quad \textit{iff} \quad X \sim \text{Exp}(\lambda). \tag{6}$$

In Section 2 and Section 3 we present the proofs of Theorem 1 and Theorem 2, respectively. The last section includes some concluding remarks.

2 Proof of Theorem 1

For simplicity and without loss of generality assume that $X \sim \text{Exp}(1)$. First, recalling that $\Phi(t) = (1+t)^{-1}$, we will show that the following linear fraction decomposition holds for $n \geq 1$ and $w \neq 1$

$$\Phi(wt)\Phi^n(t) = \frac{1}{(1+wt)(1+t)^n} \tag{7}$$
$$= \left(\frac{w}{w-1}\right)^n\frac{1}{1+wt} - \frac{w^n}{(w-1)^{n+1}}\sum_{j=1}^{n}\left(\frac{w-1}{w}\right)^j\frac{1}{(1+t)^j}.$$

If $n = 1$, then

$$
\begin{aligned}
\frac{1}{(1+wt)(1+t)} &= \frac{w}{(w-1)(1+wt)} - \frac{1}{(w-1)(1+t)} \\
&= \frac{w}{(w-1)(1+wt)} - \frac{w}{(w-1)^2}\left(\frac{w-1}{w}\right)\frac{1}{1+t}.
\end{aligned}
$$

Assuming that (7) holds for n, we obtain for the $(n+1)^{th}$ term

$$
\begin{aligned}
&\frac{1}{(1+wt)(1+t)^{n+1}} = \left[\frac{1}{(1+wt)(1+t)^n}\right]\frac{1}{1+t} \\
&= \left[\left(\frac{w}{w-1}\right)^n \frac{1}{1+wt} - \frac{w^n}{(w-1)^{n+1}}\sum_{j=1}^{n}\left(\frac{w-1}{w}\right)^j \frac{1}{(1+t)^j}\right]\frac{1}{1+t} \\
&= \left(\frac{w}{w-1}\right)^n \frac{1}{(1+wt)(1+t)} - \frac{w^n}{(w-1)^{n+1}}\sum_{j=1}^{n}\left(\frac{w-1}{w}\right)^j \frac{1}{(1+t)^{j+1}} \\
&= \left(\frac{w}{w-1}\right)^n \left[\frac{w}{(w-1)(1+wt)} - \frac{1}{(w-1)(1+t)}\right] \\
&\quad - \frac{w^{n+1}}{(w-1)^{n+2}}\sum_{j=1}^{n}\left(\frac{w-1}{w}\right)^{j+1} \frac{1}{(1+t)^{j+1}} \\
&= \left(\frac{w}{w-1}\right)^{n+1} \frac{1}{1+wt} - \frac{w^{n+1}}{(w-1)^{n+2}}\sum_{j=1}^{n+1}\left(\frac{w-1}{w}\right)^j \frac{1}{(1+t)^j},
\end{aligned}
$$

which completes the proof of (7). Multiplying both sides of (7) by $(w-1)^{n+1}/w^n$, we obtain (5), i.e., (4) is true for $r=1$ and any integer $n \geq 1$.

Next, we will prove (4) for any integer $r \geq 1$. Assuming (4) holds for r, we will prove it for $r+1$. Indeed,

$$
\begin{aligned}
&(w-1)^{r+1}\Phi^{r+1}(wt)\left(\frac{w-1}{w}\right)^n \Phi^n(t) \qquad (8) \\
&= (w-1)^r\Phi^r(wt)\left(\frac{w-1}{w}\right)^n \Phi^n(t)(w-1)\Phi(wt) \\
&= \left(\sum_{i=1}^{r} S_{n,r-i}(-1)^{r-i}(w-1)^i\Phi^i(wt) + (-1)^r\sum_{j=1}^{n} S_{n-j+1,r-1}\left(\frac{w-1}{w}\right)^j \Phi^j(t)\right) \\
&\qquad \times (w-1)\Phi(wt).
\end{aligned}
$$

