2-2022

Euclidean quantum field formulation of p-adic open string amplitudes

A. R. Fuquen-Tibatá

H. García-Compeán

Wilson A. Zuniga-Galindo
The University of Texas Rio Grande Valley

Recommended Citation

Fuquen-Tibatá, A. R., H. García-Compeán, and W. A. Zúñiga-Galindo. "Euclidean quantum field formulation of p-adic open string amplitudes." Nuclear Physics B (2022): 115684.

Available online at www.sciencedirect.com

ScienceDirect

Nuclear Physics B 975 (2022) 115684

NUCLEAR PHYSICS B

www.elsevier.com/locate/nuclphysb

Euclidean quantum field formulation of p-adic open string amplitudes

A.R. Fuquen-Tibatá [a], H. García-Compeán [b], W.A. Zúñiga-Galindo [c,a,*]

[a] *Centro de Investigación y de Estudios Avanzados del Instituto Politécnico Nacional, Departamento de Matemáticas, Unidad Querétaro, Libramiento Norponiente #2000, Fracc. Real de Juriquilla, Santiago de Querétaro, Qro. 76230, Mexico*

[b] *Centro de Investigacion y de Estudios Avanzados del I.P.N., Departamento de Física, Av. Instituto Politecnico Nacional 2508, Col. San Pedro Zacatenco, Mexico D.F., C.P. 07360, Mexico*

[c] *University of Texas Rio Grande Valley, School of Mathematical & Statistical Sciences, One West University Blvd, Brownsville, TX 78520, United States*

Received 15 October 2021; received in revised form 18 December 2021; accepted 20 January 2022
Available online 25 January 2022
Editor: Hubert Saleur

Abstract

We study in a rigorous mathematical way p-adic quantum field theories whose N-point amplitudes are the expectation of products of vertex operators. We show that this type of amplitudes admit a series expansion where each term is an Igusa's local zeta function. The lowest term in this series is a regularized version of the p-adic open Koba-Nielsen string amplitude.

Contents

1. Introduction . . . 2
2. Basic facts on p-adic analysis . . . 5
   2.1. The field of p-adic numbers . . . 5
   2.2. Topology of $\mathbb{Q}_p^N$ . . . 6

* Corresponding author.
E-mail addresses: arfuquen@math.cinvestav.mx (A.R. Fuquen-Tibatá), compean@fis.cinvestav.mx (H. García-Compeán), wilson.zunigagalindo@utrgv.edu, wazuniga@math.cinvestav.edu.mx (W.A. Zúñiga-Galindo).

https://doi.org/10.1016/j.nuclphysb.2022.115684

2.3. The Bruhat-Schwartz space . . . 6
2.4. L^{ρ} spaces . . . 7
2.5. The Fourier transform . . . 7
2.6. Distributions . . . 8
2.7. The Fourier transform of a distribution . . . 8
3. A naive Euclidean version of the p-adic open string amplitudes . . . 8
3.1. The action and the Vladimirov operator . . . 9
4. Gaussian processes and free quantum fields . . . 11
5. N-point amplitudes . . . 12
5.1. A rigorous definition of the N-point amplitudes . . . 12
5.2. Some technical results . . . 15
6. Regularization of p-adic open string amplitudes, and multivariate local zeta functions . . . 22
6.1. The p-adic Koba-Nielsen local zeta functions . . . 22
6.2. The limit $\lim_{R\to\infty}\mathcal{A}_R^{(N)}(\boldsymbol{k})$ . . . 24
6.3. $\boldsymbol{\varphi}^4$-theories . . . 25
CRediT authorship contribution statement . . . 25
Declaration of competing interest . . . 25
Acknowledgement . . . 26
References . . . 26

1. Introduction

The string amplitudes were introduced by Veneziano in the 60s, [1], further generalizations were obtained by Virasoro [2], Koba and Nielsen [3], among others. In the 80s, Freund, Witten and Volovich, among others, studied string amplitudes at the tree level over different number fields, and suggested the existence of connections between these amplitudes, see e.g. [4]-[5]. In this framework the connections with number theory, specifically with local zeta functions, appear naturally, see e.g. [6–9], and the survey [10], see also [11–13].

The p-adic string theories have been studied over time with some periodic fluctuations in their interest (for some reviews, see [14], [15], [16], [17]). Recently a considerable amount of work has been performed on this topic in the context of the AdS/CFT correspondence [18–21]. String theory with a p-adic world-sheet was proposed and studied for the first time in [22]. Later this theory was formally known as p-adic string theory. The p-adic strings are related to ordinary strings at least in two different ways. First, connections through the adelic relations [23], and second, through the limit p tends to 1 [24–26].

The tree-level string amplitudes were explicitly computed in the case of p-adic string world-sheet in [27] and [28]. Since the 80s there has been interest in constructing field theories whose correlators are the p-adic tree-level string amplitudes (or p-adic Koba-Nielsen amplitudes). Spokoiny [25] and Zhang [29], see also [30], constructed formally quantum field theories whose amplitudes are expectation values of products of vertex operators. In [31] Zabrodin established that the tree-level string amplitudes may be obtained starting with a discrete field theory on a Bruhat-Tits tree. These ideas have been used by Ghoshal and Kawano in the study of p-adic strings in constant B-fields [32]. This article aims to provide a rigorous mathematical construction of a class of quantum field theories whose amplitudes are expectations of products of vertex operators. By using this approach, we carry out a mathematically rigorous derivation of the N-point Koba-Nielsen amplitudes, thus our approach is completely different from the one followed in [25,29,31].

The naive Euclidean version of the p-adic N-point amplitudes is given by

$$\mathcal{A}^{(N)}(\boldsymbol{k}) = \left\langle \prod_{j=1}^{N} \int_{\mathbb{Q}_p} dx_j \; e^{\boldsymbol{k}_j \cdot \boldsymbol{\varphi}(x_j)} \right\rangle \tag{1.1}$$

$$= \frac{1}{Z_0^{phys}} \int D\boldsymbol{\varphi} e^{-S(\boldsymbol{\varphi})} \left\{ \int_{\mathbb{Q}_p^N} d^N x \; e^{\sum_{j=1}^{N} \boldsymbol{k}_j \cdot \boldsymbol{\varphi}(x_j)} \right\},$$

where $\int_{\mathbb{Q}_p} dx_j \; e^{\boldsymbol{k}_j \cdot \boldsymbol{\varphi}(x_j)}$ is the tachyonic vertex operator of the j-th tachyon, with momentum $\boldsymbol{k}_j = \left(k_{0,j}, \ldots, k_{D-1,j}\right)$, and field $\boldsymbol{\varphi}(x_j)$, the dot denotes the standard Euclidean scalar product, and the action is given by

$$S(\boldsymbol{\varphi}) = \frac{T_0}{2} \sum_{j=1}^{N} \int_{\mathbb{Q}_p} \int_{\mathbb{Q}_p} \left\{ \frac{\varphi_j(x_j) - \varphi_j(y_j)}{\left|x_j - y_j\right|_p} \right\}^2 dx_j dy_j. \tag{1.2}$$

It is important to note that in (1.1) the tachyonic fields must be functions not distributions. These amplitudes are exactly the ones considered in [25], [29], [32]. Since the integral $\int_{\mathbb{Q}_p^N} d^N x$ in the right-hand side of (1.1) is always divergent, it is necessary to introduce a cut-off, and to define the amplitude by a limit process. The key observation is that the action (1.2) corresponds to a free quantum field. In the Archimedean and non-Archimedean cases, free quantum fields correspond to Gaussian probability measures on suitable infinite dimensional spaces. The reader may consult [33, Section 6.2] for the Archimedean case, and [34, Section 5.5], [35], [36] for the p-adic case. We construct Gaussian probability measure $\mathbb{P}_D$ on suitable function space ($\mathcal{L}_{\mathbb{R}}^{D}\left(\mathbb{Q}_p\right)$) and propose that $\mathcal{A}^{(N)}(\boldsymbol{k}) = \lim_{R\to\infty} \mathcal{A}_R^{(N)}(\boldsymbol{k})$, where

$$\mathcal{A}_R^{(N)}(\boldsymbol{k}) = \frac{1}{Z_0} \int_{B_R^N} \left\{ \int_{\mathcal{L}_{\mathbb{R}}^{D}(\mathbb{Q}_p)} e^{\sum_{j=1}^{N} \boldsymbol{k}_j \cdot \boldsymbol{\varphi}(x_j)} d\mathbb{P}_D(\boldsymbol{\varphi}) \right\} \prod_{j=1}^{N} dx_j, \tag{1.3}$$

and B_R^N denotes an N-dimensional ball of radius p^R. Following the standard approach in QFT, we expand the right-hand side of (1.3) around a suitable solution of the equations of motion. The main difficulty is that the solutions of this equation are distributions, and we are restricted to work with functions. We show that there is a change of variables in (1.3) such that

$$\mathcal{A}_R^{(N)}(\boldsymbol{k}) = \frac{1}{Z_0} \int_{B_R^N} \prod_{j<i}^{N} \left|x_j - x_i\right|_p^{2\frac{(p-1)}{p \ln p} \boldsymbol{k}_i \cdot \boldsymbol{k}_j} \left\{ \int_{\mathcal{L}_{\mathbb{R}}^{D}(\mathbb{Q}_p)} e^{\sum_{j=1}^{N} \boldsymbol{k}_j \cdot \widetilde{\boldsymbol{\varphi}}(x_j)} d\widetilde{\mathbb{P}}_D(\widetilde{\boldsymbol{\varphi}}) \right\} \prod_{j=0}^{N} dx_j, \tag{1.4}$$

where $\widetilde{\mathbb{P}}_D$ is a probability measure. Here is an important difference with respect to the classical QFT, which is that the $\boldsymbol{k}$ cannot be considered as a coupling constant, and thus there is no a standard perturbative expansion for (1.4). By taking the classical normalization

$$x_1 = 0, x_{N-1} = 1, x_N = \infty,$$

and using the expansion of the exponential function, we show that (1.4) admits a series expansion of the form

$$\mathcal{A}_R^{(N)}(\boldsymbol{k}) = \frac{C_0}{Z_0} \sum_{l=0}^{\infty} \int_{B_R^{N-3}} \prod_{i=2}^{N-2} |x_i|_p^{2\frac{(p-1)}{p\ln p}\boldsymbol{k}_1\cdot\boldsymbol{k}_i} |1-x_i|_p^{2\frac{(p-1)}{p\ln p}\boldsymbol{k}_{N-1}\cdot\boldsymbol{k}_i} \times \prod_{2\le i,j\le N-2} \left|x_j - x_i\right|_p^{2\frac{(p-1)}{p\ln p}\boldsymbol{k}_i\cdot\boldsymbol{k}_j} G_l(\boldsymbol{k},\boldsymbol{x}) \prod_{j=2}^{N-2} dx_j,$$

where $G_0(\boldsymbol{k},\boldsymbol{x})$ is a constant and the $G_l(\boldsymbol{k},\boldsymbol{x})$s are continuous functions in $\boldsymbol{x}$, for $l \geq 1$. The product $1_{B_R^{N-3}}(\boldsymbol{x})\ G_l(\boldsymbol{k},\boldsymbol{x})$ can be approximated by a test function in $\boldsymbol{x}$ depending of $\boldsymbol{k}$, for $l \geq 1$, without altering the analytic dependence of integral $\mathcal{A}_R^{(N)}(\boldsymbol{k})$ with respect to $\boldsymbol{k}$.

An integral of the form

$$Z_\Phi^{(N)}(\boldsymbol{k}) = \int_{\mathbb{Q}_p^{N-3}} \prod_{i=2}^{N-2} |x_i|_p^{2\frac{(p-1)}{p\ln p}\boldsymbol{k}_1\cdot\boldsymbol{k}_i} |1-x_i|_p^{2\frac{(p-1)}{p\ln p}\boldsymbol{k}_{N-1}\cdot\boldsymbol{k}_i} \times \prod_{2\le i,j\le N-2} \left|x_j - x_i\right|_p^{2\frac{(p-1)}{p\ln p}\boldsymbol{k}_i\cdot\boldsymbol{k}_j} \Phi(\boldsymbol{x}) \prod_{j=2}^{N-2} dx_j,$$

where Φ is a test function is a particular case of a multivariate Igusa zeta function [37].

In [6–9] was established that the integral $Z_\Phi^{(N)}(\boldsymbol{k})$ is holomorphic in a certain domain and that if $\Phi = 1_{B_R^{N-3}}(\boldsymbol{x})$, then

$$\lim_{R\to\infty} Z_R^{(N)}(\boldsymbol{k}) = \int_{\mathbb{Q}_p^{N-3}} \prod_{i=2}^{N-2} |x_i|_p^{2\frac{(p-1)}{p\ln p}\boldsymbol{k}_1\cdot\boldsymbol{k}_i} |1-x_i|_p^{2\frac{(p-1)}{p\ln p}\boldsymbol{k}_{N-1}\cdot\boldsymbol{k}_i} \times \prod_{2\le i,j\le N-2} \left|x_j - x_i\right|_p^{2\frac{(p-1)}{p\ln p}\boldsymbol{k}_i\cdot\boldsymbol{k}_j} \prod_{j=2}^{N-2} dx_j =: Z^{(N)}(\boldsymbol{k}),$$

where $Z^{(N)}(\boldsymbol{k})$ is a meromorphic function which is a regularized version of the p-adic Koba-Nielsen amplitude, [6].

Therefore

$$\mathcal{A}^{(N)}(\boldsymbol{k}) = \lim_{R\to\infty} \mathcal{A}_R^{(N)}(\boldsymbol{k}) = A^{(N)}(\boldsymbol{k}) + \lim_{R\to\infty} \left\{ \sum_{l=1}^{\infty} \int_{B_R^{N-3}} \prod_{i=2}^{N-2} |x_i|_p^{2\frac{(p-1)}{p\ln p}\boldsymbol{k}_1\cdot\boldsymbol{k}_i} |1-x_i|_p^{2\frac{(p-1)}{p\ln p}\boldsymbol{k}_{N-1}\cdot\boldsymbol{k}_i} \right.$$

$$\left. \times \frac{C_0}{Z_0} \prod_{2 \leq i, j \leq N-2} \left|x_j - x_i\right|_p^{2\frac{(p-1)}{p \ln p} \boldsymbol{k}_i \cdot \boldsymbol{k}_j} G_l(\boldsymbol{k}, \boldsymbol{x}) \prod_{j=2}^{N-2} dx_j \right\},$$

where $A^{(N)}(\boldsymbol{k})$ is the p-adic Koba-Nielsen string amplitude in the Euclidean signature, $\frac{C_0}{Z_0}$ is a positive constant. We know that there is a common domain of convergence in $\boldsymbol{k}$ for $A^{(N)}(\boldsymbol{k})$ and all the integrals appearing in the series, but we do not know if the series converges. The study of the limit $R \to \infty$ in the previous formula is an open problem.

