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Using restrictions to accept or reject solutions of radical equations.

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The standard technique for solving equations with radicals is to square both sides of the equation as many times as necessary to eliminate all radicals. Because the procedure violates logical equivalence, it results in extraneous solutions that do not satisfy the original equation, making it necessary to check all solutions against the original equation. We propose alternative solution procedures that are rigorous and simple to execute where the extraneous solutions can be identified without verification against the original equation. In this article, we review previous literature, establish and illustrate rigorous solution procedures for radical equations of depth 1 (i.e. equations where all radicals can be eliminated in one step), and deal with an ambiguity concerning the definition of real-valued solutions to radical equations. An application to defining the inverse function, resulting in a parametric radical equation, is also explained.

1. Introduction

This article presents and proves the rigorous theory underlying the solution of three simple types of radical equations and illustrates it with several solved examples. For our discussion here we define a radical equation as an equation where the unknown variable appears at least once inside a square root. We shall not concern ourselves with cubic roots or any other higher-order roots.

In contemporary textbooks the procedure that is typically presented is to square both sides of the equation, combined with other manipulations, as many times as necessary to eliminate all square roots. The resulting polynomial equation is solved, and then we are instructed to expect that some of the roots of the polynomial are *extraneous solutions* that do not satisfy the original equation. To eliminate these extraneous solutions, each root of the polynomial equation has to be substituted to the original radical equation to check whether it satisfies it. This approach has persisted in textbooks for more than a century, even though it is unsatisfactory for at least the following three reasons: (1) From a theoretical standpoint, it is fair to say that a correct procedure should not result in "extraneous" solutions that don't work; (2) from a practical standpoint, unless the candidate solutions are integers, it can be quite tedious to verify them directly on the original equation; (3) with parametric radical equations, a solution that is rejected for some values of the parameter, may be a valid solution for other values of the parameter, and working that out by direct verification is also impractical. The goal of this article is to present and illustrate more rigorous procedures for solving radical equations where all extraneous solutions are identified and eliminated during the solution process and not with backsubstitution to the original equation. We will state and prove the properties that justify the proposed procedures. Our scope will be limited to finding only real-valued solutions.

The problem of extraneous solutions, which afflicts both fractional and radical equations, has a rich and interesting history that has been reviewed in detail by Manning [1]. In short, a rigorous treatment of extraneous solutions (in theorem-proof style) first emerged in textbooks published during the 1890s. Radical equations themselves first appeared as

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a distinct textbook topic during the 1860s, however awareness of the problem of extraneous solutions in radical equations emerged as a result of the consensus to define $\sqrt{a}$ for $a \in [0, +\infty)$ as the positive solution of the equation $x^2 - a = 0$, which happened during the 1880s. Previously, $\sqrt{a}$ had a multi-valued interpretation that has survived to the contemporary era only in the context of complex analysis, for roots of complex numbers. Unfortunately, the rigor of the 1890s textbooks was forgotten in later editions of the same textbooks during the 1900s. Attention to the extraneous solutions problem was rekindled by Taylor in 1910 [2], Hegeman in 1922 [3], Bruce in 1931 [4], and Allendoerfer in 1966 [5], but a lot of their insights have been ignored, at least by American contemporary textbooks and current teaching practices.

Nevertheless, throughout the course of the twentieth century there has been some limited but interesting literature on radical equations. Huff and Barrow stated and proved a theorem that provides the general solution for equations that follow the form $\sqrt{ax+v} + \sqrt{cx+d} = A$ [6]. Nagase gave a similar result [7] for equations that follow the form $\sqrt{ax+b} = cx + d$, based on a solution technique by Bompart [8] and Roberti [9] that will be reviewed in this article. There has also been some interest in the inverse problem of constructing a radical equation from the solutions that we would like it to have. Beach published a short note [10] showing how to construct radical equations following the forms $\sqrt{ax^2+bx+c} = Ax+B$, and $\sqrt{ax+b} \pm \sqrt{cx+d} = A$, and $\sqrt{ax+b} + \sqrt{cx+d} = \sqrt{Ax+B}$ from the roots r_1, r_2 of the quadratic equation to which they eventually reduce. The third equation form was later revisited by Schwarz [11] in a paper that he co-authored with two of his high-school students, where they showed how to construct radical equations folowing that form without any extraneous solutions.

In this article we will present, prove, and illustrate the theory that underlies the rigorous solution of radical equations that follow one of the following forms:

$$\sqrt{f(x)} = \sqrt{g(x)}, \tag{1}$$

$$\sqrt{f(x)} = g(x), \tag{2}$$

$$\sqrt{f_1(x)} + \sqrt{f_2(x)} + \cdots + \sqrt{f_n(x)} = 0. \tag{3}$$

Here, the functions listed are either polynomial or rational functions. Because in all of the above equation forms, all radicals can be eliminated with one equivalence step, we will designate these as *radical equations with depth 1*. A few additional radical equation forms that require two equivalence steps to eliminate all radicals will be designated as *radical equations with depth 2*, and they will be considered in future work.