Applying (5) with $r = 1$ and $n = j$, for the second term in the right-hand side, after multiplying it by $(w-1)\Phi(wt)$, we have

$$\sum_{j=1}^{n} S_{n-j+1,r-1} \left(\frac{w-1}{w}\right)^{j} \Phi^{j}(t)(w-1)\Phi(wt) \tag{9}$$
$$= \sum_{j=1}^{n} S_{n-j+1,r-1} \left((w-1)\Phi(wt) - \sum_{i=1}^{j} \left(\frac{w-1}{w}\right)^{i} \Phi^{i}(t)\right)$$
$$= S_{n,r}(w-1)^{i}\Phi^{i}(wt) - \sum_{j=1}^{n} S_{n-j+1,r-1} \sum_{i=1}^{j} \left(\frac{w-1}{w}\right)^{i} \Phi^{i}(t)$$
$$= S_{n,r}(w-1)^{i}\Phi^{i}(wt) - \sum_{i=1}^{n} S_{n-i+1,r} \left(\frac{w-1}{w}\right)^{i} \Phi^{i}(t).$$

Now, (8) and (9) imply (4) and thus the proof is complete.

3 Proof of Theorem 2

3.1 Auxiliary results

We will use the standard notation for the binomial coefficient: $\binom{k}{j}$ when $k \geq j$ and 0 when $j > k$, and $(x)_r := x(x-1)\dots(x-r+1)$ for the falling factorial.

Lemma 1 For any integers $n \geq 1$ and $1 \leq j \leq n$, and positive real $w \neq 1$

$$\binom{n}{j}\left(\frac{w}{w-1}\right)^{n} - \sum_{k=1}^{n-1}\left(\frac{w}{w-1}\right)^{k+1}\binom{k}{j-1} - \frac{1}{w-1}\sum_{k=1}^{n-1}\left(\frac{w}{w-1}\right)^{k}\binom{k}{j} = 0. \tag{10}$$

Proof. The identity (10) is equivalent to

$$\binom{n}{j}\left(\frac{w}{w-1}\right)^{n} - \left(\frac{w}{w-1}\right)^{j}\frac{1}{(j-1)!}\sum_{k=1}^{n-1}\left(\frac{w}{w-1}\right)^{k-j+1}(k)_{j-1} \tag{11}$$
$$-\frac{1}{w-1}\left(\frac{w}{w-1}\right)^{j}\frac{1}{j!}\sum_{k=1}^{n-1}\left(\frac{w}{w-1}\right)^{k-j}(k)_{j}$$
$$= \binom{n}{j}\left(\frac{w}{w-1}\right)^{n} - \left(\frac{w}{w-1}\right)^{j}\frac{1}{(j-1)!}\frac{\mathrm{d}^{j-1}}{\mathrm{d}x^{j-1}}\left(\sum_{k=1}^{n-1}x^{k}\right)\Bigg|_{x=w/w-1}$$
$$-\frac{1}{w-1}\left(\frac{w}{w-1}\right)^{j}\frac{1}{j!}\frac{\mathrm{d}^{j}}{\mathrm{d}x^{j}}\left(\sum_{k=1}^{n-1}x^{k}\right)\Bigg|_{x=w/w-1}.$$

Applying Leibniz formula for the m^{th} derivative of a product of functions (e.g., [11]) we obtain

$$\begin{aligned} \frac{x^m}{m!}\frac{\mathrm{d}^m}{\mathrm{d}x^m}\left(\sum_{k=1}^{n-1}x^k\right) &= \frac{x^m}{m!}\frac{\mathrm{d}^m}{\mathrm{d}x^m}\left(\frac{x^n-x}{x-1}\right) \\ &= \frac{x^m}{m!}\sum_{r=0}^{m}\binom{m}{r}(x^n-x)^{(m-r)}\left(\frac{1}{x-1}\right)^{(r)} \\ &= \frac{x^m}{m!}\sum_{r=0}^{m}\binom{m}{r}(-1)^r(n)_{m-r}r!\frac{x^{n-m+r}}{(x-1)^{r+1}} \\ &= x^n\sum_{r=0}^{m}(-1)^r\binom{n}{m-r}\frac{x^r}{(x-1)^{r+1}}. \end{aligned} \tag{12}$$