In a forthcoming article, we plan to study the p-adic quantum field theories [36] attached to a non-Archimedean version of the open string action in a background gauge field [38]. This action has cubic and quartic terms in the dynamical fields, which generate interesting non-trivial one-loop quantum corrections which determine the beta functions and the effective action for the gauge fields. We would like to find the corresponding non-Archimedean version for this case. Finally, we expect that the results presented in this work have a natural counterpart in the case of standard Koba-Nielsen amplitudes.

2. Basic facts on p-adic analysis

In this Section, we collect some basic results on p-adic analysis that we use through the article. For a detailed exposition on p-adic analysis the reader may consult [39], [40], [16].

2.1. The field of p-adic numbers

Throughout this article p will denote a prime number. The field of p-adic numbers $\mathbb{Q}_p$ is defined as the completion of the field of rational numbers $\mathbb{Q}$ with respect to the p-adic norm $|\cdot|_p$, which is defined as

$$|x|_p = \begin{cases} 0 & \text{if } x = 0 \\ p^{-\gamma} & \text{if } x = p^{\gamma} \dfrac{a}{b}, \end{cases}$$

where a and b are integers coprime with p. The integer $\gamma = ord_p(x) := ord(x)$, with $ord(0) := +\infty$, is called the *p-adic order of x*. We extend the p-adic norm to $\mathbb{Q}_p^N$ by taking

$$||x||_p := \max_{1 \leq i \leq N} |x_i|_p, \qquad \text{for } x = (x_1, \ldots, x_N) \in \mathbb{Q}_p^N.$$

We define $ord(x) = \min_{1 \leq i \leq N}\{ord(x_i)\}$, then $||x||_p = p^{-ord(x)}$. The metric space $\left(\mathbb{Q}_p^N, ||\cdot||_p\right)$ is a complete ultrametric space. As a topological space $\mathbb{Q}_p$ is homeomorphic to a Cantor-like subset of the real line, see e.g. [39], [16].

Any p-adic number $x \neq 0$ has a unique expansion of the form

$$x = p^{ord(x)} \sum_{j=0}^{\infty} x_j p^j,$$

where $x_j \in \{0, 1, 2, \ldots, p-1\}$ and $x_0 \neq 0$. By using this expansion, we define *the fractional part $\{x\}_p$ of $x \in \mathbb{Q}_p$* as the rational number

$$\{x\}_p = \begin{cases} 0 & \text{if } x = 0 \text{ or } ord(x) \geq 0 \\ p^{ord(x)} \sum_{j=0}^{-ord(x)-1} x_j p^j & \text{if } ord(x) < 0. \end{cases}$$

In addition, any $x \in \mathbb{Q}_p^N \smallsetminus \{0\}$ can be represented uniquely as $x = p^{ord(x)} v(x)$ where $\|v(x)\|_p = 1$.

2.2. *Topology of $\mathbb{Q}_p^N$*

For $r \in \mathbb{Z}$, denote by $B_r^N(a) = \{x \in \mathbb{Q}_p^N; ||x-a||_p \leq p^r\}$ *the ball of radius p^r with center at* $a = (a_1, \dots, a_N) \in \mathbb{Q}_p^N$, and take $B_r^N(0) := B_r^N$. Note that $B_r^N(a) = B_r(a_1) \times \cdots \times B_r(a_N)$, where $B_r(a_i) := \{x \in \mathbb{Q}_p; |x_i - a_i|_p \leq p^r\}$ is the one-dimensional ball of radius p^r with center at $a_i \in \mathbb{Q}_p$. The ball B_0^N equals the product of N copies of $B_0 = \mathbb{Z}_p$, *the ring of $p-$adic integers*. We also denote by $S_r^N(a) = \{x \in \mathbb{Q}_p^N; ||x-a||_p = p^r\}$ *the sphere of radius p^r with center at* $a = (a_1, \dots, a_N) \in \mathbb{Q}_p^N$, and take $S_r^N(0) := S_r^N$. We notice that $S_0^1 = \mathbb{Z}_p^\times$ (the group of units of $\mathbb{Z}_p$), but $\left(\mathbb{Z}_p^\times\right)^N \subsetneq S_0^N$. The balls and spheres are both open and closed subsets in $\mathbb{Q}_p^N$. In addition, two balls in $\mathbb{Q}_p^N$ are either disjoint or one is contained in the other.

As a topological space $\left(\mathbb{Q}_p^N, ||\cdot||_p\right)$ is totally disconnected, i.e. the only connected subsets of $\mathbb{Q}_p^N$ are the empty set and the points. A subset of $\mathbb{Q}_p^N$ is compact if and only if it is closed and bounded in $\mathbb{Q}_p^N$, see e.g. [16, Section 1.3], or [39, Section 1.8]. The balls and spheres are compact subsets. Thus $\left(\mathbb{Q}_p^N, ||\cdot||_p\right)$ is a locally compact topological space.

Since $(\mathbb{Q}_p^N, +)$ is a locally compact topological group, there exists a Haar measure $d^N x$, which is invariant under translations, i.e. $d^N(x+a) = d^N x$. If we normalize this measure by the condition $\int_{\mathbb{Z}_p^N} dx = 1$, then $d^N x$ is unique.

Notation 1. We will use $\Omega\left(p^{-r}||x-a||_p\right)$ to denote the characteristic function of the ball $B_r^N(a)$. For more general sets, we will use the notation 1_A for the characteristic function of a set A.

2.3. *The Bruhat-Schwartz space*

A complex-valued function φ defined on $\mathbb{Q}_p^N$ is *called locally constant* if for any $x \in \mathbb{Q}_p^N$ there exist an integer $l(x) \in \mathbb{Z}$ such that

$$\varphi(x + x') = \varphi(x) \text{ for any } x' \in B_{l(x)}^N. \tag{2.1}$$

A function $\varphi : \mathbb{Q}_p^N \to \mathbb{C}$ is called a *Bruhat-Schwartz function (or a test function)* if it is locally constant with compact support. Any test function can be represented as a linear combination, with complex coefficients, of characteristic functions of balls. The $\mathbb{C}$-vector space of Bruhat-Schwartz functions is denoted by $\mathcal{D}(\mathbb{Q}_p^N) := \mathcal{D}$. We denote by $\mathcal{D}_{\mathbb{R}}(\mathbb{Q}_p^N) := \mathcal{D}_{\mathbb{R}}$ the $\mathbb{R}$-vector space of Bruhat-Schwartz functions. For $\varphi \in \mathcal{D}(\mathbb{Q}_p^N)$, the largest number $l = l(\varphi)$ satisfying (2.1) is called *the exponent of local constancy (or the parameter of constancy) of φ*.

We denote by $\mathcal{D}_m^l(\mathbb{Q}_p^N)$ the finite-dimensional space of test functions from $\mathcal{D}(\mathbb{Q}_p^N)$ having supports in the ball B_m^N and with parameters of constancy $\geq l$. We now define a topology on

$\mathcal{D}$ as follows. We say that a sequence $\{\varphi_j\}_{j\in\mathbb{N}}$ of functions in $\mathcal{D}$ converges to zero, if the two following conditions hold:

(1) there are two fixed integers k_0 and m_0 such that each $\varphi_j \in \mathcal{D}_{m_0}^{k_0}$;

(2) $\varphi_j \to 0$ uniformly.

$\mathcal{D}$ endowed with the above topology becomes a topological vector space.

2.4. L^ρ spaces

Given $\rho \in [1,\infty)$, we denote by $L^\rho := L^\rho\left(\mathbb{Q}_p^N\right) := L^\rho\left(\mathbb{Q}_p^N, d^N x\right)$, the $\mathbb{C}$-vector space of all the complex valued functions g satisfying

$$\int_{\mathbb{Q}_p^N} |g(x)|^\rho \, d^N x < \infty,$$

where $d^N x$ is the normalized Haar measure on $\left(\mathbb{Q}_p^N, +\right)$. The corresponding $\mathbb{R}$-vector spaces are denoted as $L_{\mathbb{R}}^\rho := L_{\mathbb{R}}^\rho\left(\mathbb{Q}_p^N\right) = L_{\mathbb{R}}^\rho\left(\mathbb{Q}_p^N, d^N x\right)$, $1 \le \rho < \infty$.

If U is an open subset of $\mathbb{Q}_p^N$, $\mathcal{D}(U)$ denotes the space of test functions with supports contained in U, then $\mathcal{D}(U)$ is dense in

$$L^\rho(U) = \left\{\varphi : U \to \mathbb{C};\ \|\varphi\|_\rho = \left\{\int_U |\varphi(x)|^\rho \, d^N x\right\}^{\frac{1}{\rho}} < \infty\right\},$$

for $1 \le \rho < \infty$, see e.g. [39, Section 4.3]. We denote by $L_{\mathbb{R}}^\rho(U)$ the real counterpart of $L^\rho(U)$.

2.5. The Fourier transform

Set $\chi_p(y) = \exp(2\pi i\{y\}_p)$ for $y \in \mathbb{Q}_p$. The map $\chi_p(\cdot)$ is an additive character on $\mathbb{Q}_p$, i.e. a continuous map from $(\mathbb{Q}_p, +)$ into S (the unit circle considered as multiplicative group) satisfying $\chi_p(x_0 + x_1) = \chi_p(x_0)\chi_p(x_1)$, $x_0, x_1 \in \mathbb{Q}_p$. The additive characters of $\mathbb{Q}_p$ form an Abelian group which is isomorphic to $(\mathbb{Q}_p, +)$. The isomorphism is given by $\kappa \to \chi_p(\kappa x)$, see e.g. [39, Section 2.3].

Given $\kappa = (\kappa_1, \dots, \kappa_N)$ and $y = (x_1, \dots, x_N) \in \mathbb{Q}_p^N$, we set $\kappa \cdot x := \sum_{j=1}^N \kappa_j x_j$. The Fourier transform of $\varphi \in \mathcal{D}(\mathbb{Q}_p^N)$ is defined as

$$(\mathcal{F}\varphi)(\kappa) = \int_{\mathbb{Q}_p^N} \chi_p(\kappa \cdot x)\varphi(x) d^N x \quad \text{for } \kappa \in \mathbb{Q}_p^N,$$

where $d^N x$ is the normalized Haar measure on $\mathbb{Q}_p^N$. The Fourier transform is a linear isomorphism from $\mathcal{D}(\mathbb{Q}_p^N)$ onto itself satisfying

$$(\mathcal{F}(\mathcal{F}\varphi))(\kappa) = \varphi(-\kappa), \tag{2.2}$$

see e.g. [39, Section 4.8]. We will also use the notation $\mathcal{F}_{x\to\kappa}\varphi$ and $\widehat{\varphi}$ for the Fourier transform of φ.

The Fourier transform extends to L^2. If $f \in L^2$, its Fourier transform is defined as

$$(\mathcal{F}f)(\kappa) = \lim_{k\to\infty} \int\limits_{||x||_p \le p^k} \chi_p(\kappa \cdot x) f(x) d^N x, \quad \text{for } \kappa \in \mathbb{Q}_p^N,$$

where the limit is taken in L^2. We recall that the Fourier transform is unitary on L^2, i.e. $||f||_2 = ||\mathcal{F}f||_2$ for $f \in L^2$ and that (2.2) is also valid in L^2, see e.g. [40, Chapter III, Section 2].

2.6. Distributions

The $\mathbb{C}$-vector space $\mathcal{D}'\left(\mathbb{Q}_p^n\right) := \mathcal{D}'$ of all continuous linear functionals on $\mathcal{D}(\mathbb{Q}_p^n)$ is called the *Bruhat-Schwartz space of distributions*. Every linear functional on $\mathcal{D}$ is continuous, i.e. $\mathcal{D}'$ agrees with the algebraic dual of $\mathcal{D}$, see e.g. [16, Chapter 1, VI.3, Lemma]. We denote by $\mathcal{D}'_{\mathbb{R}}\left(\mathbb{Q}_p^n\right) := \mathcal{D}'_{\mathbb{R}}$ the dual space of $\mathcal{D}_{\mathbb{R}}$.

We endow $\mathcal{D}'$ with the weak topology, i.e. a sequence $\left\{T_j\right\}_{j\in\mathbb{N}}$ in $\mathcal{D}'$ converges to T if $\lim_{j\to\infty} T_j(\varphi) = T(\varphi)$ for any $\varphi \in \mathcal{D}$. The map

$$\begin{aligned} \mathcal{D}' \times \mathcal{D} &\rightarrow \mathbb{C} \\ (T, \varphi) &\rightarrow T(\varphi) \end{aligned}$$

is a bilinear form which is continuous in T and φ separately. We call this map the pairing between $\mathcal{D}'$ and $\mathcal{D}$. From now on we will use (T, φ) instead of $T(\varphi)$.

Every f in L^1_{loc} defines a distribution $f \in \mathcal{D}'\left(\mathbb{Q}_p^N\right)$ by the formula

$$(f, \varphi) = \int\limits_{\mathbb{Q}_p^N} f(x)\varphi(x)\, d^N x.$$

Such distributions are called *regular distributions*. Notice that for $f \in L^2_{\mathbb{R}}$, $(f, \varphi) = \langle f, \varphi\rangle$, where $\langle\cdot,\cdot\rangle$ denotes the scalar product in $L^2_{\mathbb{R}}$.

2.7. The Fourier transform of a distribution

The Fourier transform $\mathcal{F}[T]$ of a distribution $T \in \mathcal{D}'\left(\mathbb{Q}_p^n\right)$ is defined by

$$(\mathcal{F}[T], \varphi) = (T, \mathcal{F}[\varphi]) \text{ for all } \varphi \in \mathcal{D}(\mathbb{Q}_p^n).$$

The Fourier transform $T \rightarrow \mathcal{F}[T]$ is a linear (and continuous) isomorphism from $\mathcal{D}'\left(\mathbb{Q}_p^n\right)$ onto $\mathcal{D}'\left(\mathbb{Q}_p^n\right)$. Furthermore, $T = \mathcal{F}[\mathcal{F}[T](-\xi)]$.