The main results presented in this article are Proposition 3.3, Proposition 4.2, and Lemma 2.2 establishing the solution procedures for radical equations following the forms of Eq. (1), Eq. (2), and Eq. (3) correspondingly. I was first exposed to these techniques by my high-school teacher, Mr. Alexandros Pistofidis, during the standard algebra course for 10th and 11th grade in Greece during the years 1990 and 1991. The course was similar to College Algebra and Precalculus taught at universities and colleges in the United States, except more rigorous and with a lot of more detail. Over a two-year period, the topics included logic, quantifiers, method of induction, applications of logic to the theory of the quadratic, proving identities and inequalities, simplifying expressions with radicals. solving equations, inequalities, systems of equations, and systems of inequalities, functions, arithmetic and geometric sequences, exponentials and logarithms. The particular techniques for the solution of radical equations, presented in this article, are not currently part of the curriculum mandated by the Greek government [12], however they were included in a set of very innovative lecture notes [13, 14] that were distributed to us in class. The solution techniques proposed in this article are described in Ref. [13], except without a formal justification. Example 4.1 first appeared in Ref. [13], whereas all other solved examples are original. An informal explanation of the technique proposed to solve radical equations that follow the form of Eq. (2) was also given much later by Gurevich [15]. The historical development of these particular solution procedures is not known to me beyond what has been detailed above.

Another original contribution of this article, going beyond Pistofidis [13, 14], that I have not seen previously addressed in textbooks or other literature, is the need to carefully define the concept of a real-valued solution to a radical equation. Typically, when we restrict our interest to real-valued solutions, we do so because we have some reason to want to avoid square roots of negative numbers. So, should we accept or reject a real-valued solution that verifies when substituted to the original radical equation but results in square roots of negative numbers during the verification?

For example, consider the equation $\sqrt{1-3x} = \sqrt{x-7}$, which follows the form of Eq. (1). Standard procedure gives the solution $x = 2$, which satisfies the original equation, except that it does so by yielding imaginary radicals: $\sqrt{1-3x} = \sqrt{x-7} = i\sqrt{5}$. Do we want to accept or reject this solution? That depends on what we want to do with it. A formalist viewpoint can be that the solution should be accepted, if our goal is to find the elements of the set $S = \{x \in \mathbb{R} \mid \sqrt{1-3x} = \sqrt{x-7}\}$. Another viewpoint can be that if we define real-valued functions $f : A \to \mathbb{R}$ and $g : B \to \mathbb{R}$ with $f(x) = \sqrt{1-3x}$ and $g(x) = \sqrt{x-7}$ and with the widest possible implied domains $A = (-\infty, 1/3]$ and $B = [7, +\infty)$, and are interested in the set S of all points where the graphs of f and g intersect, then the formal definition of S reads $S = \{x \in A \cap B \mid f(x) = g(x)\}$ and the solution $x = 2$ should be rejected, because $A \cap B = \emptyset$, which implies that no solutions at all can be accepted. Typically, the equations we solve tend to arise from problems that concern functions, so this second more restrictive viewpoint is oftentimes the relevant one.

For the purposes of this article we will designate all real numbers that satisfy the equation in the formal sense, allowing for roots of negative numbers, as *formal solutions*. From amongst the formal solutions, those solutions that do not result in roots of negative numbers, when substituted to the original radical equation, will be designated, for lack of appropriate standard terminology, as *strong solutions*. We will see in this article that the set of all strong solutions is always equal to the set of all formal solutions for the radical equations that follow the form of Eq. (2) and Eq. (3). However, the two sets may be different for radical equations that follow the form of Eq. (1) and for other more complicated forms with depth 2 that will be studied in Part 2 of this article series.

The article is organized as follows. Preliminary results and notation are introduced in Section 2. Section 3 discusses the solution of radical equations following the form of Eq. (1), Section 4 discusses the solution of radical equations following the form of Eq. (2), and Section 5 discusses the solution of radical equations following the form of Eq. (3). In Section 6 we show an interesting application of the theory of radical equations to the problem of finding the definition and the domain of the inverse of a function whose definition involves square roots of polynomials or rational functions. The article concludes with Section 7.

2. Preliminaries

Let $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$ be the set of all natural numbers and for any $n \in \mathbb{N} - \{0\}$ introduce the notation $[n] = \{k \in \mathbb{N} \mid 1 \leq k \leq n\} = \{1, 2, 3, \ldots, n\}$. Our point of departure is the following well-known property of real numbers, which we state without proof:

$$\forall a, b \in \mathbb{R} : (ab = 0 \Longleftrightarrow (a = 0 \lor b = 0)). \tag{4}$$

Note that we will use explicitly written quantifier and boolean algebra notation for all mathematical statements in this article. The proofs will be presented in detail and can serve as good examples of proofwriting technique that are accessible with very basic algebra background. From Eq. (4) we derive Lemma 2.1 and Lemma 2.2, which we then use to establish the propositions given in Section 3 and Section 4. Furthermore, Lemma 2.2 is used directly in the solution procedure described in Section 5.