Setting $m=j$ and $x=w/(w-1)$, for the last term in (11) we have

$$\begin{aligned} &\frac{1}{w-1}\left(\frac{w}{w-1}\right)^j\frac{1}{j!}\frac{\mathrm{d}^j}{\mathrm{d}x^j}\left(\sum_{k=1}^{n-1}x^k\right)\Bigg|_{x=w/w-1} \\ &= \frac{1}{w-1}x^n\sum_{r=0}^{j}(-1)^r\binom{n}{j-r}\frac{x^r}{(x-1)^{r+1}}\Bigg|_{x=w/w-1} \\ &= \left(\frac{w}{w-1}\right)^n\sum_{r=0}^{j}(-1)^r\binom{n}{j-r}w^r \\ &= \left(\frac{w}{w-1}\right)^n\binom{n}{j}+\left(\frac{w}{w-1}\right)^n\sum_{r=1}^{j}(-1)^r\binom{n}{j-r}w^r. \end{aligned} \tag{13}$$

Similarly, for the other term in (11), one can obtain

$$\left(\frac{w}{w-1}\right)^j\frac{1}{(j-1)!}\frac{\mathrm{d}^{j-1}}{\mathrm{d}x^{j-1}}\left(\sum_{k=1}^{n-1}x^k\right)\Bigg|_{x=w/w-1} = \left(\frac{w}{w-1}\right)^n\sum_{r=1}^{j}(-1)^{r-1}\binom{n}{j-r}w^r. \tag{14}$$

Substituting (13) and (14) into (11), we obtain (10) and complete the proof.

Lemma 2 For any integers $n\geq 2$ and $j\geq 1$, and positive real $w\neq 1$

$$Q_{j,n}(w) := n\left(\frac{w}{w-1}\right)^n - \frac{1}{w-1}\sum_{k=0}^{n-1}\left(\frac{w}{w-1}\right)^k(w^j+k)\begin{cases} =0 & j=1, \\ \neq 0 & j\geq 2. \end{cases} \tag{15}$$

Proof. We have

$$
\begin{aligned}
Q_{j,n}(w) &= n\left(\frac{w}{w-1}\right)^n - \sum_{k=0}^{n-1}\frac{w^{k+j}}{(w-1)^{k+1}} - \sum_{k=1}^{n-1}\frac{kw^k}{(w-1)^{k+1}} \\
&= n\left(\frac{w}{w-1}\right)^n - w^{j-1}\sum_{k=0}^{n}\frac{w^{k+1}}{(w-1)^{k+1}} - \frac{w}{(w-1)^2}\sum_{k=1}^{n-1}\frac{kw^{k-1}}{(w-1)^{k-1}} \\
&= n\left(\frac{w}{w-1}\right)^n + w^{j-1}\left(w - \frac{w^{n+1}}{(w-1)^n}\right) - \frac{w}{(w-1)^2}\frac{d}{\mathrm{d}x}\left(\frac{x-x^n}{1-x}\right)\Bigg|_{x=w/(w-1)} \\
&= n\left(\frac{w}{w-1}\right)^n - \frac{w^{j+n}}{(w-1)^n} - w - \frac{nw^n}{(w-1)^n} + \frac{w^{n+1}}{(w-1)^n}
\end{aligned}
$$

and after some algebra, we obtain

$$
Q_{j,n}(w) = \frac{w(w^{j-1}-1)}{(w-1)^n}((w-1)^n - w^n),
$$

which implies (15).

3.2 Proof of the theorem

If $X \sim \mathrm{Exp}(\lambda)$, then $Y_{1,n} \sim \mathrm{HypoE}\left(w^{-1}\lambda, \lambda, \ldots, \lambda\right)$ by the definition of hypoexponential distribution. We will proceed with the proof of the opposite direction in (6). The case $n = 1$ is a particular case of (2) included in [7]. Let $n \geq 2$. Consider the function Ψ with the following series expansion

$$
\Psi(t) := \frac{1}{\Phi(t)} = \sum_{j=0}^{\infty} a_j t^j, \qquad t > 0. \tag{16}
$$

Note that, as a consequence of Cramér's condition, the above series is uniformly convergent in a proper neighborhood of $t = 0$ (see [10], p.240). To prove the theorem, it is sufficient to show that for some $\lambda > 0$

$$
\Psi(t) = 1 + \lambda^{-1}t, \tag{17}
$$

i.e., the coefficients of the series in (16) are

$$
a_0 = 1, \quad a_1 = \lambda^{-1} > 0, \quad a_j = 0, \quad j \geq 2. \tag{18}
$$