3. A naive Euclidean version of the p-adic open string amplitudes

We set $\boldsymbol{k} := (\boldsymbol{k}_1, \ldots, \boldsymbol{k}_N)$, where $\boldsymbol{k}_j = \left(k_{0,j}, \ldots, k_{D-1,j}\right) \in \mathbb{R}^D$ is the momentum of a tachyon, $j = 1, \ldots, N$. The dimension $D \geq 1$ is fixed along this work. We also set

$$\boldsymbol{\varphi}(\cdot) = (\varphi_0(\cdot), \ldots, \varphi_{D-1}(\cdot)) \in \left(\mathcal{D}_{\mathbb{R}}\left(\mathbb{Q}_p\right)\right)^D.$$

For $\boldsymbol{a} = (a_0, a_1, \ldots, a_{D-1})$, $\boldsymbol{b} = (b_0, \ldots, b_{D-1}) \in \mathbb{R}^D$, $\boldsymbol{a} \cdot \boldsymbol{b}$ denotes the standard scalar product in $\mathbb{R}^D$.

The naive Euclidean version of the p-adic N-point amplitudes is given by

$$\mathcal{A}^{(N)}(\boldsymbol{k}) = \frac{1}{Z_0^{phys}} \int D\boldsymbol{\varphi} e^{-S(\boldsymbol{\varphi})} \int\limits_{\mathbb{Q}_p^N} d^N x\, e^{\sum_{j=1}^N \boldsymbol{k}_j \cdot \boldsymbol{\varphi}(x_j)} \tag{3.1}$$

where $d^N x = \prod_{j=1}^N dx_j$, $S(\boldsymbol{\varphi}) = \frac{T_0}{2} \sum_{j=0}^{D-1} S_j(\varphi_j)$, with

$$S_j(\varphi_j) = \int\limits_{\mathbb{Q}_p} \int\limits_{\mathbb{Q}_p} \left\{ \frac{\varphi_j(x_j) - \varphi_j(y_j)}{|x_j - y_j|_p} \right\}^2 dx_j dy_j,$$

and

$$Z_0^{phys} = \int D\boldsymbol{\varphi} e^{-S(\boldsymbol{\varphi})}.$$

The amplitudes (3.1) are just expectation values of products of vertex operators. These amplitudes were proposed by Spokoiny [25] and Zhang [29], see also [30], [31]. In these articles the authors obtain the p-adic open Koba-Nielsen amplitudes from amplitudes (3.1) by a formal calculation. The central goal of this work is to provide a mathematical framework to understand these calculations.

Since there is $l \in \mathbb{Z}$ such that $\varphi_j(x_j) = 0$ for $|x_j|_p > p^l$, for some $l \in \mathbb{Z}$,

$$\int\limits_{\mathbb{Q}_p^N} d^N x e^{\sum_{j=1}^N \boldsymbol{k}_j \cdot \boldsymbol{\varphi}(x_j)} = \infty.$$

To fix this problem, it is necessary to introduce a cut-off and set

$$\mathcal{A}_R^{(N)}(\boldsymbol{k}) = \frac{1}{Z_0^{phys}} \int D\boldsymbol{\varphi} e^{-S(\boldsymbol{\varphi})} \int\limits_{B_R^N} d^N x e^{\sum_{j=1}^N \boldsymbol{k}_j \cdot \boldsymbol{\varphi}(x_j)},$$

where R is a positive integer and $B_R^N = \left\{ x \in \mathbb{Q}_p^N; \|x\|_p \leq p^R \right\}$.

3.1. *The action and the Vladimirov operator*

3.1.1. *The Vladimirov operator*

The Vladimirov operator $\boldsymbol{D} : \mathcal{D}(\mathbb{Q}_p) \to L^2(\mathbb{Q}_p)$ is defined as

$$\begin{aligned} \boldsymbol{D}\theta(x) &= \frac{p^2}{p+1} \int\limits_{\mathbb{Q}_p} \frac{\theta(x) - \theta(y)}{|x-y|_p^2} dy = \frac{p^2}{p+1} \int\limits_{\mathbb{Q}_p} \frac{\theta(x) - \theta(x-z)}{|z|_p^2} dz \\ &= \mathcal{F}^{-1}_{\xi \to x}\left[|\xi|_p \mathcal{F}_{x \to \xi} \theta \right]. \end{aligned}$$

This operator satisfies

$$\boldsymbol{D}\theta(x) = -\frac{p^2}{p+1} |x|_p^{-2} * \theta(x), \text{ for } \theta \in \mathcal{D}(\mathbb{Q}_p),$$

see e.g. [16, Chapter 2, Section IX.1].

3.1.2. The action

We now express the action in terms of the Vladimirov operator. For $\varphi_j \in \mathcal{D}\left(\mathbb{Q}_p\right)$,

$$
\begin{aligned}
S_j\left(\varphi_j\right) &= \int_{\mathbb{Q}_p}\int_{\mathbb{Q}_p} \left\{ \frac{\varphi_j\left(x_j\right)-\varphi_j\left(y_j\right)}{\left|x_j-y_j\right|_p} \right\}^2 dx_j dy_j \\
&= 2\int_{\mathbb{Q}_p}\int_{\mathbb{Q}_p} \frac{\varphi_j\left(x_j\right)\left(\varphi_j\left(x_j\right)-\varphi_j\left(y_j\right)\right)}{\left|x_j-y_j\right|_p^2} dy_j dx_j = 2\frac{(p+1)}{p^2}\int_{\mathbb{Q}_p} \varphi_j\left(x_j\right) \boldsymbol{D}\varphi_j\left(x_j\right) dx_j \\
&= 2\frac{p+1}{p^2}\int_{\mathbb{Q}_p} \varphi_j\left(x_j\right) \mathcal{F}^{-1}_{\xi_j\to x_j}\left[\left|\xi_j\right|_p \mathcal{F}_{x_j\to\xi_j}\varphi_j\right] dx_j \\
&= 2\frac{p+1}{p^2}\int_{\mathbb{Q}_p} \overline{\widehat{\varphi_j}\left(\xi_j\right)} \left|\xi_j\right|_p \widehat{\varphi_j}\left(\xi_j\right) d\xi_j = 2\frac{p+1}{p^2}\int_{\mathbb{Q}_p} \left|\xi_j\right|_p \left|\widehat{\varphi_j}\left(\xi_j\right)\right|^2 d\xi_j .
\end{aligned}
$$

Then

$$
S\left(\boldsymbol{\varphi}\right) = \frac{T_0\left(p+1\right)}{p^2}\sum_{j=0}^{D-1}\int_{\mathbb{Q}_p} \varphi_j\left(x_j\right) \boldsymbol{D}\varphi_j\left(x_j\right) dx_j .
$$

3.1.3. The inverse of the Vladimirov operator

We set

$$
\mathcal{L}\left(\mathbb{Q}_p\right) = \left\{\theta \in \mathcal{D}\left(\mathbb{Q}_p\right); \widehat{\theta}\left(0\right)=0\right\}.
$$

The complex vector space $\mathcal{L}\left(\mathbb{Q}_p\right)$ endowed with the topology inherited from $\mathcal{D}\left(\mathbb{Q}_p\right)$ is called the *p-adic Lizorkin space of test functions of the second kind*, see [39, Chapter 7]. We set $\mathcal{L}_{\mathbb{R}}\left(\mathbb{Q}_p\right) := \mathcal{D}_{\mathbb{R}}\left(\mathbb{Q}_p\right) \cap \mathcal{L}\left(\mathbb{Q}_p\right)$.

We define the inverse of **D** as

$$
\begin{aligned}
\boldsymbol{D}^{-1}:\ \mathcal{L}\left(\mathbb{Q}_p\right) &\to \mathcal{L}\left(\mathbb{Q}_p\right) \\
\theta &\to \boldsymbol{D}^{-1}\theta,
\end{aligned}
$$

where $\boldsymbol{D}^{-1}\theta\left(x\right) = \mathcal{F}^{-1}_{\xi\to x}\left[\left|\xi\right|_p^{-1} \mathcal{F}_{x\to\xi}\theta\right]$. Since $\left(\mathcal{F}_{x\to\xi}\theta\right)\left(0\right)=0$, we have $\boldsymbol{D}^{-1}\theta\left(x\right) \in \mathcal{L}\left(\mathbb{Q}_p\right)$.

Consider the equation

$$
\boldsymbol{D}\psi\left(x\right) = \theta\left(x\right) \text{ for } \theta \in \mathcal{L}\left(\mathbb{Q}_p\right).
$$

This equation has a unique solution $\psi \in \mathcal{L}\left(\mathbb{Q}_p\right)$. Set

$$
\left(f_1, \theta\right) = \frac{-\left(p-1\right)}{p\ln p}\int_{\mathbb{Q}_p} \theta\left(x\right) \ln\left|x\right|_p dx, \text{ for } \theta \in \mathcal{L}\left(\mathbb{Q}_p\right). \tag{3.2}
$$

Then

$$
\widehat{f_1}\left(\xi\right) = \frac{1}{\left|\xi\right|_p} \text{ in } \mathcal{L}'\left(\mathbb{Q}_p\right),
$$

and

$$\psi(x) = \boldsymbol{D}^{-1}\theta(x) = f_1(x) * \theta(x),$$

see e.g. [16, Chapter 2, Section IX.2].

4. Gaussian processes and free quantum fields

We define the bilinear form $\mathbb{B}$

$$\begin{aligned}\mathbb{B} : \mathcal{L}_{\mathbb{R}}\left(\mathbb{Q}_p\right) \times \mathcal{L}_{\mathbb{R}}\left(\mathbb{Q}_p\right) &\to \quad \mathbb{R} \\ (\varphi, \theta) \quad &\to \langle \varphi, \boldsymbol{D}^{-1}\theta\rangle\end{aligned}$$

where $\langle \cdot, \cdot \rangle$ denotes the scalar product in $L^2\left(\mathbb{Q}_p\right)$.

Lemma 1. $\mathbb{B}$ *is a positive, continuous bilinear form from* $\mathcal{L}_{\mathbb{R}}\left(\mathbb{Q}_p\right) \times \mathcal{L}_{\mathbb{R}}\left(\mathbb{Q}_p\right)$ *into* $\mathbb{R}$.

Proof. We first notice that for $\varphi \in \mathcal{L}_{\mathbb{R}}\left(\mathbb{Q}_p\right)$, we have

$$\mathbb{B}(\varphi, \varphi) = \langle \varphi, \boldsymbol{D}^{-1}\varphi\rangle = \langle \mathcal{F}^{-1}\varphi, \frac{\mathcal{F}\varphi}{|\xi|_p}\rangle = \int_{\mathbb{Q}_p} \frac{|\widehat{\varphi}(\xi)|^2\, d\xi}{|\xi|_p} \geq 0.$$

Then $\mathbb{B}(\varphi, \varphi) = 0$ implies that φ is zero almost everywhere and since φ is continuous $\varphi = 0$. Let $(\varphi_n, \theta_n) \in \mathcal{L}_{\mathbb{R}}\left(\mathbb{Q}_p\right) \times \mathcal{L}_{\mathbb{R}}\left(\mathbb{Q}_p\right)$ be two sequences such that $\varphi_n \to 0$ and $\theta_n \to 0$ in $\mathcal{L}_{\mathbb{R}}\left(\mathbb{Q}_p\right)$. We recall that the topology of $\mathcal{L}_{\mathbb{R}}\left(\mathbb{Q}_p\right)$ agrees with the topology of $\mathcal{D}_{\mathbb{R}}\left(\mathbb{Q}_p\right)$. Now,

$$\begin{aligned}\mathbb{B}(\theta_n, \varphi_n) &= \int_{\mathbb{Q}_p} \frac{\widehat{\theta}_n(\xi)\,\overline{\widehat{\varphi}}_n(\xi)}{|\xi|_p} d\xi = \int_{\mathbb{Z}_p} \frac{\widehat{\theta}_n(\xi)\,\overline{\widehat{\varphi}}_n(\xi)}{|\xi|_p} d\xi + \int_{\mathbb{Q}_p \smallsetminus \mathbb{Z}_p} \frac{\widehat{\theta}_n(\xi)\,\overline{\widehat{\varphi}}_n(\xi)}{|\xi|_p} d\xi \\ &=: I_1(\theta_n, \varphi_n) + I_2(\theta_n, \varphi_n).\end{aligned}$$

Since $\theta_n \in \mathcal{D}_{\mathbb{R}}\left(\mathbb{Q}_p\right)$ there exist two integers m_0, l_0, independent of n, such that

$$\operatorname{supp} \widehat{\theta}_n \subset p^{l_0}\mathbb{Z}_p \text{ and } \widehat{\theta}_n(\xi)\mid_{\xi_0 + p^{m_0}\mathbb{Z}_p} = \widehat{\theta}_n(\xi_0)$$

for each $n \in \mathbb{N}$. Without loss of generality, we may assume that m_0 is a positive integer. Then $\widehat{\theta}_n(\xi)\mid_{p^{m_0}\mathbb{Z}_p} = \widehat{\theta}_n(0) = 0$ for each $n \in \mathbb{N}$, and

$$\begin{aligned}|I_1(\varphi_n, \theta_n)| &\leq \|\widehat{\varphi}_n\|_\infty \int_{p^{-m_0} < |\xi|_p \leq 1} \frac{\left|\widehat{\theta}_n(\xi)\right|}{|\xi|_p} d\xi \leq \|\varphi_n\|_1 \|\widehat{\theta}_n\|_\infty \int_{p^{-m_0} < |\xi|_p \leq 1} \frac{1}{|\xi|_p} d\xi \\ &\leq C_1 \|\varphi_n\|_1 \|\theta_n\|_1.\end{aligned}$$

For the second integral,

$$\begin{aligned}|I_2(\varphi_n, \theta_n)| &\leq \|\widehat{\varphi}_n\|_\infty \int_{|\xi|_p > 1} \frac{\left|\widehat{\theta}_n(\xi)\right|}{|\xi|_p} d\xi \leq \|\widehat{\varphi}_n\|_\infty \int_{|\xi|_p > 1} \left|\widehat{\theta}_n(\xi)\right| d\xi \\ &\leq \|\varphi_n\|_1 \|\widehat{\theta}_n\|_1.\end{aligned}$$

Therefore,

$$\mathbb{B}(\varphi_n, \theta_n) \leq C_1 \|\varphi_n\|_1 \|\theta_n\|_1 + \|\varphi_n\|_1 \|\widehat{\theta_n}\|_1 .$$

Now, the continuity of $\mathbb{B}$ follows from the fact that $\varphi_n \underrightarrow{\text{uniform}} 0$ and $\theta_n \underrightarrow{\text{uniform}} 0$ imply that $\|\varphi_n\|_1 \to 0$, $\|\theta_n\|_1 \to 0$, and $\|\widehat{\theta_n}\|_1 \to 0$ as n tends to infinity. The convergence of the last sequence follows from

$$\left\|\widehat{\theta_n}\right\|_1 = \int\limits_{p^{l_0}\mathbb{Z}_p} \left|\widehat{\theta_n}(\xi)\right| d\xi \leq p^{-l_0} \left\|\widehat{\theta_n}\right\|_\infty \leq p^{-l_0} \|\theta_n\|_1 . \quad \square$$

We recall that $\mathcal{D}\left(\mathbb{Q}_p\right)$ is a nuclear space cf. [41, Section 4], and since any subspace of a nuclear space is also nuclear, $\mathcal{L}_{\mathbb{R}}\left(\mathbb{Q}_p\right)$ is a nuclear space that is dense and continuously embedded in $L^2_{\mathbb{R}}\left(\mathbb{Q}_p\right)$, cf. [39, theorem 7.4.4]. Then we have the following Gel'fand triple:

$$\mathcal{L}_{\mathbb{R}}\left(\mathbb{Q}_p\right) \hookrightarrow L^2_{\mathbb{R}}\left(\mathbb{Q}_p\right) \hookrightarrow \mathcal{L}'_{\mathbb{R}}\left(\mathbb{Q}_p\right).$$

We denote by $\mathcal{B} := \mathcal{B}\left(\mathcal{L}'_{\mathbb{R}}\left(\mathbb{Q}_p\right)\right)$ the σ-algebra generated by the cylinder subsets of $\mathcal{L}'_{\mathbb{R}}\left(\mathbb{Q}_p\right)$.