Lemma 2.1: $\forall a, b \in [0, +\infty) : (a^2 = b^2 \iff a = b)$.

Proof: Let $a, b \in [0, +\infty)$ be given. We note that

$$a^2 = b^2 \iff a^2 - b^2 = 0 \iff (a-b)(a+b) = 0 \iff \iff a - b = 0 \lor a + b = 0 \iff a = b \lor a = -b. \tag{5}$$

We will now show the equivalence

$$(a = b \lor a = -b) \iff a = b. \tag{6}$$

($\Longrightarrow$): We assume that $a = b \lor a = -b$ and distinguish between the following cases:
Case 1: We assume that $a = b$. Then the conclusion is given immediately by hypothesis.
Case 2: We assume that $a = -b$. Then it follows that $b \in [0, +\infty) \Longrightarrow b \geq 0 \Longrightarrow a = -b \leq 0 \Longrightarrow a \leq 0$ and $a \in [0, +\infty) \Longrightarrow a \geq 0$. We conclude that $a = 0$ and therefore $b = -a = -0 = 0 = a$.

In both cases we conclude that $a = b$.

($\Longleftarrow$): Assume that $a = b$. It trivially follows that $a = b \lor a = -b$.

This proves the claim given by Eq. (6). Combining Eq. (5) and Eq. (6) it follows that $a^2 = b^2 \iff a = b$. □

Lemma 2.2: *Let $n \in \mathbb{N} - \{0\}$ be a natural number. Then, it follows that*

$$\forall a_1, \ldots, a_n \in \mathbb{R} : \left(\sum_{k=1}^{n} \sqrt{a_k} = 0 \iff \forall k \in [n] : a_k = 0\right). \tag{7}$$

Proof: Let $a_1, \ldots, a_n \in \mathbb{R}$ be given. We define

$$A = \{k \in [n] \mid a_k \geq 0\} \quad \text{and} \quad B = \{k \in [n] \mid a_k \leq 0\}, \tag{8}$$

and note that $\forall k \in A : \sqrt{a_k} \geq 0$ and $\forall k \in B : \sqrt{|a_k|} \geq 0$.
($\Longrightarrow$): Assume that $\sum_{k=1}^{n} \sqrt{a_k} = 0$. Let $k \in [n]$ be given. To show that $a_k = 0$, we assume that $a_k \neq 0$ in order to derive a contradiction. We distinguish between the following cases.
Case 1: Assume that $a_k > 0$. It follows that $k \in A$, and we note that

$$\sum_{m \in [n]} \sqrt{a_m} = 0 \Longrightarrow \mathrm{Re}\Big(\sum_{m \in [n]} \sqrt{a_m}\Big) = 0 \Longrightarrow \sum_{m \in A} \sqrt{a_m} = 0, \tag{9}$$

and

$$a_k > 0 \Longrightarrow \sqrt{a_k} > 0 \Longrightarrow \sum_{m \in A} \sqrt{a_m} \geq \sqrt{a_k} > 0 \Longrightarrow \sum_{m \in A} \sqrt{a_m} > 0, \tag{10}$$

which leads to a contradiction between Eq. (9) and Eq. (10).
Case 2: Assume that $a_k < 0$. It follows that $k \in B$, and we note that

$$\sum_{m \in [n]} \sqrt{a_m} = 0 \Longrightarrow \mathrm{Im}\Big(\sum_{m \in [n]} \sqrt{a_m}\Big) = 0 \Longrightarrow \sum_{m \in B} \sqrt{a_m} = 0$$

$$\Longrightarrow \sum_{m\in B} i\sqrt{|a_m|} = 0 \Longrightarrow \sum_{m\in B} \sqrt{|a_m|} = 0, \tag{11}$$

and

$$\begin{aligned} a_k < 0 \Longrightarrow |a_k| > 0 \Longrightarrow \sqrt{|a_k|} > 0 \Longrightarrow \sum_{m\in B} \sqrt{|a_m|} \geq \sqrt{|a_k|} > 0 \\ \Longrightarrow \sum_{m\in B} \sqrt{|a_m|} > 0, \end{aligned} \tag{12}$$

which also leads to a contradiction between Eq. (11) and Eq. (12).

From both cases above we conclude that $a_k = 0$. It follows overall that $\forall k \in [n] : a_k = 0$.