Clearly,

$$
a_0 = \frac{1}{\Phi(0)} = 1. \tag{19}
$$

Dividing both sides of (5) by $(w-1)^{n+1}w^{-n}\Phi(wt)\Phi^n(t)$ and changing the summation index, we obtain

$$H_{1,n}(t) := \left(\frac{w}{w-1}\right)^n \Psi^n(t) - \sum_{k=0}^{n-1} \frac{1}{w-1}\Psi(wt)\left(\frac{w}{w-1}\right)^k \Psi^k(t) = 1. \quad (20)$$

To calculate the coefficients a_j, we differentiate both sides of (20) with respect to t at $t=0$. It follows from (20) that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} H_{1,n}(t)|_{t=0} &= \left[n\left(\frac{w}{w-1}\right)^n - \frac{1}{w-1}\sum_{k=0}^{n-1}\left(\frac{w}{w-1}\right)^k (w+k)\right] a_1 \\ &=: c_1(n)a_1 = 0.\end{aligned}$$

It follows from (15) that $c_1(n) = 0$ and thus there exists $\lambda > 0$ such that

$$a_1 = \lambda^{-1}. \quad (21)$$

Differentiating (20) twice with respect to t at $t=0$, we have

$$\begin{aligned}&\frac{1}{2!}\frac{\mathrm{d}^2}{\mathrm{d}t^2} H_{1,n}(t)|_{t=0} \\ &= \left[\binom{n}{2}\left(\frac{w}{w-1}\right)^n - \frac{1}{w-1}\sum_{k=0}^{n-1}\left(\frac{w}{w-1}\right)^k \left(\binom{k}{1}w + \binom{k}{2}\right)\right] a_1^2 \\ &\quad + \left[\left(\frac{w}{w-1}\right)^n - \frac{1}{w-1}\sum_{k=0}^{n-1}\left(\frac{w}{w-1}\right)^k (w^2+k)\right] a_2 \\ &=: c_{1,2}(n)a_1^2 + c_2(n)a_2 = 0.\end{aligned}$$

Lemma 1 and Lemma 2 with $j = 2$ yield $c_{1,2}(n) = 0$ and $c_2(n) \neq 0$, respectively. Therefore,

$$a_2 = 0. \quad (22)$$

It remains to prove that $a_j = 0$ for all $j \geq 3$. We will need the general Leibniz rule for differentiating a product of functions. Denote by $v^{(j)}(x)$ the j^{th} derivative of $v(x)$; $v^{(0)}(x) := v(x)$. Define a multi-index set $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ as a n-tuple of non-negative integers. Denote $\|\boldsymbol{\alpha}\| = \alpha_1 + \alpha_2 + \ldots + \alpha_n$ and $\Lambda_j := \{\boldsymbol{\alpha} : \|\boldsymbol{\alpha}\| = j\}$. The j^{th} derivative (when exists) of the product $v_1(t)v_2(t)\cdots v_n(t)$ is given by (e.g. [11])

$$\frac{\mathrm{d}^j}{\mathrm{d}t^j}\prod_{i=1}^n v_i(t) = \sum_{\Lambda_j}\left(\frac{j!}{\alpha_1!\alpha_2!\cdots\alpha_n!}\prod_{i=1}^n v_i^{(\alpha_i)}(t)\right). \quad (23)$$

Let us write Λ_j as union of three disjoint subsets as follows:

$$\Lambda_j = \Lambda'_j \cup \Lambda''_j \cup \Lambda'''_j,$$

where

$$\begin{aligned}
\Lambda'_j &= \{\|\boldsymbol{\alpha}\| = j : \text{ only one of } \{\alpha_1, \alpha_2, \dots, \alpha_n\} \text{ equals } j \text{ (others are zeros)}\}\\
\Lambda''_j &= \{\|\boldsymbol{\alpha}\| = j : \text{ exactly } j \text{ of } \{\alpha_1, \alpha_2, \dots, \alpha_n\} \text{ equal } 1 \text{ (others are zeros)}\}\\
\Lambda'''_j &= \{\|\boldsymbol{\alpha}\| = j : \text{ there is an index } \alpha_i \text{ with } 2 \le \alpha_i < j\}.
\end{aligned}$$

Notice that by definition, Λ''_j is not empty only if $j \le n$.