Consider the mapping

$$\begin{array}{cccc} \mathcal{C}: & \mathcal{L}_{\mathbb{R}}\left(\mathbb{Q}_p\right) & \rightarrow & \mathbb{C} \\ & f & \rightarrow & e^{\frac{-1}{2}\mathbb{B}(f,f)}. \end{array}$$

This functional is a continuous, positive definite mapping, cf. Lemma 1, and $\mathcal{C}(0) = 1$. Then $\mathcal{C}$ defines a characteristic functional in $\mathcal{L}_{\mathbb{R}}\left(\mathbb{Q}_p\right)$. By Bochner-Minlos theorem, there exists a unique probability measure $\mathbb{P}$ called the canonical Gaussian measure on $\left(\mathcal{L}'_{\mathbb{R}}\left(\mathbb{Q}_p\right), \mathcal{B}\right)$ given by its characteristic functional as

$$\int\limits_{\mathcal{L}'_{\mathbb{R}}(\mathbb{Q}_p)} e^{\sqrt{-1}(W,f)} d\mathbb{P}(W) = e^{-\frac{1}{2}\mathbb{B}(f,f)}, \quad f \in \mathcal{L}_{\mathbb{R}}\left(\mathbb{Q}_p\right), \tag{4.1}$$

where $(\cdot,\cdot)$ is the pairing between $\mathcal{L}'_{\mathbb{R}}\left(\mathbb{Q}_p\right)$ and $\mathcal{L}_{\mathbb{R}}\left(\mathbb{Q}_p\right)$. The measure $\mathbb{P}$ corresponds to a free quantum field on $\mathcal{L}'_{\mathbb{R}}\left(\mathbb{Q}_p\right)$. This identification is well-known in the Archimedean and non-Archimedean settings, see e.g. [33, Section 6.2], [34, Section 5.5].

5. N-point amplitudes

5.1. A rigorous definition of the N-point amplitudes

We denote by

$$\bigotimes_{j=0}^{D-1} \mathbb{P}\left(\varphi_j\right) = \mathbb{P}_D(\boldsymbol{\varphi}),$$

the product probability measure on the product σ-algebra $\mathcal{B}^D$. We set

$$\mathcal{L}^D_{\mathbb{R}}\left(\mathbb{Q}_p\right) = \mathcal{L}_{\mathbb{R}}\left(\mathbb{Q}_p\right) \times \cdots \times \mathcal{L}_{\mathbb{R}}\left(\mathbb{Q}_p\right), \ D\text{-times}.$$

The probability measure

$$\frac{1_{\mathcal{L}^D_{\mathbb{R}}(\mathbb{Q}_p)}(\boldsymbol{\varphi})\, d\mathbb{P}_D(\boldsymbol{\varphi})}{Z_0}, \tag{5.1}$$

where $Z_0 = \int_{\mathcal{L}^D_{\mathbb{R}}(\mathbb{Q}_p)} d\mathbb{P}_D(\boldsymbol{\varphi})$ represents a free quantum field in $\mathcal{L}^D_{\mathbb{R}}(\mathbb{Q}_p)$.

Intuitively, the N-point amplitudes are the expectation values of the products of the vertex operators with respect to the measure (5.1):

$$\left\langle \prod_{j=1}^{N} \int_{\mathbb{Q}_p} dx_j e^{\boldsymbol{k}_j \cdot \boldsymbol{\varphi}(x_j)} \right\rangle_{\mathbb{P}_D} = \frac{1}{Z_0} \int_{\mathcal{L}^D_{\mathbb{R}}(\mathbb{Q}_p)} \int_{\mathbb{Q}_p^N} d^N x e^{\sum_{j=1}^{N} \boldsymbol{k}_j \cdot \boldsymbol{\varphi}(x_j)} d\mathbb{P}_D(\boldsymbol{\varphi}). \tag{5.2}$$

It is important to mention that $e^{\sum_{j=1}^{N} \boldsymbol{k}_j \cdot \boldsymbol{\varphi}(x_j)}$ requires that each entry of $\boldsymbol{\varphi}(x_j)$ be a function, for this reason, the factor $1_{\mathcal{L}^D_{\mathbb{R}}(\mathbb{Q}_p)}$ is completely necessary in (5.1).

Due to the divergence of the second integral in the right-hand side of (5.2), we define the N-point amplitudes as follows.

Definition 1. For a positive integer R, we define the p-adic N-point amplitudes as $\mathcal{A}^{(N)}(\boldsymbol{k}) = \lim_{R\to\infty} \mathcal{A}^{(N)}_R(\boldsymbol{k})$, where

$$\mathcal{A}^{(N)}_R(\boldsymbol{k}) := \frac{1}{Z_0} \int_{B_R^N} \left\{ \int_{\mathcal{L}^D_{\mathbb{R}}(\mathbb{Q}_p)} e^{\sum_{j=1}^{N} \boldsymbol{k}_j \cdot \boldsymbol{\varphi}(x_j)} d\mathbb{P}_D(\boldsymbol{\varphi}) \right\} \prod_{j=1}^{N} dx_j.$$

Our central goal is to show (in a rigorous mathematical way) that the ansatz proposed in the above definition allow us to obtain the p-adic open Koba-Nielsen amplitudes as the constant term of a series expansion of $\lim_{R\to\infty} \mathcal{A}^{(N)}_R(\boldsymbol{k})$ in functions depending on $\boldsymbol{k}$. The precise statements of our main results are given in Theorems 1, 2.

By using that

$$\sum_{j=1}^{N} \boldsymbol{k}_j \cdot \boldsymbol{\varphi}(x_j) = \sum_{j=1}^{N} \sum_{l=0}^{D-1} k_{l,j} \varphi_l(x_j)$$

we have

$$\begin{aligned} \mathcal{A}^{(N)}_R(\boldsymbol{k}) &= \frac{1}{Z_0} \int_{B_R^N} \left\{ \int_{\mathcal{L}^D_{\mathbb{R}}(\mathbb{Q}_p)} e^{\sum_{j=1}^{N} \sum_{l=0}^{D-1} k_{l,j} \varphi_l(x_j)} \prod_{l=0}^{D-1} d\mathbb{P}(\varphi_l) \right\} \prod_{j=1}^{N} dx_j \\ &= \frac{1}{Z_0} \int_{B_R^N} \left\{ \prod_{l=0}^{D-1} \int_{\mathcal{L}_{\mathbb{R}}(\mathbb{Q}_p)} e^{\sum_{j=1}^{N} k_{l,j} \varphi_l(x_j)} d\mathbb{P}(\varphi_l) \right\} \prod_{j=1}^{N} dx_j. \end{aligned} \tag{5.3}$$

We now introduce the notation

$$\sum_{j=1}^{N} k_{l,j} \varphi_l(x_j) := \sum_{j=1}^{N} v_j \varphi(x_j) \tag{5.4}$$

taking advantage that l is fixed. Here $v_j \in \mathbb{R}$ and $\varphi \in \mathcal{L}_{\mathbb{R}}\left(\mathbb{Q}_p\right)$.

We set

$$\widetilde{\mathcal{A}}_R^{(N)}(\boldsymbol{x}, \boldsymbol{v}) := \frac{1}{Z_0^{1/D}} \int\limits_{\mathcal{L}_{\mathbb{R}}(\mathbb{Q}_p)} e^{\sum_{j=1}^N v_j \varphi(x_j)} d\mathbb{P}(\varphi), \tag{5.5}$$

where $\boldsymbol{x} = (x_1, \ldots, x_N) \in \mathbb{Q}_p^N$, $\boldsymbol{v} = (v_1, \ldots, v_N) \in \mathbb{R}^N$. Notice that

$$\sum_{j=1}^N v_j \varphi\left(x_j\right) = \sum_{j=1}^N v_j \left(\delta\left(x - x_j\right), \varphi(x)\right),$$

where $\delta\left(\cdot - x_j\right)$ denotes the Dirac distribution centered at x_j.

Lemma 2. *$\widetilde{\mathcal{A}}_R^{(N)}(\boldsymbol{x}, \boldsymbol{v}) < \infty$ for any $R, N, \boldsymbol{x}, \boldsymbol{v}$. Furthermore, for $R, N, \boldsymbol{v}$ fixed, $\widetilde{\mathcal{A}}_R^{(N)}(\boldsymbol{x}, \boldsymbol{v})$ is a continuous function in $\boldsymbol{x}$.*

Proof. We first recall that

$$\int\limits_{\mathcal{L}'_{\mathbb{R}}(\mathbb{Q}_p)} e^{(W,\theta)} d\mathbb{P}(W) < \infty \tag{5.6}$$

for any $\theta \in \mathcal{L}_{\mathbb{R}}\left(\mathbb{Q}_p\right)$, here (W, θ) denotes the pairing between the space of distributions $\mathcal{L}'_{\mathbb{R}}\left(\mathbb{Q}_p\right)$ and the space of Lizorkin test functions $\mathcal{L}_{\mathbb{R}}\left(\mathbb{Q}_p\right)$, cf. [42, Theorem 1.7].

By using that $\sum_{j=1}^N \left|v_j\right| \left|\varphi\left(x_j\right)\right| \delta\left(x - x_j\right) \in \mathcal{L}'_{\mathbb{R}}\left(\mathbb{Q}_p\right)$, for any $\varphi \in \mathcal{L}_{\mathbb{R}}\left(\mathbb{Q}_p\right)$, and by fixing $\theta \in \mathcal{L}_{\mathbb{R}}\left(\mathbb{Q}_p\right)$ such that $\theta\left(x_j\right) > 1$ for $j = 1, \ldots, N$, we have

$$\begin{aligned}
&\sum_{j=1}^N v_j \varphi\left(x_j\right) \le \sum_{j=1}^N \left|v_j\right| \left|\varphi\left(x_j\right)\right| \le \sum_{j=1}^N \left|v_j\right| \left|\varphi\left(x_j\right)\right| \theta\left(x_j\right) \\
&= \left(\sum_{j=1}^N \left|v_j\right| \left|\varphi\left(x_j\right)\right| \delta\left(x - x_j\right), \theta(x)\right),
\end{aligned}$$

and thus

$$\begin{aligned}
&\int\limits_{\mathcal{L}_{\mathbb{R}}(\mathbb{Q}_p)} e^{\sum_{j=1}^N v_j \varphi(x_j)} d\mathbb{P}(\varphi) \le \int\limits_{\mathcal{L}_{\mathbb{R}}(\mathbb{Q}_p)} e^{\sum_{j=1}^N |v_j| |\varphi(x_j)|} d\mathbb{P}(\varphi) \\
&\qquad \le \int\limits_{\mathcal{L}_{\mathbb{R}}(\mathbb{Q}_p)} e^{\left(\sum_{j=1}^N |v_j| |\varphi(x_j)| \delta(x - x_j), \theta(x)\right)} d\mathbb{P}(\varphi) \le \int\limits_{\mathcal{L}'_{\mathbb{R}}(\mathbb{Q}_p)} e^{(W,\theta)} d\mathbb{P}(W) < \infty
\end{aligned}$$

Finally, the continuity in $\boldsymbol{x}$ follows from the dominated convergence theorem by using that

$$\int\limits_{\mathcal{L}'_{\mathbb{R}}(\mathbb{Q}_p)} 1_{\mathcal{L}_{\mathbb{R}}(\mathbb{Q}_p)}(\varphi)\, e^{\sum_{j=1}^N v_j \varphi(x_j)} d\mathbb{P}(\varphi) \le \int\limits_{\mathcal{L}'_{\mathbb{R}}(\mathbb{Q}_p)} e^{(W,\theta)} d\mathbb{P}(W). \quad \square$$

Corollary 1. *For R fixed, $\mathcal{A}_R^{(N)}(\boldsymbol{k}) < \infty$ for any $\boldsymbol{k}$. Furthermore,*

$$\mathcal{A}_R^{(N)}(\boldsymbol{k}) = \frac{1}{Z_0}\int_{B_R^N}\left\{\int_{\mathcal{L}_{\mathbb{R}}^D(\mathbb{Q}_p)} e^{\sum_{j=1}^N \boldsymbol{k}_j\cdot\boldsymbol{\varphi}(x_j)} d\mathbb{P}_D(\boldsymbol{\varphi})\right\}\prod_{j=1}^N dx_j$$
$$= \frac{1}{Z_0}\int_{\mathcal{L}_{\mathbb{R}}^D(\mathbb{Q}_p)}\left\{\int_{B_R^N} e^{\sum_{j=1}^N \boldsymbol{k}_j\cdot\boldsymbol{\varphi}(x_j)}\prod_{j=1}^N dx_j\right\} d\mathbb{P}_D(\boldsymbol{\varphi}).$$

Proof. By Lemma 2, for $R, N, \boldsymbol{k}$ given,

$$\boldsymbol{x} \to \prod_{l=0}^{D-1}\int_{\mathcal{L}_{\mathbb{R}}(\mathbb{Q}_p)} e^{\sum_{j=1}^N k_{l,j}\varphi_l(x_j)} d\mathbb{P}(\varphi_l) = \int_{\mathcal{L}_{\mathbb{R}}^D(\mathbb{Q}_p)} e^{\sum_{j=1}^N\sum_{l=0}^{D-1} k_{l,j}\varphi_l(x_j)}\prod_{l=0}^{D-1} d\mathbb{P}(\varphi_l) < \infty$$

is a well-defined and continuous function. Now, the announced formula is a consequence of Fubini's theorem. □

5.2. Some technical results

We set

$$\delta_n(x) = \begin{cases} p^n & |x|_p \le p^{-n} \\ 0 & |x|_p > p^{-n}, \end{cases}$$

for a positive integer n, and recall that $\delta_n(x) \xrightarrow{\mathcal{D}'(\mathbb{Q}_p)} \delta(x)$, the Dirac distribution, as $n \to \infty$. We now introduce an approximation for $\widetilde{\mathcal{A}}_R^{(N)}(\boldsymbol{x}, \boldsymbol{v})$ given by

$$\widetilde{\mathcal{A}}_R^{(N)}(\boldsymbol{x}, \boldsymbol{v}; I) := \frac{1}{Z_0^{1/D}}\int_{\mathcal{L}_{\mathbb{R}}(\mathbb{Q}_p)} e^{\sum_{j=1}^N v_j(\delta_I(x-x_j),\varphi(x))} d\mathbb{P}(\varphi), \tag{5.7}$$

where I is a positive integer.