($\Longleftarrow$): Assume that $\forall k \in [n] : a_k = 0$. Then it trivially follows that $\sum_{k=1}^{n} \sqrt{a_k} = 0$. □

3. Equations with two equal radicals

The solution technique, based on the restriction set method, for finding the strong solutions for equations of the form $\sqrt{f(x)} = \sqrt{g(x)}$ is as follows:

(1) We find the domain A of the equation by requiring that $f(x) \geq 0$ and $g(x) \geq 0$:

$$\begin{cases} f(x) \geq 0 \\ g(x) \geq 0 \end{cases} \Longleftrightarrow \cdots \Longleftrightarrow x \in A. \tag{13}$$

(2) We solve the equation by squaring both sides:

$$\sqrt{f(x)} = \sqrt{g(x)} \Longleftrightarrow f(x) = g(x) \Longleftrightarrow \cdots \Longleftrightarrow x \in S_0. \tag{14}$$

(3) We accept the solutions in S_0 that also belong to A. Consequently, the solution set for all strong solutions is given by $S = S_0 \cap A$.

The solutions rejected by this procedure are still formal solutions of the original equation, however they are not strong solutions. We illustrate the solution technique with Example 3.1 and Example 3.2. The theoretical justification is given by Proposition 3.3. Note that the braces used in Eq. (15) and in the exposition below represent the boolean conjunction operation (i.e. the logical "and"). Furthermore, in all of the example solutions and proposition proofs given in the following, we use boolean notation throughout the solution process in conjunction with the appropriate amount of narrative that should be present in a well-written mathematical argument.

Example 3.1 Find all strong solutions of the equation $\sqrt{3x+1} = \sqrt{2-x}$.

Solution: We require that

$$\begin{cases} 3x+1 \geq 0 \\ 2-x \geq 0 \end{cases} \Longleftrightarrow \begin{cases} 3x \geq -1 \\ x \leq 2 \end{cases} \Longleftrightarrow \begin{cases} x \geq -1/3 \\ x \leq 2 \end{cases} \Longleftrightarrow x \in [-1/3, 2], \tag{15}$$

therefore the domain of the equation is the interval $A = [-1/3, 2]$. Solving the equation gives:

$$\sqrt{3x+1} = \sqrt{2-x} \Longleftrightarrow 3x+1 = 2-x \Longleftrightarrow 3x + x = 2 - 1 \tag{16}$$

$$\Longleftrightarrow 4x = 1 \Longleftrightarrow x = 1/4. \tag{17}$$

The solution is accepted, since $1/4 \in A$. Consequently, the set of all strong solutions to the equation is $S = \{1/4\}$. □

Example 3.2 Find all strong solutions of the equation $\sqrt{2x+3} = \sqrt{3x+5}$

Solution : We require that

$$\begin{cases} 2x+3 \geq 0 \\ 3x+5 \geq 0 \end{cases} \iff \begin{cases} 2x \geq -3 \\ 3x \geq -5 \end{cases} \iff \begin{cases} x \geq -3/2 \\ x \geq -5/3 \end{cases} \tag{18}$$

$$\iff x \in [-3/2, +\infty) \cap [-5/3, +\infty) \iff x \in [-3/2, +\infty), \tag{19}$$

and therefore the domain of the equation is $A = [-3/2, +\infty)$. Solving the equation gives:

$$\sqrt{2x+3} = \sqrt{3x+5} \iff 2x+3 = 3x+5 \iff 3x-2x = 3-5 \iff x = -2. \tag{20}$$

This solution is rejected, because $-2 \notin A$, consequently the equation has no strong solutions □

In connection with Example 3.2, it should be emphasized that the rejected solution is, in fact, a formal solution of the radical equation, and as such it would have been accepted if the problem was to find the set of all formal solutions.

Proposition 3.3: *Consider the equation $\sqrt{f(x)} = \sqrt{g(x)}$ with $f : A \to \mathbb{R}$ and $g : B \to \mathbb{R}$ polynomial or rational functions with $A \subseteq \mathbb{R}$ and $B \subseteq \mathbb{R}$. The set S_1 of all strong solutions is given by*

$$S_1 = S_0 \cap A, \tag{21}$$

$$S_0 = \{x \in A \cap B \mid f(x) = g(x)\}, \tag{22}$$

$$A_1 = \{x \in A \cap B \mid |f(x) \geq 0 \wedge g(x) \geq 0\}, \tag{23}$$

and the set S_2 of all formal solutions is given by $S_2 = S_0$.

Proof : *Strong solutions:* We begin with the observation that in order for x to be a strong solution of the equation $\sqrt{f(x)} = \sqrt{g(x)}$ it has to satisfy the condition $f(x) \geq 0 \wedge g(x) \geq 0$ which is equivalent to $x \in A_1$. We can therefore rule out all $x \notin A_1$ as possible strong solutions to our equation.

Let $x \in A_1$ be given. It follows that

$$\begin{cases} f(x) \geq 0 \\ g(x) \geq 0 \end{cases} \implies \begin{cases} \sqrt{f(x)} \geq 0 \\ \sqrt{g(x)} \geq 0, \end{cases} \tag{24}$$

and therefore, using Lemma 2.1, we have

$$\sqrt{f(x)} = \sqrt{g(x)} \iff (\sqrt{f(x)})^2 = (\sqrt{g(x)})^2 \qquad \text{[via Lemma 2.1]} \tag{25}$$

$$\iff f(x) = g(x) \iff x \in S_0. \tag{26}$$

We conclude that the set of all strong solutions to our equation is given by $S_1 = S_0 \cap A$.