We will proceed by induction with respect to the index $j \ge 2$ of a_j. For $j = 2$ we have already proved that $a_2 = 0$. Assuming $a_i = 0$ for $2 \le i \le j-1$, we will show that $a_j = 0$. Since $a_0 = 1$, applying (23), we obtain

$$\begin{aligned}
\frac{1}{j!}\frac{\mathrm{d}^j}{\mathrm{d}t^j}\Psi^n(t)|_{t=0} &= \sum_{\Lambda_j}\left(\prod_{i=1}^n a_{\alpha_i}\right) = \sum_{\Lambda'_j}(\cdot) + \sum_{\Lambda''_j}(\cdot) + \sum_{\Lambda'''_j}(\cdot)\\
&= \binom{n}{1} a_j a_0^{n-1} + \binom{n}{j} a_1^j a_0^{n-j}\\
&= n a_j + \binom{n}{j} a_1^j.
\end{aligned}$$

Notice that $\sum_{\Lambda'''_j}(\cdot) = 0$ by the induction assumption. Also,

$$\begin{aligned}
&\frac{1}{j!}\frac{\mathrm{d}^j}{\mathrm{d}t^j}\Psi(wt)\Psi^k(t)|_{t=0} = \sum_{\Lambda_j}\left(w^{\alpha_{k+1}} a_{\alpha_{k+1}} \prod_{i=1}^k a_{\alpha_i}\right) = \sum_{\Lambda'}(\cdot) + \sum_{\Lambda''}(\cdot) + \sum_{\Lambda'''}(\cdot)\\
&= \left(w^j a_0^k a_j + \binom{k}{1} a_j a_0^{k-1} a_0\right) + \left(\binom{k}{j-1} a_1^{j-1} a_0^{k-j+1} w a_1 + \binom{k}{j} a_1^j a_0^{k-j} a_0\right)\\
&= \left(\binom{k}{j-1} w + \binom{k}{j}\right) a_1^j + \left(w^j + k\right) a_j.
\end{aligned}$$

Therefore, grouping the coefficients in front of a_1^j and a_j, we write

$$\begin{aligned}
&\frac{1}{j!}\frac{\mathrm{d}^j}{\mathrm{d}t^j} H_{1,n}(t)|_{t=0}\\
&= \left[\binom{n}{j}\left(\frac{w}{w-1}\right)^n - \frac{1}{w-1}\sum_{k=0}^{n-1}\left(\frac{w}{w-1}\right)^k \left(\binom{k}{j-1} w + \binom{k}{j}\right)\right] a_1^j\\
&\quad + \left[n\left(\frac{w}{w-1}\right)^n - \frac{1}{w-1}\sum_{k=0}^{n-1}\left(\frac{w}{w-1}\right)^k (w^j + k)\right] a_j\\
&=: c_{1,j}(n) a_1^j + c_j(n) a_j = 0.
\end{aligned}$$

It follows from Lemma 1 that $c_{1,j}(n) = 0$ and hence

$$c_{1,j}(n)a_1^j + c_j(n)a_j = c_j(n)a_j = 0.$$

Finally, according to Lemma 2, $c_j(n) \neq 0$ when $j \geq 2$, which implies

$$a_j = 0, \qquad j \geq 2. \tag{24}$$

Now, (19), (21), (22), and (24) lead to (18), which completes the proof.

4 Concluding remarks

In this paper we continue the study of the relation between the exponential and hypoexponential distributions, initiated in [6] and extended in [7]. The obtained characterization complements those in the above papers. Here we deal with a situation where the rate parameters λ_i's in a convolution of exponential variables are not all different from each other. First, we obtain a representation for the LT of the hypoexponential distribution in the case of two coinciding parameters' values. Applying this decomposition, we prove a characterization of the exponential distribution. The obtained result is of interest itself, however it can also serve as a basis for further investigations of more complex compositions of the rate parameters. In particular, the question whether or not equation (4) with $r > 1$ is a sufficient condition for Φ to be a LT of the exponential distribution is still open.

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