Lemma 3.

$$\lim_{I\to\infty}\widetilde{\mathcal{A}}_R^{(N)}(\boldsymbol{x}, \boldsymbol{v}; I) = \widetilde{\mathcal{A}}_R^{(N)}(\boldsymbol{x}, \boldsymbol{v}).$$

Proof. The proof is similar to the one given for Lemma 2. The result follows from

$$1_{\mathcal{L}_{\mathbb{R}}(\mathbb{Q}_p)}(\varphi)\, e^{\sum_{j=1}^N v_j(\delta_I(z-x_j),\varphi(x))} \le 1_{\mathcal{L}_{\mathbb{R}}(\mathbb{Q}_p)}(\varphi)\, e^{(W,\theta)},$$

where $W \in \mathcal{L}'_{\mathbb{R}}(\mathbb{Q}_p)$ is distribution depending on x_j, v_j, for $j = 1, \dots, N$, but not on I, and where $\theta \in \mathcal{L}_{\mathbb{R}}(\mathbb{Q}_p)$ is a fixed positive function. Now, by using (5.6) and the dominated convergence theorem, we have

$$\left|\sum_{j=1}^{N} v_j\left(\delta_I\left(z-x_j\right), \varphi(z)\right)\right| = \left|p^I \sum_{j=1}^{N} v_j \int_{x_j+p^I\mathbb{Z}_p} \varphi(y)\, dy\right|$$
$$\leq p^I \sum_{j=1}^{N}\left|v_j\right| \int_{x_j+p^I\mathbb{Z}_p} |\varphi(y)|\, dy.$$

We denote by l_φ index of local constancy of φ. We pick $I_\varphi = \max\left\{I, l_\varphi\right\}$, then $p^{I_\varphi}\mathbb{Z}_p$ is a subgroup of $p^I\mathbb{Z}_p$ and

$$G_j := \left(x_j + p^I\mathbb{Z}_p\right)/p^{I_\varphi}\mathbb{Z}_p$$

is a finite set such that $x_j + p^I\mathbb{Z}_p = \bigsqcup_{\widetilde{x}\in G_j}\left(\widetilde{x} + p^{I_\varphi}\mathbb{Z}_p\right)$ (disjoint union). We pick a function $\theta \in \mathcal{L}_{\mathbb{R}}\left(\mathbb{Q}_p\right)$ satisfying $\theta(\widetilde{x}) \geq 1$ for $\widetilde{x} \in G_j$, now

$$p^I \sum_{j=1}^{N}\left|v_j\right| \int_{x_j+p^I\mathbb{Z}_p} |\varphi(y)|\, dy = p^I \sum_{j=1}^{N}\sum_{\widetilde{x}\in G_j}\left|v_j\right| \int_{\widetilde{x}+p^{I_\varphi}\mathbb{Z}_p} |\varphi(y)|\, dy$$
$$= p^{I-I_\varphi}\sum_{j=1}^{N}\sum_{\widetilde{x}\in G_j}\left|v_j\right||\varphi(\widetilde{x})| \leq \sum_{j=1}^{N}\sum_{\widetilde{x}\in G_j}\left|v_j\right||\varphi(\widetilde{x})|$$
$$\leq \sum_{j=1}^{N}\sum_{\widetilde{x}\in G_j}\left|v_j\right||\varphi(\widetilde{x})|\theta(\widetilde{x}) = \sum_{j=1}^{N}\sum_{\widetilde{x}\in G_j}\left|v_j\right|\left(|\varphi(\widetilde{x})|\delta(z-\widetilde{x}), \theta(z)\right). \quad \square$$

5.2.1. A change of variables

Let $\varphi_{L,m}$, $\widetilde{\varphi}$ be functions in $\mathcal{L}_{\mathbb{R}}\left(\mathbb{Q}_p\right)$, for $L \geq 1$ and $m \in \mathbb{Q}_p^\times$. We now use the measurable mapping

$$\begin{aligned} \mathcal{L}_{\mathbb{R}}\left(\mathbb{Q}_p\right) &\rightarrow \mathcal{L}_{\mathbb{R}}\left(\mathbb{Q}_p\right) \\ \widetilde{\varphi} - \varphi_{L,m} &\rightarrow \quad \varphi, \end{aligned}$$

as a change of variables in (5.7). There exists a measure $\widetilde{\mathbb{P}}_{L,m}$ such that

$$\begin{aligned} \widetilde{\mathcal{A}}_R^{(N)}(\boldsymbol{x}, \boldsymbol{v}; I) &= \frac{1}{Z_0^{1/D}} \int_{\mathcal{L}_{\mathbb{R}}(\mathbb{Q}_p)} e^{\sum_{j=1}^{N} v_j\left(\delta_I\left(x-x_j\right), \widetilde{\varphi}-\varphi_{L,m}\right)} d\widetilde{\mathbb{P}}_{L,m}(\widetilde{\varphi}) \\ &= \frac{1}{Z_0^{1/D}} e^{\sum_{j=1}^{N} v_j\left(\delta_I\left(x-x_j\right), -\varphi_{L,m}\right)} \int_{\mathcal{L}_{\mathbb{R}}(\mathbb{Q}_p)} e^{\sum_{j=1}^{N} v_j\left(\delta_I\left(x-x_j\right), \widetilde{\varphi}\right)} d\widetilde{\mathbb{P}}_{L,m}(\widetilde{\varphi}). \end{aligned} \tag{5.8}$$

Our next goal is to compute the limits $|m|_p \rightarrow \infty$, $L \rightarrow \infty$, $I \rightarrow \infty$ in (5.8) to obtain a formula for $\mathcal{A}_R^{(N)}(\boldsymbol{k})$. This calculation is carried out in two steps.

5.2.2. Calculation of the first limit

Define for $L \geq 1$ and $m \in \mathbb{Q}_p^{\times}$,

$$J_{L,m}(x) = \sum_{j=1}^{N} v_j \delta_L\left(x - x_j\right) - \sum_{j=1}^{N} v_j \left|m\right|_p^{-1} \Omega\left(\frac{|x|_p}{|m|_p}\right) * \delta_L\left(x - x_j\right).$$

Lemma 4. *With the above notation, the following holds true:*
(i) $J_{L,m}(x) \in \mathcal{L}_{\mathbb{R}}\left(\mathbb{Q}_p\right)$ *for any* $L \geq 1$, $m \in \mathbb{Q}_p^{\times}$*;*
(ii) $J_{L,m}(x) \to J_L(x) := \sum_{j=1}^{N} v_j \delta_L\left(x - x_j\right)$ *in* $L^{\rho}\left(\mathbb{Q}_p\right)$, $1 < \rho < \infty$ *as* $|m|_p \to \infty$*;*
(iii) $J_{L,m}(x) \to J_L(x)$ *in* $\mathcal{L}'_{\mathbb{R}}\left(\mathbb{Q}_p\right)$ *as* $|m|_p \to \infty$*;*
(iv) The equation $\boldsymbol{D}\varphi_{L,m} = J_{L,m}$ *has a unique solution* $\varphi_{L,m} \in \mathcal{L}_{\mathbb{R}}\left(\mathbb{Q}_p\right)$ *given by* $\varphi_{L,m} = f_1 * J_{L,m}$*, where* f_1 *is defined in (3.2);*
(v) $\varphi_{L,m} \to f_1 * J_L$ *in* $\mathcal{L}'_{\mathbb{R}}\left(\mathbb{Q}_p\right)$ *as* $|m|_p \to \infty$*;*
(vi) $f_1 * J_L = \frac{1-p}{p \ln p} \sum_{j=1}^{N} v_j \ln\left|x - x_j\right|_p$*, if* $\left|x - x_j\right|_p > p^{-L}$ *for* $j = 1, \ldots, N$*.*

Proof. (i) Denote by $\Delta_m(\xi) = \Omega\left(|m\xi|_p\right)$, $m \in \mathbb{Q}_p^{\times}$, the characteristic function of the ball $B_{\log_p |m|_p^{-1}}$. Then

$$\widehat{J}_{L,m}(\xi) = \sum_{j=1}^{N} v_j \chi_p\left(\xi \cdot x_j\right) \Delta_L(\xi)\left(1 - \Delta_m(\xi)\right),$$

where $\Delta_L(\xi) = \Omega\left(p^{-L}|\xi|_p\right)$, which implies that $\widehat{J}_{L,m}$ is a test function satisfying $\widehat{J}_{L,m}(0) = 0$ for $|m|_p > 1$.

(ii) Notice that

$$J_{L,m}(x) - \sum_{j=1}^{N} v_j \delta_L\left(x - x_j\right) = -\sum_{j=1}^{N} v_j \left|m\right|_p^{-1} \Omega\left(\frac{|x|_p}{|m|_p}\right) * \delta_L\left(x - x_j\right).$$

By using that $|m|_p^{-1} \Omega\left(|m|_p^{-1}|x|_p\right) \in L^1\left(\mathbb{Q}_p\right)$ and $\delta_L(x) \in L^{\rho}$, $1 < \rho < \infty$, and applying [39, Lemma 7.4.2] we have $\Omega\left(\frac{|x|_p}{|m|_p}\right) * \delta_L\left(x - x_j\right) \underset{\rightarrow}{L^{\rho}} 0$ as $|m|_p \to \infty$.

(iii) Take $\theta \in \mathcal{L}_{\mathbb{R}}\left(\mathbb{Q}_p\right)$, by using the Cauchy–Schwarz inequality,

$$\left|\int_{\mathbb{Q}_p} J_{L,m}(x)\theta(x)\,dx - \int_{\mathbb{Q}_p} J_L(x)\theta(x)\,dx\right| = \left|\int_{\mathbb{Q}_p} \theta(x)\left(J_{L,m}(x) - J_L(x)\right)dx\right| \leq \|\theta\|_2 \|J_{L,m} - J_L\|_2.$$

By the second part $\|J_{L,m} - J_L\|_2 \to 0$ as $|m|_p \to \infty$.

(iv) See [16, Chapter 2, Section IX.2] or [39, Theorem 9.2.6]. (v) It follows from the third part by using the continuity of the convolution.

(vi) If $\left|x - x_j\right|_p > p^{-L}$ for any $j = 1, \ldots, N$,

$$f_1 * J_L(x) = \frac{1-p}{p \ln p} \sum_{j=1}^{N} v_j \ln|x|_p * \delta_L\left(x - x_j\right)$$

$$=\frac{1-p}{p\ln p}\sum_{j=1}^{N}v_j p^L \int\limits_{x-x_j+p^L\mathbb{Z}_p} \ln|z|_p\,dz = \frac{1-p}{p\ln p}\sum_{j=1}^{N}v_j \ln\left|x-x_j\right|_p. \tag{5.9}$$

□

Lemma 5.

$$\lim_{I\to\infty}\ \lim_{L\to\infty}\ \lim_{|m|_p\to\infty} e^{\sum_{j=1}^{N}v_j\left(\delta_I\left(x-x_j\right),-\varphi_{L,m}\right)} = e^{\dfrac{p-1}{p\ln p}\sum_{j=1}^{N}\sum_{i=1,i\neq j}^{N}v_j v_i \ln\left|x_j-x_i\right|_p}.$$

Proof. By using the formula for $f_1 * J_L(x)$, in the case $\left|x-x_j\right|_p > p^{-L}$ for any x_j, see (5.9), and the continuity of the pairing and the continuity of the convolution,

$$\sum_{j=1}^{N}v_j\left(\delta_I\left(x-x_j\right),-\varphi_{L,m}\right)\to\sum_{j=1}^{N}v_j\left(\delta_I\left(x-x_j\right),\frac{p-1}{p\ln p}\sum_{i=1}^{N}v_i\ln|x-x_i|_p\right)$$

in $\mathcal{L}'_{\mathbb{R}}\left(\mathbb{Q}_p\right)$ as $|m|_p\to\infty$, which implies that

$$e^{\sum_{j=1}^{N}v_j\left(\delta_I\left(x-x_j\right),-\varphi_{L,m}\right)}\to e^{\sum_{j=1}^{N}v_j\left(\delta_I\left(x-x_j\right),\frac{p-1}{p\ln p}\sum_{i=1}^{N}v_i\ln|x-x_i|_p\right)}$$

as $|m|_p\to\infty$. Now since $\ln|x|_p$ is locally constant in $\mathbb{Q}_p^{\times}$, and the $\lim_{t\to-\infty}e^t=0$, we have for I sufficiently large that

$$\left(\delta_I\left(x-x_j\right),\frac{p-1}{p\ln p}\sum_{i=1}^{N}v_i\ln|x-x_i|_p\right)=\frac{p-1}{p\ln p}\sum_{i=1}^{N}v_i\ln\left|x_j-x_i\right|_p,$$

if $x_j\neq x_i$, and $-\infty$ otherwise. Therefore,

$$e^{\sum_{j=1}^{N}v_j\left(\delta_I\left(x-x_j\right),\frac{p-1}{p\ln p}\sum_{i=1}^{N}v_i\ln|x-x_i|_p\right)}=e^{\frac{p-1}{p\ln p}\sum_{j=1}^{N}\sum_{i=1,i\neq j}^{N}v_j v_i\ln\left|x_j-x_i\right|_p},$$

for I sufficiently large. □

5.2.3. Calculation of the second limit

We now describe the measure $\widetilde{\mathbb{P}}_{L,m}$. Take $\varphi_{L,m}\in\mathcal{L}'_{\mathbb{R}}\left(\mathbb{Q}_p\right)$, $\widetilde{W}\in\mathcal{L}'_{\mathbb{R}}\left(\mathbb{Q}_p\right)$, by using (4.1) and changing variables as $W=\widetilde{W}-\varphi_{L,m}$, we have

$$\int\limits_{\mathcal{L}'_{\mathbb{R}}\left(\mathbb{Q}_p\right)} e^{\sqrt{-1}\left(\widetilde{W}-\varphi_{L,m},g\right)}d\widetilde{\mathbb{P}}_{L,m}\left(\widetilde{W}\right)=e^{-\frac{1}{2}\mathbb{B}(g,g)},$$

i.e.

$$\int\limits_{\mathcal{L}'_{\mathbb{R}}\left(\mathbb{Q}_p\right)} e^{\sqrt{-1}\left(\widetilde{W},g\right)}d\widetilde{\mathbb{P}}_{L,m}\left(\widetilde{W}\right)=e^{\sqrt{-1}\left(\varphi_{L,m},g\right)-\frac{1}{2}\mathbb{B}(g,g)}=:\mathcal{C}_{L,m}(g) \tag{5.10}$$

Notice that by Lemma 4,

$$\lim_{L\to\infty}\ \lim_{|m|_p\to\infty}\mathcal{C}_{L,m}(g)=e^{\sqrt{-1}\left(\frac{p-1}{p\ln p}\sum_{j=1}^{N}v_j\ln\left|x-x_j\right|_p,g\right)-\frac{1}{2}\mathbb{B}(g,g)}:=\mathcal{C}(g).$$

We denote $\widetilde{\mathbb{P}}$ the measure corresponding to $\mathcal{C}(g)$.