Formal solutions: We now look for additional solutions that violate the requirement $f(x) \geq 0 \wedge g(x) \geq 0$. Let $x \in A \cap B$ be given. We distinguish between the following cases.

Case 1: Assume that $f(x) < 0 \wedge g(x) \geq 0$. Then

$$\mathrm{Im}(\sqrt{f(x)}) = \mathrm{Im}(\sqrt{g(x)}) \qquad \text{[via the equation]} \tag{27}$$

$$= 0, \qquad [\text{via } g(x) \geq 0] \tag{28}$$

which is a contradiction because $f(x) < 0 \Longrightarrow \text{Im}(\sqrt{f(x)}) \neq 0$ consequently this case contributes no additional solutions.
Case 2: Assume that $f(x) \geq 0 \wedge g(x) < 0$. Then, with similar argument, this also leads to the same contradiction, and therefore this case too contributes no additional solutions.
Case 3: Assume that $f(x) < 0 \wedge g(x) < 0$. Then

$$\sqrt{f(x)} = \sqrt{g(x)} \Longleftrightarrow i\sqrt{-f(x)} = i\sqrt{-g(x)} \tag{29}$$

$$\Longleftrightarrow \sqrt{-f(x)} = \sqrt{-g(x)} \tag{30}$$

$$\Longleftrightarrow (\sqrt{-f(x)})^2 = (\sqrt{-g(x)})^2 \qquad [\text{via Lemma 2.1}] \tag{31}$$

$$\Longleftrightarrow -f(x) = -g(x) \Longleftrightarrow f(x) = g(x), \tag{32}$$

consequently this case contributes the solutions

$$S_0' = \{x \in A \cap B \mid f(x) = g(x) < 0\}. \tag{33}$$

From the above argument, we conclude that the set of all formal solutions is given by

$$S_2 = \{x \in A \cap B \mid f(x) = g(x) \geq 0\} \cup \{x \in A \cap B \mid f(x) = g(x) < 0\} \tag{34}$$

$$= \{x \in A \cap B \mid f(x) = g(x)\} = S_0. \tag{35}$$

□

4. Equations with one radical

For equations with one radical that take the form $\sqrt{f(x)} = g(x)$, an interesting result is that the set of all strong solutions is always equal to the set of all formal solutions. Furthermore, it is not necessary to require the restriction $f(x) \geq 0$ when determining the domain of the equation.

The solution technique is as follows:

(1) The domain of the equation is determined by requiring that the left-hand-side of the equation be greater or equal to zero:

$$g(x) \geq 0 \Longleftrightarrow \cdots \Longleftrightarrow x \in A. \tag{36}$$

(2) We solve the equation by squaring both sides:

$$\sqrt{f(x)} = g(x) \Longleftrightarrow f(x) = [g(x)]^2 \Longleftrightarrow \cdots \Longleftrightarrow x \in S_0. \tag{37}$$

(3) We accept only those solutions of S_0 that belong also to the domain A of the equation. Consequently, the solution set is given by $S = S_0 \cap A$.

We stress again that the set of all strong solutions and the set of all formal solutions coincide and are both given by $S = S_0 \cap A$. Unlike the case previously discussed in Section 2, all solutions that are rejected by the above procedure do not satisfy the original equation, either in the strong sense or in the formal sense. An intuitive way to explain all this is to note that any solution obtained from solving the equation $f(x) = [g(x)]^2$ will satisfy $f(x) \geq 0$ since $f(x)$ is set equal to the square of a real number. That in turn retroactively justifies squaring both sides of the original radical equation after imposing the restriction $g(x) \geq 0$. The proof of Proposition 4.2 also considers why no solutions are possible with $g(x) < 0$. The solution technique is illustrated via Example 4.1 and is formally justified by Proposition 4.2, both given in the following.

Example 4.1 Solve the equation $\sqrt{x^2 - 2x + 6} + 3 = 2x$.

Solution: We note that

$$\sqrt{x^2 - 2x + 6} + 3 = 2x \iff \sqrt{x^2 - 2x + 6} = 2x - 3. \tag{38}$$

To determine the domain of the equation we require that

$$2x - 3 \geq 0 \iff 2x \geq 3 \iff x \geq 3/2 \iff x \in [3/2, +\infty) \equiv A. \tag{39}$$

Solving the equation for all $x \in A$ gives

$$\text{Eq. (38)} \iff x^2 - 2x + 6 = (2x - 3)^2 \iff x^2 - 2x + 6 = 4x^2 - 12x + 9 \tag{40}$$

$$\iff (4 - 1)x^2 + (-12 + 2)x + (9 - 6) = 0 \iff 3x^2 - 10x + 3 = 0 \tag{41}$$

$$\iff \cdots \iff x = 3 \vee x = 1/3. \tag{42}$$

The solution $x = 3 \in A$ is accepted and the solution $x = 1/3 \notin A$ is rejected. It follows that the solution set of all strong solutions or all formal solutions is given by $S = \{3\}$. □

