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The problem of finding the interest rate of an investment is one that often confronts the business man and, in the absence of sufficiently extended tables, is apt to give him no little trouble. For the functions which give the present value and the amount of annuities, straight-line interpolation is untrustworthy. Thus in *Accountancy and Investment,* by Sprague, revised by Perrine, on page 221 is given a solution by interpolative methods of the problem to find the income rate of a 4 per cent. bond for $1,000,000 due in 100 years, and bought for 1,264,806.66. The author finds the income rate to be 3.1318509 per cent. and calculates the value of the bond at the rate 3.131851 per cent. to be 1,264,806.6645. Such extraordinary accuracy in the use of the tables seemed to warrant investigation and the writer found that only the first four figures of the income rate were correct, and the value of the bond, computed at the rate of 3.131851 per cent. is $1,262,003.60, which is about $3,000 too small. This error, which has likely enough been noted before, illustrates the danger of depending on interpolative methods in problems of this sort.

The determination of the income rate is found to depend on an equation of degree at least as high as n where n is the number of years covered by the transaction, which accounts for the inadequacy of the straight-line approximation. There is no theoretical difficulty involved in the actual solution by Horner's method of these equations of high degrees, but this method becomes intolerable for equations of degree higher than three or four. The usual method of attack seems to be to simplify the equation by the omission of terms of higher degree than the second, the interest rate being usually small enough to warrant such simplification. Further approximations are made by methods easily available to one familiar with the processes of the theory of equations.

It occurred to the writer that the method of Newton might be used to furnish a "correction formula" that could be applied without much difficulty to better the approximation obtained from tables. The method is particularly valuable in that the trial value of the root may be above or below the true value, and so the closest tabular value is available. The sign of the correction is taken care of by the formula itself. The method requires no mathematical equipment beyond the ability to substitute numbers in a formula.

Consider first the simple problem of computing the rate of interest on a bond redeemable at par and bought for a certain amount. We use the customary notation:

C = the redemption value.
A = the price paid for the bond.
n = the number of years before redemption.
r = the dividend rate.
v = $1/(1+i)$, where i is the unknown rate of interest.
a_n = present value of an annuity for n years, at the unknown rate i.

We then have to solve the equation

$$A = C(v^n + ra_n)$$

(See Skinner, *Mathematical Theory of Investment,* p. 129.)

Put now $A/C = M$ and write

$$y = v^n + ra_n - M;$$

then we seek to determine i so that $y = O$. Newton's method is to choose an approximate value of i and take for correction

$$c = -y/y'$$

where y' is the first derivative of y with respect to i. Taking the well-known formula for a_n (Skinner, p. 82) it is not difficult to get the following general formula for the correction.

$$c = i\,\frac{(i-r)v^n - iM + r}{ni(i-r)v^{n+1} + r(1-v^n)} \qquad \ldots\ldots\ldots\ldots\ldots(1)$$

In applying the method we find from the tables or otherwise an approximate value of i. We substitute this value in (1) and determine the correction. If this corrected value is not accurate enough, substitute it again in the correction formula and determine again the further correction. Proceed with this corrected value as before. The accuracy at any stage is determined by the value of c.

As an example we may take the problem mentioned in the first paragraph. We have here

M = 1.264,806,66
n = 100
r = .04

The tables indicate that the rate lies somewhere between 3 per cent. and 4 per cent. We enter the correction formula with $i = .03$. The tables give for this value of i

$$v^{100} = .0520328$$

from which we get, on dividing by 1.03

$$v^{101} = .05058728$$

Entering with these values into the formula we find

$c = +\ .00126$

which added to the assumed rate, .03, gives for the corrected rate

$i = .03126$

We enter the correction formula now with this value of i and get for correction

$c = .0000505$

which added to .03126 gives for a new trial value

$i = .0313105$

The correction obtained from this value of i turns out to be

$c = -\ .00000008$

so that the value of i true to one part in one hundred million is

$i = .03131041$

Using this value of i the amount of the bond comes out

$A = \$1,264,813.78$

which is still $7.12 too large.

A slightly more complicated problem arises when the redemption price is different from the face of the bond. If the redemption price is R and the other letters have the same meaning as before, we get the correction formula

$$c = i\frac{(iR - rC)v^n - iA + rC}{in(iR - rC)v^{n+1} + rC(1 - v^n)}$$

The application of this formula is as before. A trial value of i is substituted in the formula and the resulting correction is added or subtracted according to its sign. As a numerical example we give the following problem which would ordinarily arise only under war or panic conditions.

A bond for $1,000 with interest at 8 per cent. nominal payable four times a year, and redeemable in seven years for $1,250, was bought for $950. Find the rate of interest on the investment.

We compute the interest per quarter year and take

$n = 28 \quad R = 1,250; \quad r = .02 \quad A = 950 \quad C = 1,000$

The interest rate will be somewhere in the neighborhood of 3 per cent. as can be easily verified from tables. Taking this value of i we get a correction

$c = -.00146,$

which gives for a new trial value of i

$i = .02854$

It is hardly worth while to correct this value further, as the value of the bond computed with this interest is found to be $950.55.

Another problem which arises is to find the rate of interest at which an annuity of C per period for n periods will amount to a given sum A. Writing, as before $M = C/A$, the equation to be solved is

$$y = s_n - M = 0$$

where s_n, the amount of the annuity, is given by the well-known formula

$$s_n = \frac{(1+i)^n - 1}{i}$$

The correction formula corresponding to this equation is easily found to be

$$c = i\,\frac{1 + Mi - (1+i)^n}{1 - (1+i)^n + ni(1+i)^{n-1}}$$

As a numerical example we may take the one found on page 109 of Skinner's *Mathematical Theory of Investment*. It is required to find the rate of interest at which an annuity of 1 per annum will amount to $12.50 in ten years. From the tables the nearest value of i is 5 per cent. Putting $i = .05$ in the correction formula we easily obtain

$$c = -.001326,$$

which gives for the next trial value of i

$$i = .04867$$

Another problem which may arise is to find the rate of interest such that an annuity of C period for n periods shall have a present value equal to A. Calling as before, $M = A/C$ we have to solve the equation

$$y = a_n - M = 0$$

where

$$a_n = \frac{1 - v^n}{i}$$

The correction formula turns out, in this case, to be

$$c = i\,\frac{1 - Mi - v^n}{1 - v^n - niv^{n+1}}$$

A numerical example of this problem is given on page 113 of Skinner. To find the rate of interest, given the present value of an annuity of $100 a year, payable annually for ten years as $780.

Here $M = 7.8$ and the tables give i as lying between 4.5 per cent. and 5 per cent. with a slight preference for \$5 per cent. We put this value in the correction formula and derive

$$c = -.00208$$

which gives

$$i = .04791$$

With this value of i an annuity of \$100 a year for ten years will be found to have a present value of \$780.07.

It is to be observed that in the problems that ordinarily arise only one correction is necessary, and that the quantities that are to be substituted in the correction formula are obtainable from the available tables on v^n and $(1 + i)^n$.