We now recall that

$$\mathcal{C}(h)=\int\limits_{\mathcal{L}'_{\mathbb{R}}(\mathbb{Q}_p)} e^{\sqrt{-1}(W,h)}d\mathbb{P}(W)=\int\limits_{\mathbb{R}} e^{\sqrt{-1}x}d\mathbb{P}_h(x),$$

where $\mathbb{P}_h(x)$ is the measure of the half-space $(W,h)\leq x$ in $\mathcal{L}'_{\mathbb{R}}(\mathbb{Q}_p)$, see e.g. [43, Chapter IV, Section 4.1]. Now if $\mathcal{C}(h_n)\to\mathcal{C}(\widetilde{h})$, and $\mathbb{P}_{h_n}(\mathbb{R})\leq 1$ for all n, then $\mathbb{P}_{h_n}\Rightarrow\mathbb{P}_{\widetilde{h}}$, $\mathcal{C}(\widetilde{h})$ is the characteristic function of $\mathbb{P}_{\widetilde{h}}$, see e.g. [44, Theorem 7.8.11]. The arrow '$\Rightarrow$' means that

$$\int\limits_{\mathbb{R}} l(x)d\mathbb{P}_{h_n}(x)\to\int\limits_{\mathbb{R}} l(x)d\mathbb{P}_{\widetilde{h}}(x)\ \text{ for any bounded continuous function } l(x). \tag{5.11}$$

Therefore

$$\widetilde{\mathbb{P}}_{L,m}\Rightarrow\widetilde{\mathbb{P}}\text{ when } |m|_p\to\infty,\, L\to\infty.$$

Now, if $l(x)\in L^1\left(\mathbb{R},\mathbb{P}_{h_n}\right)$ for any n and $l(x)\in L^1\left(\mathbb{R},\mathbb{P}_{\widetilde{h}}\right)$, by using the fact that the bounded continuous functions are dense in $L^1\left(\mathbb{R},\mathbb{P}_{h_n}\right)$ and $L^1\left(\mathbb{R},\mathbb{P}_{\widetilde{h}}\right)$, see e.g. [45, Proposition 1.3.22], in (5.11) we can assume that $l(x)$ is an integrable function.

In conclusion, we have the following result.

Lemma 6.

$$\lim_{L\to\infty}\lim_{|m|_p\to\infty}\int\limits_{\mathcal{L}_{\mathbb{R}}(\mathbb{Q}_p)} e^{\sum_{j=1}^N v_j\left(\delta_I\left(x-x_j\right),\widetilde{\varphi}\right)}d\widetilde{\mathbb{P}}_{L,m}(\widetilde{\varphi})=\int\limits_{\mathcal{L}_{\mathbb{R}}(\mathbb{Q}_p)} e^{\sum_{j=1}^N v_j\left(\delta_I\left(x-x_j\right),\widetilde{\varphi}\right)}d\widetilde{\mathbb{P}}(\widetilde{\varphi}),$$

and

$$\lim_{I\to\infty}\int\limits_{\mathcal{L}_{\mathbb{R}}(\mathbb{Q}_p)} e^{\sum_{j=1}^N v_j\left(\delta_I\left(x-x_j\right),\widetilde{\varphi}\right)}d\widetilde{\mathbb{P}}(\widetilde{\varphi})=\int\limits_{\mathcal{L}_{\mathbb{R}}(\mathbb{Q}_p)} e^{\sum_{j=1}^N v_j\widetilde{\varphi}(x_j)}d\widetilde{\mathbb{P}}(\widetilde{\varphi}).$$

5.2.4. *A formula for* $\mathcal{A}_R^{(N)}(\boldsymbol{k})$

Now, we recall that by using the change of variables (5.8), we have

$$\widetilde{\mathcal{A}}_R^{(N)}(\boldsymbol{x},\boldsymbol{v};I)=\frac{1}{Z_0^{1/D}}e^{\sum_{j=1}^N v_j\left(\delta_I\left(x-x_j\right),-\varphi_{L,m}\right)}\int\limits_{\mathcal{L}_{\mathbb{R}}(\mathbb{Q}_p)} e^{\sum_{j=1}^N v_j\left(\delta_I\left(x-x_j\right),\widetilde{\varphi}\right)}d\widetilde{\mathbb{P}}_{L,m}(\widetilde{\varphi}),$$

taking $\varphi_{L,m}$ to be the unique solution of $\boldsymbol{D}\varphi_{L,m}=J_{L,m}$, for each $m\in\mathbb{Q}_p^{\times}$. Then by applying Lemmas 3, 5, 6,

$$\lim_{I\to\infty}\lim_{L\to\infty}\lim_{|m|_p\to\infty,}\widetilde{\mathcal{A}}_R^{(N)}(\boldsymbol{x},\boldsymbol{v};I)=\widetilde{\mathcal{A}}_R^{(N)}(\boldsymbol{x},\boldsymbol{v})$$
$$=\frac{1}{Z_0^{1/D}}e^{\frac{p-1}{p\ln p}\sum_{j=1}^N\sum_{i=1,i\neq j}^N v_jv_i\ln\left|x_j-x_i\right|_p}\int\limits_{\mathcal{L}_{\mathbb{R}}(\mathbb{Q}_p)} e^{\sum_{j=1}^N v_j\widetilde{\varphi}(x_j)}d\widetilde{\mathbb{P}}(\widetilde{\varphi}).$$

By using this formula and the definition $\mathcal{A}_R^{(N)}(\boldsymbol{k})$, we establish the following result.

Proposition 1. *The amplitude* $\mathcal{A}_R^{(N)}(\boldsymbol{k})$ *satisfies*

$$\mathcal{A}_R^{(N)}(\boldsymbol{k}) = \frac{1}{Z_0}\int_{B_R^N} \prod_{j<i}^{N} \left|x_j - x_i\right|_p^{2\frac{(p-1)}{p\ln p}\boldsymbol{k}_i\cdot\boldsymbol{k}_j} \int_{\mathcal{L}_{\mathbb{R}}^D(\mathbb{Q}_p)} e^{\sum_{j=1}^N \boldsymbol{k}_j\cdot\widetilde{\boldsymbol{\varphi}}(x_j)} d\widetilde{\mathbb{P}}_D(\widetilde{\boldsymbol{\varphi}}) \prod_{j=0}^{N} dx_j.$$

We now introduce the 'convention' that the insertion points $x_1, x_2, \ldots, x_{N-1}, x_N$, with $N \geq 4$, belong to the p-adic projective line, and then by using the Möbius group, we may take the normalization

$$x_1 = 0, x_{N-1} = 1, x_N = \infty.$$

In our framework, the convention $x_N = \infty$ means that the N-point amplitudes do not depend on x_N, then $\mathcal{A}_R^{(N)}(\boldsymbol{k})$ takes the form

$$\mathcal{A}_R^{(N)}(\boldsymbol{k}) = \frac{C_0}{Z_0}\int_{B_R^{N-3}} \prod_{i=2}^{N-2} |x_i|_p^{2\frac{(p-1)}{p\ln p}\boldsymbol{k}_1\cdot\boldsymbol{k}_i} |1-x_i|_p^{2\frac{(p-1)}{p\ln p}\boldsymbol{k}_{N-1}\cdot\boldsymbol{k}_i}$$
$$\times \prod_{2\leq i,j\leq N-2} \left|x_j - x_i\right|_p^{2\frac{(p-1)}{p\ln p}\boldsymbol{k}_i\cdot\boldsymbol{k}_j} \int_{\mathcal{L}_{\mathbb{R}}^D(\mathbb{Q}_p)} e^{\sum_{j=2}^{N-2} \boldsymbol{k}_j\cdot\widetilde{\boldsymbol{\varphi}}(x_j)} d\widetilde{\mathbb{P}}_D(\widetilde{\boldsymbol{\varphi}}) \prod_{j=2}^{N-2} dx_j,$$

where the momenta vectors satisfy $\sum_{i=1}^N \boldsymbol{k}_i = \boldsymbol{0}$ and

$$C_0 = \int_{\mathcal{L}_{\mathbb{R}}^D(\mathbb{Q}_p)} e^{\boldsymbol{k}_1\cdot\widetilde{\boldsymbol{\varphi}}(0)+\boldsymbol{k}_{N-1}\cdot\widetilde{\boldsymbol{\varphi}}(1)} d\widetilde{\mathbb{P}}_D(\widetilde{\boldsymbol{\varphi}}).$$

We now consider the function

$$\Theta(\boldsymbol{k},\boldsymbol{x}) := \Theta(\boldsymbol{k}, x_2, \ldots, x_{N-2}) = \int_{\mathcal{L}_{\mathbb{R}}^D(\mathbb{Q}_p)} e^{\sum_{j=2}^{N-2} \boldsymbol{k}_j\cdot\widetilde{\boldsymbol{\varphi}}(x_j)} d\widetilde{\mathbb{P}}_D(\widetilde{\boldsymbol{\varphi}}).$$

By using that

$$e^{\sum_{j=2}^{N-2} \boldsymbol{k}_j\cdot\widetilde{\boldsymbol{\varphi}}(x_j)} = \lim_{M\to\infty}\sum_{r=0}^{M} \frac{\left(\sum_{j=2}^{N-2} \boldsymbol{k}_j\cdot\widetilde{\boldsymbol{\varphi}}(x_j)\right)^r}{r!}$$
$$= \lim_{M'\to\infty}\sum_{r=0}^{M'} F_r(\boldsymbol{k}, \widetilde{\boldsymbol{\varphi}}(x_2), \ldots, \widetilde{\boldsymbol{\varphi}}(x_{N-2})),$$

where $F_r(\boldsymbol{k}, \widetilde{\boldsymbol{\varphi}}(x_2), \ldots, \widetilde{\boldsymbol{\varphi}}(x_{N-2}))$ is a homogeneous polynomial of degree r in the variables $k_{l,j}$, $l = 0, \ldots, D-1$, $j = 2, \ldots, N-2$, whose coefficients are polynomials in the $\widetilde{\boldsymbol{\varphi}}(x_2), \ldots, \widetilde{\boldsymbol{\varphi}}(x_{N-2})$. By the dominated convergence theorem, Corollary 1, and

$$\sum_{r=0}^{M'} |F_r(\boldsymbol{k}, \widetilde{\boldsymbol{\varphi}}(x_2), \ldots, \widetilde{\boldsymbol{\varphi}}(x_{N-2}))| \leq e^{\sum_{j=2}^{N-2}\sum_{l=0}^{D-1}|k_{l,j}||\widetilde{\varphi}_l(x_j)|} \in L^1\left(\mathcal{L}_{\mathbb{R}}^D(\mathbb{Q}_p), \widetilde{\mathbb{P}}_D\right),$$

we have

$$\Theta(\boldsymbol{k},\boldsymbol{x}) = \int\limits_{\mathcal{L}_{\mathbb{R}}^{D}(\mathbb{Q}_p)} \left\{ \lim_{M'\to\infty} \sum_{r=0}^{M'} F_r\left(\boldsymbol{k}, \widetilde{\boldsymbol{\varphi}}(x_2), \ldots, \widetilde{\boldsymbol{\varphi}}(x_{N-2})\right) \right\} d\widetilde{\mathbb{P}}_D(\boldsymbol{\varphi})$$

$$= \lim_{M'\to\infty} \sum_{r=0}^{M'} \int\limits_{\mathcal{L}_{\mathbb{R}}^{D}(\mathbb{Q}_p)} F_r\left(\boldsymbol{k}, \widetilde{\boldsymbol{\varphi}}(x_2), \ldots, \widetilde{\boldsymbol{\varphi}}(x_{N-2})\right) d\widetilde{\mathbb{P}}_D(\widetilde{\boldsymbol{\varphi}})$$

$$= \int\limits_{\mathcal{L}_{\mathbb{R}}^{D}(\mathbb{Q}_p)} d\widetilde{\mathbb{P}}_D(\widetilde{\boldsymbol{\varphi}}) + \sum_{r=1}^{\infty} \int\limits_{\mathcal{L}_{\mathbb{R}}^{D}(\mathbb{Q}_p)} F_r\left(\boldsymbol{k}, \widetilde{\boldsymbol{\varphi}}(x_2), \ldots, \widetilde{\boldsymbol{\varphi}}(x_{N-2})\right) d\widetilde{\mathbb{P}}_D(\widetilde{\boldsymbol{\varphi}}),$$

where $\boldsymbol{x} = (x_2, \ldots, x_{N-2})$. Now by using that $F_r\left(\boldsymbol{k}, \widetilde{\boldsymbol{\varphi}}(x_2), \ldots, \widetilde{\boldsymbol{\varphi}}(x_{N-2})\right)$ are integrable continuous functions in $\boldsymbol{x}$ for $\boldsymbol{k}$ fixed, we conclude that

$$G_r(\boldsymbol{k},\boldsymbol{x}) := \int\limits_{\mathcal{L}_{\mathbb{R}}^{D}(\mathbb{Q}_p)} F_r\left(\boldsymbol{k}, \widetilde{\boldsymbol{\varphi}}(x_2), \ldots, \widetilde{\boldsymbol{\varphi}}(x_{N-2})\right) d\widetilde{\mathbb{P}}_D(\widetilde{\boldsymbol{\varphi}})$$

are continuous functions in $\boldsymbol{x}$. Therefore

$$\Theta(\boldsymbol{k},\boldsymbol{x}) = C + \sum_{r=1}^{\infty} G_r(\boldsymbol{k},\boldsymbol{x}).$$

Now by using the formula given in Proposition 1, and Fubini's theorem to interchange $\int_{B_R^{N-3}}$ and $\sum_{r=1}^{\infty}$, we obtain the following result.