Proposition 4.2: *Consider the equation* $\sqrt{f(x)} = g(x)$ *with* $f : A \to \mathbb{R}$ *and* $g : B \to \mathbb{R}$ *polynomial or rational functions with* $A \subseteq \mathbb{R}$ *and* $B \subseteq \mathbb{R}$. *The set* S_1 *of all strong solutions and the set* S_2 *of all formal solutions to the equation are given by*

$$S_1 = S_2 = S_0 \cap A_1, \tag{43}$$

$$S_0 = \{x \in A \cap B \mid f(x) = [g(x)]^2\}, \tag{44}$$

$$A_1 = \{x \in A \cap B \mid g(x) \geq 0\}. \tag{45}$$

Proof: First, we note that the equation does not have any formal solutions with $f(x) < 0$. To show a contradiction, let us assume that $x \in A \cap B$ is a formal solution such that $f(x) < 0$. Then

$$\text{Im}(g(x)) = \text{Im}(\sqrt{f(x)}) \qquad \text{[via the equation]} \tag{46}$$

$$\neq 0. \qquad \text{[via } f(x) < 0\text{]} \tag{47}$$

This is a contradiction, since g has been defined as a real-valued mapping $g : B \to \mathbb{R}$, and therefore $\text{Im}(g(x)) = 0$. It follows that all formal solutions of the original equation will satisfy the inequality $f(x) \geq 0$. This also implies that any formal solutions that may exist will be found under the next case, and as such, they will also be strong solutions.

To solve the equation, let $x \in A \cap B$ be given such that $f(x) \geq 0$. We distinguish between the following cases:
Case 1: Assume that $g(x) < 0$. Since $f(x) \geq 0 \implies \sqrt{f(x)} \geq 0$, it follows that $\sqrt{f(x)} \neq g(x)$, consequently the equation does not have any formal solutions with $g(x) < 0$.
Case 2: Assume that $g(x) \geq 0$. Since we also have $f(x) \geq 0 \implies \sqrt{f(x)} \geq 0$, we can use

Lemma 2.1 and write

$$\sqrt{f(x)} = g(x) \iff f(x) = [g(x)]^2 \iff x \in S_0. \tag{48}$$

From the above argument we conclude that the set S_1 of all strong solutions and the set S_2 of all formal solutions are given by

$$S_1 = S_2 = \{x \in A \cap B \mid |f(x) = [g(x)]^2 \wedge g(x) \geq 0\} = S_0 \cap A_1. \tag{49}$$

□

It is worth mentioning that for equations of the form $\sqrt{ax+b} = g(x)$, where $g(x)$ is a polynomial or rational function, an alternate method by Bompart [8] reduces them to equations of the form $\sqrt{Ax+B} = C$ with A, B, C constants. Then, it is clear that if $C < 0$, then there are no solutions, and when $C \geq 0$, we are justified by Lemma 2.1 to square both sides of the equation without introducing any extraneous solutions by doing so. The idea is to apply an auxiliary change of variables $y = \sqrt{ax+b}$, solve for x (with the implicit assumption that $y \geq 0$) to obtain $x = (1/a)(y^2 - b)$, and substitute to the original equation thereby obtaining an equivalent equation $y = g((1/a)(y^2 - b))$ which is going simplify either to a polynomial or to a rational equation. Once we find all y solutions, we solve the corresponding equations $\sqrt{ax+b} = y$ with respect to x for all solutions y that satisfy $y \geq 0$.

A similar technique was later proposed by Roberti [9] that bypasses the auxiliary substitution, but works only when $g(x)$ is also a linear function with respect to x. Nagase [7] then used the technique of Roberti [9] to give the general solution to equations of the form $\sqrt{ax+b} = cx + d$. As for the original technique by Bompart [8], it is very nice when it works but ineffective when the radical contains a quadratic or a polynomial of higher order. On the other hand, although not noted by either Bompart [8] or Roberti [9], this substitution technique will also work with radical equations that follow the form $\sqrt{(ax+b)/(cx+d)} = g(x)$ with $g(x)$ a polynomial or rational function. Example 4.3 illustrates the technique.

Example 4.3 Solve the equation $\sqrt{2x+3} = 3x - 6$.

Solution : We define $y = \sqrt{2x+3}$ and note that

$$y = \sqrt{2x+3} \iff 2x + 3 = y^2 \iff 2x = y^2 - 3 \iff x = (1/2)(y^2 - 3).$$

It follows that

$$\begin{aligned} \sqrt{2x+3} = 3x - 6 &\iff y = 3(1/2)(y^2 - 3) - 6 \iff 2y = 3(y^2 - 3) - 12 \\ &\iff 2y = 3y^2 - 9 - 12 \iff 3y^2 - 2y - 21 = 0 \\ &\iff \cdots \iff y = 3 \vee y = -7/3 \\ &\iff \sqrt{2x+3} = 3 \vee \sqrt{2x+3} = -7/3. \end{aligned} \tag{50}$$

Noting that the second equation has no real solutions, it follows that

$$\text{Eq. (50)} \iff \sqrt{2x+3} = 3 \iff 2x + 3 = 9 \iff 2x = 9 - 3 = 6 \iff x = 3$$

□

5. Equations with a sum of roots equal to zero

Equations that simplify to a sum of square roots set equal to zero can be solved directly using Lemma 2.2. For $n = 2$ and $n = 3$, Lemma 2.2 reduces to the following equivalencies:

$$\sqrt{f(x)} + \sqrt{g(x)} = 0 \Longleftrightarrow f(x) = 0 \wedge g(x) = 0, \tag{51}$$

$$\sqrt{f(x)} + \sqrt{g(x)} + \sqrt{h(x)} = 0 \Longleftrightarrow f(x) = 0 \wedge g(x) = 0 \wedge h(x) = 0. \tag{52}$$

Similar statements follow for all natural numbers $n > 3$. These equivalencies immediately remove the square roots, so there is no need to square both sides of the equation in order to eliminate the roots. Furthermore, there is no need to impose any restrictions in order to accept or reject solutions, as in the equation types discussed in Section 2 and Section 3. The solution technique is to simply use the equivalencies that follow from Lemma 2.2 to eliminate the square roots. We illustrate it with Example 5.1 and Example 5.2 given in the following:

Example 5.1 Solve the equation $\sqrt{x^2 - 9} + \sqrt{x^2 + 5x + 6} = 0$.

Solution : Since,

$$\sqrt{x^2 - 9} + \sqrt{x^2 + 5x + 6} = 0 \Longleftrightarrow \begin{cases} x^2 - 9 = 0 \\ x^2 + 5x + 6 = 0 \end{cases} \Longleftrightarrow \begin{cases} (x-3)(x+3) = 0 \\ (x+2)(x+3) = 0 \end{cases} \tag{53}$$

$$\Longleftrightarrow \begin{cases} x - 3 = 0 \vee x + 3 = 0 \\ x + 2 = 0 \vee x + 3 = 0 \end{cases} \Longleftrightarrow \begin{cases} x = 3 \vee x = -3 \\ x = -2 \vee x = -3 \end{cases} \tag{54}$$

$$\Longleftrightarrow x \in \{3, -3\} \cap \{-2, -3\} \Longleftrightarrow x = -3, \tag{55}$$

it follows that the set of all formal or strong solutions is given by $S = \{-3\}$. □

Example 5.2 Solve the equation $\sqrt{x^2 - 1} + \sqrt{x^2 + 3x + 2} = -\sqrt{x^2 + 4x + 3}$.

Solution : Since,

$$\sqrt{x^2 - 1} + \sqrt{x^2 + 3x + 2} = -\sqrt{x^2 + 4x + 3} \tag{56}$$

$$\Longleftrightarrow \sqrt{x^2 - 1} + \sqrt{x^2 + 3x + 2} + \sqrt{x^2 + 4x + 3} = 0 \tag{57}$$

$$\Longleftrightarrow \begin{cases} x^2 - 1 = 0 \\ x^2 + 3x + 2 = 0 \\ x^2 + 4x + 3 = 0 \end{cases} \Longleftrightarrow \begin{cases} (x-1)(x+1) = 0 \\ (x+1)(x+2) = 0 \\ (x+1)(x+3) = 0 \end{cases} \tag{58}$$

$$\Longleftrightarrow \begin{cases} x - 1 = 0 \vee x + 1 = 0 \\ x + 1 = 0 \vee x + 2 = 0 \\ x + 1 = 0 \vee x + 3 = 0 \end{cases} \Longleftrightarrow \begin{cases} x = 1 \vee x = -1 \\ x = -1 \vee x = -2 \\ x = -1 \vee x = -3 \end{cases} \tag{59}$$

$$\Longleftrightarrow x \in \{-1, 1\} \cap \{-1, -2\} \cap \{-1, -3\} \Longleftrightarrow x = -1, \tag{60}$$

it follows that the set of all formal or strong solutions is given by $S = \{-1\}$. □

6. Application to inverse functions

Radical equations arise naturally when attempting to find the inverse function f^{-1} of a function f whose definition involves an expression with square roots of polynomials or rational functions. Recall that given a function $f : A \to f(A)$ with $A \subseteq \mathbb{R}$, we find the inverse function $f^{-1} : f(A) \to A$ by solving the equation $f(y) = x$ with respect to y for all $x \in \mathbb{R}$. The domain of f^{-1} is the range $f(A)$ of f, and can be calculated by finding all $x \in \mathbb{R}$ such that the equation $f(y) = x$ has at least one solution, when solving it with respect to y. Furthermore, showing that whenever at least one solution exists it is also unique, establishes the existence of the inverse function. In order to carry out such an argument rigorously and effectively, the rigorous understanding of radical equations promoted in this article is essential. We illustrate this procedure with Example 6.1 given below.