Theorem 1. *The amplitude $\mathcal{A}_R^{(N)}(\boldsymbol{k})$ admits the following expansion in the momenta:*

$$\mathcal{A}_R^{(N)}(\boldsymbol{k}) = \frac{CC_0}{Z_0} \int\limits_{B_R^{N-3}} \prod_{i=2}^{N-2} |x_i|_p^{2\frac{(p-1)}{p\ln p}\boldsymbol{k}_1\cdot\boldsymbol{k}_i} |1-x_i|_p^{2\frac{(p-1)}{p\ln p}\boldsymbol{k}_{N-1}\cdot\boldsymbol{k}_i}$$

$$\times \prod_{2\le i,j\le N-2} \left|x_j - x_i\right|_p^{2\frac{(p-1)}{p\ln p}\boldsymbol{k}_i\cdot\boldsymbol{k}_j} \prod_{j=2}^{N-2} dx_j$$

$$+ \frac{C_0}{Z_0} \sum_{r=1}^{\infty} \int\limits_{B_R^{N-3}} \prod_{i=2}^{N-2} |x_i|_p^{2\frac{(p-1)}{p\ln p}\boldsymbol{k}_1\cdot\boldsymbol{k}_i} |1-x_i|_p^{2\frac{(p-1)}{p\ln p}\boldsymbol{k}_{N-1}\cdot\boldsymbol{k}_i}$$

$$\times \prod_{2\le i,j\le N-2} \left|x_j - x_i\right|_p^{2\frac{(p-1)}{p\ln p}\boldsymbol{k}_i\cdot\boldsymbol{k}_j} G_r(\boldsymbol{k},\boldsymbol{x}) \prod_{j=2}^{N-2} dx_j.$$

To continue the study of the amplitudes $\mathcal{A}_R^{(N)}(\boldsymbol{k})$, we introduce the following notation:

$$A_R^{(N)}(\boldsymbol{k}) = \frac{CC_0}{Z_0} \int_{B_R^{N-3}} \prod_{i=2}^{N-2} |x_i|_p^{2\frac{(p-1)}{p\ln p}\boldsymbol{k}_1\cdot\boldsymbol{k}_i} |1-x_i|_p^{2\frac{(p-1)}{p\ln p}\boldsymbol{k}_{N-1}\cdot\boldsymbol{k}_i} \times \prod_{2\le i,j\le N-2} |x_j - x_i|_p^{2\frac{(p-1)}{p\ln p}\boldsymbol{k}_i\cdot\boldsymbol{k}_j} \prod_{j=2}^{N-2} dx_j,$$

$$Z_{G_r,R}^{(N)}(\boldsymbol{k}) = \frac{C_0}{Z_0} \int_{\mathbb{Q}_p^{N-3}} \prod_{i=2}^{N-2} |x_i|_p^{2\frac{(p-1)}{p\ln p}\boldsymbol{k}_1\cdot\boldsymbol{k}_i} |1-x_i|_p^{2\frac{(p-1)}{p\ln p}\boldsymbol{k}_{N-1}\cdot\boldsymbol{k}_i} \times \prod_{2\le i,j\le N-2} |x_j - x_i|_p^{2\frac{(p-1)}{p\ln p}\boldsymbol{k}_i\cdot\boldsymbol{k}_j} 1_{B_R^{N-3}}(\boldsymbol{x}) G_r(\boldsymbol{k},\boldsymbol{x}) \prod_{j=2}^{N-2} dx_j.$$

Notice that $1_{B_R^{N-3}}(\boldsymbol{x})G_r(\boldsymbol{k},\boldsymbol{x})$ is a continuous function in $\boldsymbol{x}$ with support contained in B_R^{N-3}.

6. Regularization of p-adic open string amplitudes, and multivariate local zeta functions

6.1. The p-adic Koba-Nielsen local zeta functions

Take $N \geq 4$ and $s_{ij} \in \mathbb{C}$ satisfying $s_{ij} = s_{ji}$ for $1 \leq i < j \leq N-1$. The p-adic Koba-Nielsen local zeta function (or p-adic open string N-point zeta function) is defined as

$$Z^{(N)}(\mathbf{s}) = \int_{\mathbb{Q}_p^{N-3} \smallsetminus \Lambda} \prod_{i=2}^{N-2} |x_i|_p^{s_{1i}} |1-x_i|_p^{s_{(N-1)i}} \prod_{2\le i<j\le N-2} |x_i - x_j|_p^{s_{ij}} \prod_{i=2}^{N-2} dx_i, \tag{6.1}$$

where $\boldsymbol{s} = (s_{ij}) \in \mathbb{C}^{D_0}$, here D_0 denotes the total number of possible subsets $\{i, j\}$, $\prod_{i=2}^{N-2} dx_i$ is the normalized Haar measure of $\mathbb{Q}_p^{N-3}$, and

$$\Lambda := \left\{ (x_2, \dots, x_{N-2}) \in \mathbb{Q}_p^{N-3};\ \prod_{i=2}^{N-2} x_i (1-x_i) \prod_{2\le i<j\le N-2} (x_i - x_j) = 0 \right\}.$$

These functions were introduced in [8], see also [6]. The functions $Z^{(N)}(\mathbf{s})$ are holomorphic in a certain domain of $\mathbb{C}^{D_0}$ and admit analytic continuations to $\mathbb{C}^{D_0}$ (denoted also as $Z^{(N)}(\mathbf{s})$) as rational functions in the variables

$$p^{-s_{ij}}, i, j \in \{1, \dots, N-1\},$$

see [8, Theorem 1], [6, Theorem 6.1].

If $\phi(x_2, \dots, x_{N-2})$ is a locally constant function with compact support, then

$$Z_\phi^{(N)}(\boldsymbol{s}) = \int_{\mathbb{Q}_p^{N-3} \smallsetminus \Lambda} \phi(x_2, \dots, x_{N-2}) \prod_{i=2}^{N-2} |x_i|_p^{s_{1i}} |1-x_i|_p^{s_{(N-1)i}} \prod_{2\le i<j\le N-2} |x_i - x_j|_p^{s_{ij}} \prod_{i=2}^{N-2} dx_i,$$

for $\mathrm{Re}(s_{ij}) > 0$ for any ij, is a multivariate Igusa local zeta function. These functions admit analytic continuations as rational functions of the variables $p^{-s_{ij}}$, [46]. If we take ϕ to be the characteristic function of B_R^{N-3}, the ball centered at the origin with radius p^R, the dominated convergence theorem and [8, Theorem 1], imply that

$$\lim_{R\to\infty} Z_R^{(N)}(\boldsymbol{s}) := \lim_{R\to\infty} \int_{B_R^{N-3}\smallsetminus\Lambda} \prod_{i=2}^{N-2} |x_i|_p^{s_{1i}} |1-x_i|_p^{s_{(N-1)i}} \prod_{2\le i<j\le N-2} \left|x_i - x_j\right|_p^{s_{ij}} \prod_{i=2}^{N-2} dx_i \tag{6.2}$$
$$= Z^{(N)}(\mathbf{s}),$$

for any $\boldsymbol{s}$ in the natural domain of $Z^{(N)}(\mathbf{s})$.

In [4], Brekke, Freund, Olson and Witten work out the N-point amplitudes in explicit form and investigate how these can be obtained from an effective Lagrangian. The p-adic open string N-point tree amplitudes are defined as

$$A_{\mathcal{M}}^{(N)}(\boldsymbol{k}) = \tag{6.3}$$
$$\int_{\mathbb{Q}_p^{N-3}} \prod_{i=2}^{N-2} |x_i|_p^{\boldsymbol{k}_1\boldsymbol{k}_i} |1-x_i|_p^{\boldsymbol{k}_{N-1}\boldsymbol{k}_i} \prod_{2\le i<j\le N-2} \left|x_i - x_j\right|_p^{\boldsymbol{k}_i\boldsymbol{k}_j} \prod_{i=2}^{N-2} dx_i,$$

where $\prod_{i=2}^{N-2} dx_i$ is the normalized Haar measure of $\mathbb{Q}_p^{N-3}$, $\boldsymbol{k} = (\boldsymbol{k}_1, \ldots, \boldsymbol{k}_N)$, $\boldsymbol{k}_i = \big(k_{0,i}, \ldots, k_{D-1,i}\big)$, $i = 1, \ldots, N$, $N \ge 4$, is the momentum vector of the i-th tachyon (with Minkowski product $\boldsymbol{k}_i\boldsymbol{k}_j = -k_{0,i}k_{0,j} + k_{1,i}k_{1,j} + \cdots + k_{D-1,i}k_{D-1,j}$) obeying

$$\sum_{i=1}^{N} \boldsymbol{k}_i = \mathbf{0}, \qquad \boldsymbol{k}_i\boldsymbol{k}_i = 2 \quad \text{for } i = 1, \ldots, N.$$

In [8], [6], the p-adic open string N-point tree integrals $Z^{(N)}(\boldsymbol{s})$ are used as regularizations of the amplitudes $A_{\mathcal{M}}^{(N)}(\boldsymbol{k})$. More precisely, the amplitude $A_{\mathcal{M}}^{(N)}(\boldsymbol{k})$ can be re-define as

$$A_{\mathcal{M}}^{(N)}(\boldsymbol{k}) = Z^{(N)}(\boldsymbol{s})\,|_{s_{ij}=\boldsymbol{k}_i\boldsymbol{k}_j} \text{ with } i \in \{1, \ldots, N-1\},\ j \in T \text{ or } i, j \in T,$$

where $T = \{2, \ldots, N-2\}$. Then the amplitudes $A_{\mathcal{M}}^{(N)}(\boldsymbol{k})$ are well-defined rational functions of the variables $p^{-\boldsymbol{k}_i\boldsymbol{k}_j}$, $i, j \in \{1, \ldots, N-1\}$, which agree with integrals (6.3) when they converge.

Remark 1. In [8], [6], the local zeta functions $Z^{(N)}(\boldsymbol{s})$ were used to regularize Koba-Nielsen amplitudes $A_{\mathcal{M}}^{(N)}(\boldsymbol{k})$, when the momenta $\boldsymbol{k}$ belong to the Minkowski space. In this article, we use the functions $Z^{(N)}(\boldsymbol{s})$ to regularize Koba-Nielsen amplitudes $A^{(N)}(\boldsymbol{k})$ when the momenta $\boldsymbol{k}$ belong to the Euclidean space. This is possible because $Z^{(N)}(\boldsymbol{s})$ is a rational function in the variables $p^{-s_{ij}}$, $s_{ij} \in \mathbb{C}$, for $i, j \in \{1, \ldots, N-1\}$.

Remark 2. We denote by $Z_{\cdot}^{(N)}(\boldsymbol{s})$ the distribution $\phi \to Z_{\phi}^{(N)}(\boldsymbol{s})$. Then the mapping

$$\begin{array}{ccc} \mathbb{C}^{D_0} & \to & \mathcal{D}'(\mathbb{Q}_p^{N-3}) \\ \boldsymbol{s} & \to & Z_{\cdot}^{(N)}(\boldsymbol{s}) \end{array} \tag{6.4}$$

is a meromorphic function of $\boldsymbol{s}$. By using the fact that $\mathcal{D}(\mathbb{Q}_p^{N-3})$ is dense in the space of continuous functions with compact support $\mathcal{C}_c(\mathbb{Q}_p^{N-3})$, the functional $\phi \to Z_\phi^{(N)}(\boldsymbol{s})$ has a unique extension to $\mathcal{C}_c(\mathbb{Q}_p^{N-3})$. Furthermore, if $\boldsymbol{s}_0$ is a pole of $Z_\phi^{(N)}(\boldsymbol{s})$, by using Gel'fand-Shilov method of analytic continuation, see e.g. [37, pgs. 65-67],

$$Z_\phi^{(N)}(\boldsymbol{s}) = \sum_{\boldsymbol{k}\in\mathbb{Z}^{D_0}} c_{\boldsymbol{k}}(\phi)\,(\boldsymbol{s}-\boldsymbol{s}_0)^{\boldsymbol{k}},$$

where the $c_{\boldsymbol{k}}$s are distributions from $\mathcal{D}'(\mathbb{Q}_p^{N-3})$. The density of $\mathcal{D}(\mathbb{Q}_p^{N-3})$ in $\mathcal{C}_c(\mathbb{Q}_p^{N-3})$ implies that $c_{\boldsymbol{k}} \neq 0$ in $\mathcal{D}'(\mathbb{Q}_p^{N-3})$ if and only if $c_{\boldsymbol{k}} \neq 0$ in $\mathcal{C}_c'(\mathbb{Q}_p^{N-3})$, the strong dual space of $\mathcal{C}_c(\mathbb{Q}_p^{N-3})$. This implies that the mapping

$$\begin{array}{ccc} \mathbb{C}^{D_0} & \to & \mathcal{C}_c'(\mathbb{Q}_p^{N-3}) \\ & & \\ \boldsymbol{s} & \to & Z_\cdot^{(N)}(\boldsymbol{s}) \end{array}$$

is a meromorphic function in $\boldsymbol{s}$ having the same poles of the mapping (6.4).