Example 6.1 Define the formula and the domain of the inverse function f^{-1} corresponding to the function f defined by $f(x) = 2 + (1/3)\sqrt{3x+1}$.

Solution : First, we note that

$$f^{-1}(x) = y \iff f(y) = x \iff 2 + (1/3)\sqrt{3y+1} = x \tag{61}$$

$$\iff 6 + \sqrt{3y+1} = 3x \iff \sqrt{3y+1} = 3x - 6 \tag{62}$$

$$\iff \begin{cases} 3y + 1 = (3x-6)^2 \\ 3x - 6 \geq 0. \end{cases} \tag{63}$$

The domain A of the inverse function f^{-1} is obtained by solving the restriction:

$$3x - 6 \geq 0 \iff x - 2 \geq 0 \iff x \geq 2 \iff x \in [2, +\infty) \equiv A. \tag{64}$$

For all $x \in A$, we have:

$$\text{Eq. (63)} \iff 3y + 1 = 9(x-2)^2 \iff 3y = 9(x-2)^2 - 1 \tag{65}$$

$$\iff y = 3(x-2)^2 - 1/3. \tag{66}$$

It follows that the inverse function f^{-1} is defined by:

$$f^{-1}(x) = 3(x-2)^2 - 1/3, \ \forall x \in [2, +\infty). \tag{67}$$

□

Here, the quantifier is used to indicate that the domain of f^{-1} is the set $A = [2, \infty)$. In reviewing Example 6.1, it is important to emphasize that one cannot find the correct domain of the inverse function f^{-1} from the resulting formula for $f^{-1}(x)$. Since $f^{-1}(x)$ is found to be equal to a polynomial, the domain implied by the formula for $f^{-1}(x)$ is the set $\mathbb{R}$ or all real numbers. The correct procedure for finding the domain of f^{-1} is by collecting all $x \in \mathbb{R}$ for which the equation $f(y) = x$ has a solution, when we solve with respect to y, also showing that the solution is always unique, when it exists.

7. Conclusion

As has been reviewed in the introduction, the problem of extraneous solutions that emerge in the course of the solution of radical equations has a long and interesting history. The prevalent approach of most contemporary textbooks is to solve the equation with equivalence violating manipulations and then identify any resulting extraneous solutions by testing all obtained solutions against the original equation. This is conceptually unsatisfactory and in certain cases also impractical. In this article, I presented more rigorous solution procedures for solving radical equations of depth 1 in a way that identifies the extraneous solutions in the course of the solution process, thereby making an explicit verification of all solutions unnecessary. I have also stated and proved Lemma 2.2, Proposition 3.3, and Proposition 4.2 that justify these solution procedures and illustrated them with solved examples. It should be noted that the assumption in Proposition 3.3 and Proposition 4.2 that the functions f, g be polynomials or rational functions is needed to justify the results about the formal solutions set, but can be removed if we only want to know the strong solutions set. The improved solution techniques can be easily taught in lower-level coursework. The propositions and the corresponding proofs are more appropriate in an honors course or in the context of teaching students the basics of proofwriting.

One obvious application of the improved solution procedures, where they are clearly superior to the standard approach, is in tackling parametric radical equations, which emerge naturally in the context of problems where want to state the definition and implicit domain of the inverse of a function whose definition involves an expression where the variable appears under at least one square root. Although the solution technique corresponding to Proposition 4.2 was also discussed informally by Gurevich [15], I learned all of these techniques much earlier from Pistofidis [13, 14]. Furthermore, the solution procedure for radical equations that follow the form of Eq. (1) raises the need to clarify the difference between formal and strong real-valued solutions, and the procedure for radical equations that follow the form of Eq. (3) do not require any squaring or the use of restriction sets. The distinction between formal and strong solutions, to the best of my knowledge, is original to this article.

We conclude with some points of comparison between the standard technique (verify solutions against the original equation) and the procedures proposed earlier. The advantage of the standard technique is that it is easy to teach, and if the problems have integer solutions or if we allow use of calculators, the verification step is not too cumbersome. As such, the standard technique may be effective in preparing students to pass easy standardized tests, but that is done at the expense of deeper conceptual understanding. The standard technique is also very ineffective against parametric radical equations. On the other side, our proposed solution procedures are also very simple to execute, at least for radical equations with depth 1, have a wide range of generality, encourage conceptual understanding and correct writing of mathematical arguments, and are a good foundation for taking on more complex problems, such as parametric radical equations. A possible disadvantage can be that students will have to be able to identify different equation forms and apply the appropriate procedure for each form. However, this can also be seen as an advantage in that confronting students with the need to choose the appropriate procedure develops mathematical maturity that will be of use in more advanced courses. Ultimately, the decision depends on whether the chosen learning objective is to just teach one more skill at a very basic level, or to teach at a deeper level and promote the development of mathematical maturity.

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