6.2. *The limit* $\lim_{R\to\infty}\mathcal{A}_R^{(N)}(\boldsymbol{k})$

We now apply the above-mentioned results to study the limit

$$\lim_{R\to\infty}\mathcal{A}_R^{(N)}(\boldsymbol{k}).$$

First, notice that by (6.2),

$$\frac{Z_0}{CC_0}\lim_{R\to\infty} A_R^{(N)}(\boldsymbol{k}) = \lim_{R\to\infty}\left(Z_R^{(N)}(\boldsymbol{s})\mid_{s_{ij}=2\frac{(p-1)}{p\ln p}\boldsymbol{k}_i\cdot\boldsymbol{k}_j}\right) = Z^{(N)}(\boldsymbol{s})\mid_{s_{ij}=2\frac{(p-1)}{p\ln p}\boldsymbol{k}_i\cdot\boldsymbol{k}_j}.$$

Now by using the fact that $Z^{(N)}(\boldsymbol{s})$ is a holomorphic function in a certain domain of $\mathbb{C}^{D_0}$, we conclude that $\lim_{R\to\infty} A_R^{(N)}(\boldsymbol{k})$ exists for $\boldsymbol{k}$ belonging a non-empty subset of $\mathbb{C}^{D_0}$.

Second, by using Remark 2, we may assume that $1_{B_R^{N-3}}(\boldsymbol{x})G_r(\boldsymbol{k},\boldsymbol{x}) = \phi$ is a test function in $\boldsymbol{x}$, and then $Z_{G_r,R}^{(N)}(\boldsymbol{k}) = \frac{C_0}{Z_0} Z_\phi^{(N)}(\boldsymbol{s})\mid_{s_{ij}=2\frac{(p-1)}{p\ln p}\boldsymbol{k}_i\cdot\boldsymbol{k}_j}$ is a multivariate local zeta function. Furthermore,

$$|Z_{G_r,R}^{(N)}(\boldsymbol{k})| \le \frac{C_0}{Z_0}\int_{\mathbb{Q}_p^{N-3}}\prod_{i=2}^{N-2}|x_i|_p^{2\frac{(p-1)}{p\ln p}\boldsymbol{k}_1\cdot\boldsymbol{k}_i}\,|1-x_i|_p^{2\frac{(p-1)}{p\ln p}\boldsymbol{k}_{N-1}\cdot\boldsymbol{k}_i}$$
$$\times\prod_{2\le i,j\le N-2}\left|x_j-x_i\right|_p^{2\frac{(p-1)}{p\ln p}\boldsymbol{k}_i\cdot\boldsymbol{k}_j}\,|G_r(\boldsymbol{k},\boldsymbol{x})|\prod_{j=2}^{N-2}dx_j,$$

which implies that $\left|Z_{G_r,R}^{(N)}(\boldsymbol{k})\right| \le \frac{C_0\,C_r(\boldsymbol{k},R)}{Z_0} Z^{(N)}(\boldsymbol{k})$, where

$$C_r(\boldsymbol{k},R) = \sup_{\boldsymbol{x}\in B_R^{N-3}}|G_r(\boldsymbol{k},\boldsymbol{x})|.$$

Since $Z^{(N)}(\boldsymbol{k})$ converges in a non-empty open set, we conclude that all the $Z_{G_r,R}^{(N)}(\boldsymbol{k})$s converges in the open set where $Z^{(N)}(\boldsymbol{k})$ converges.

In conclusion we have the following result.

Theorem 2. *The amplitudes* $\mathcal{A}_R^{(N)}(\boldsymbol{k})$ *satisfy the following. For R fixed,*

$$\mathcal{A}_R^{(N)}(\boldsymbol{k}) = A_R^{(N)}(\boldsymbol{k}) + \sum_{r=1}^{\infty} Z_{G_r,R}^{(N)}(\boldsymbol{k}),$$

where $A_R^{(N)}(\boldsymbol{k})$*, and all the* $Z_{G_r,R}^{(N)}(\boldsymbol{k})$*s are multivariate Igusa's local zeta functions, all of them converging in a common non-empty open set. Furthermore,*

$$\lim_{R\to\infty} A_R^{(N)}(\boldsymbol{k}) = \frac{CC_0}{Z_0} Z^{(N)}(\boldsymbol{k}),$$

which is the p*-adic Koba-Nielsen open string amplitude.*

6.3. $\boldsymbol{\varphi}^4$-theories

Consider the family of $\boldsymbol{\varphi}^4$-interacting quantum field theories:

$$\frac{1_{\mathcal{L}_{\mathbb{R}}^D(\mathbb{Q}_p)}(\boldsymbol{\varphi})\, e^{-\lambda E_{int}(\boldsymbol{\varphi})} d\mathbb{P}_D(\boldsymbol{\varphi})}{Z}, \text{ for } \lambda > 0,$$

where

$$E_{int}(\boldsymbol{\varphi}) = \sum_{j=0}^{D-1} \int_{\mathbb{Q}_p} \varphi_j^4(x)dx, \quad \text{and} \quad Z = \int_{\mathcal{L}_{\mathbb{R}}^D(\mathbb{Q}_p)} e^{-\lambda E_{int}(\boldsymbol{\varphi})} d\mathbb{P}_D(\boldsymbol{\varphi}).$$

The amplitudes of such theories are defined as

$$\mathcal{A}_R^{(N)}(\boldsymbol{k},\lambda) = \frac{1}{Z}\int_{B_R^{N-3}} \left\{ \int_{\mathcal{L}_{\mathbb{R}}^D(\mathbb{Q}_p)} e^{\sum_{j=2}^{N-2} \boldsymbol{k}_j\cdot\boldsymbol{\varphi}(x_j) - \lambda E_{int}(\boldsymbol{\varphi})} d\mathbb{P}_D(\boldsymbol{\varphi}) \right\} \prod_{j=2}^{N-2} dx_j.$$

These amplitudes admit expansions of the type given in Proposition 1, where the functions $G_r(\boldsymbol{k},\boldsymbol{x})$ are replaced by continuous functions in $\boldsymbol{x}$ depending on $\boldsymbol{k}$ and λ. The behavior of these quantum field theories is completely different from the standard ones due to the fact that we are computing the correlation functions for a very particular class of observables, which are products of vertex operators.

CRediT authorship contribution statement

All the authors contributed to the manuscript equally. All authors have read and agreed to the published version of the manuscript.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgement

The authors wish to thank the referee for his/her careful reading of the original manuscript and the suggestions made.

References

[1] G. Veneziano, Construction of a crossing-symmetric, Reggeon-behaved amplitude for linearly rising trajectories, Il Nuovo Cimento A 57 (1968) 190–197.
[2] M. Virasoro, Alternative constructions of crossing-symmetric amplitudes with Regge behavior, Phys. Rev. 177 (5) (1969) 2309–2311.
[3] Z. Koba, H. Nielsen, Reaction amplitude for N-mesons: a generalization of the Veneziano-Bardakçi-Ruegg-Virasoro model, Nucl. Phys. B 10 (4) (1969) 633–655.
[4] Lee Brekke, Peter G.O. Freund, Mark Olson, Edward Witten, Non-Archimedean string dynamics, Nucl. Phys. B 302 (3) (1988) 365–402.
[5] I.V. Volovich, p-Adic string, Class. Quantum Gravity 4 (4) (1987) L83–L87.
[6] M. Bocardo-Gaspar, Willem Veys, W.A. Zuñiga-Galindo, Meromorphic continuation of Koba-Nielsen string amplitudes, J. High Energy Phys. 138 (2020).
[7] H. García-Compeán, Edgar Y. López, W.A. Zúñiga-Galindo, p-Adic open string amplitudes with Chan-Paton factors coupled to a constant B-field, Nucl. Phys. B 951 (2020) 114904.
[8] M. Bocardo-Gaspar, H. Garcia-Compean, W.A. Zúñiga-Galindo, Regularization of p-adic string amplitudes, and multivariate local zeta functions, Lett. Math. Phys. 109 (5) (2019) 1167–1204.
[9] M. Bocardo-Gaspar, H. Garcia-Compean, W.A. Zúñiga-Galindo, p-Adic string amplitudes in the limit p approaches to one, J. High Energy Phys. 43 (2018).
[10] M. Bocardo-Gaspar, H. García-Compeán, E.Y. López, W.A. Zúñiga-Galindo, Local zeta functions and Koba–Nielsen string amplitudes, Symmetry 13 (2021) 967, https://doi.org/10.3390/sym13060967.
[11] I.Ya. Aref'eva, B.G. Dragović, I.V. Volovich, On the adelic string amplitudes, Phys. Lett. B 209 (4) (1988) 445–450.
[12] I.Ya. Aref'eva, B.G. Dragović, I.V. Volovich, Open and closed p-adic strings and quadratic extensions of number fields, Phys. Lett. B 212 (3) (1988) 283–291.
[13] I.Ya. Aref'eva, B.G. Dragović, I.V. Volovich, p-Adic superstrings, Phys. Lett. B 214 (3) (1988) 339–349.
[14] Lee Brekke, Peter G.O. Freund, p-Adic numbers in physics, Phys. Rep. 233 (1) (1993) 1–66.
[15] Zvonimir Hloušek, Donald spector, p-adic string theories, Ann. Phys. 189 (2) (1989) 370–431.
[16] V.S. Vladimirov, I.V. Volovich, E.I. Zelenov, p-Adic Analysis and Mathematical Physics, World Scientific, Singapore, 1994.
[17] B. Dragovich, A.Y. Khrennikov, S.V. Kozyrev, I.V. Volovich, E.I. Zelenov, p-Adic mathematical physics: the first 30 years, P-Adic Numb. Ultrametr. Anal. Appl. 9 (2017) 87–121.
[18] S.S. Gubser, J. Knaute, S. Parikh, A. Samberg, P. Witaszczyk, p-Adic AdS/CFT, Commun. Math. Phys. 352 (3) (2017) 1019.
[19] M. Heydeman, M. Marcolli, I. Saberi, B. Stoica, Tensor networks, p-adic fields, and algebraic curves: arithmetic and the $\mathrm{AdS}_3/\mathrm{CFT}_2$ correspondence, Adv. Theor. Math. Phys. 22 (1) (2018) 93–176.
[20] S.S. Gubser, M. Heydeman, C. Jepsen, M. Marcolli, S. Parikh, I. Saberi, B. Stoica, B. Trundy, Edge length dynamics on graphs with applications to p-adic AdS/CFT, J. High Energy Phys. 157 (2017).
[21] P. Dutta, D. Ghoshal, A. Lala, Notes on the exchange interactions in holographic p-adic CFT, Phys. Lett. B 773 (2017) 283–289.
[22] P.G.O. Freund, M. Olson, Nonarchimedean strings, Phys. Lett. B 199 (2) (1987) 186–190.
[23] P.G.O. Freund, E. Witten, Adelic string amplitudes, Phys. Lett. B 199 (2) (1987) 191–194.
[24] A.A. Gerasimov, S.L. Shatashvili, On exact Tachyon potential in open string field theory, J. High Energy Phys. 5 (2013).
[25] B.L. Spokoiny, Quantum geometry of nonarchimedean particles and strings, Phys. Lett. B 208 (3–4) (1988) 401–405.
[26] D. Ghoshal, p-Adic string theories provide lattice discretization to the ordinary string worldsheet, Phys. Rev. Lett. 97 (2006) 151601.
[27] L. Brekke, P.G.O. Freund, M. Olson, E. Witten, Non-archimedean string dynamics, Nucl. Phys. B 302 (3) (1988) 365–402.
[28] P.H. Frampton, Y. Okada, The p-adic string N-point function, Phys. Rev. Lett. 60 (1988) 484–486.

[29] R.B. Zhang, Lagrangian formulation of open and closed p-adic strings, Phys. Lett. B 209 (2–3) (1988) 229–232.
[30] G. Parisi, On p-adic functional integrals, Mod. Phys. Lett. A 3 (1988) 639–643.
[31] A.V. Zabrodin, Nonarchimedean strings and Bruhat-Tits trees, Commun. Math. Phys. 123 (1989) 463–483.
[32] D. Ghoshal, T. Kawano, Towards p-adic string in constant B-field, Nucl. Phys. B 710 (2005) 577–598.
[33] J. Glimm, A. Jaffe, Quantum Physics. A Functional Integral Point of View, second edition, Springer, New York, 1987.
[34] Edilberto Arroyo-Ortiz, W.A. Zúñiga-Galindo, Construction of p-adic covariant quantum fields in the framework of white noise analysis, Rep. Math. Phys. 84 (1) (2019) 1–34.
[35] W.A. Zúñiga-Galindo, Non-Archimedean white noise, pseudodifferential stochastic equations, and massive Euclidean fields, J. Fourier Anal. Appl. 23 (2) (2017) 288–323.
[36] W.A. Zúñiga-Galindo, Non-Archimedean statistical field theory, arXiv:2006.05559.
[37] J.-I. Igusa, An Introduction to the Theory of Local Zeta Functions, AMS/IP Studies in Advanced Mathematics, vol. 14, American Mathematical Society, Providence, RI, International Press, Cambridge, MA, 2000.
[38] A. Abouelsaood, C.G. Callan Jr., C.R. Nappi, S.A. Yost, Open strings in background gauge fields, Nucl. Phys. B 280 (1987) 599–624, https://doi.org/10.1016/0550-3213(87)90164-7.
[39] S. Albeverio, A.Yu. Khrennikov, V.M. Shelkovich, Theory of p-Adic Distributions: Linear and Nonlinear Models, London Mathematical Society Lecture Note Series, vol. 370, Cambridge University Press, Cambridge, 2010.
[40] M.H. Taibleson, Fourier Analysis on Local Fields, Princeton University Press, Princeton, N.J., 1975.
[41] François Bruhat, Distributions sur un groupe localement compact et applications à l'étude des représentations des groupes p-adiques, Bull. Soc. Math. Fr. 89 (1961) 43–75.
[42] Hida Takeyuki, Kuo Hui-Hsiung, Potthoff Jürgen, Streit Ludwig, White Noise. An Infinite Dimensional Calculus, Kluwer Academic Publishers, Dordrecht, 1993.
[43] I.M. Gel'fand, N.Ya. Vilenkin, Generalized Functions. Vol. 4. Applications of Harmonic Analysis, Academic Press, New York - London, 1964.
[44] Robert B. Ash, Probability and measure theory, in: With Contributions by Catherine Doléans-Dade, second edition, Harcourt/Academic Press, Burlington, MA, 2000.
[45] Sergio Albeverio, Yuri Kondratiev, Yuri Kozitsky, Michael Röckner, The Statistical Mechanics of Quantum Lattice Systems. A Path Integral Approach, EMS Tracts in Mathematics, vol. 8, European Mathematical Society (EMS), Zürich, 2009.
[46] F. Loeser, Fonctions zêta locales d'Igusa à plusieurs variables, intégration dans les fibres, et discriminants, Ann. Sci. Éc. Norm. Supér. 22 (3) (1989) 435–471.