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Rashad M. EL-Sagheer
Mathematics Department, Faculty of Science, Al-Azhar University, Nasr city 11884, Cairo, Egypt,
Rashadmath27@gmail.com

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A Simple Algorithm for Generating a New Two Sample Type-II Progressive Censoring with Applications

E. M. Shokr[1,*], *R. M. EL-Sagheer*[2,3], *M. M. Mansour*[4], *H. M. Faied*[5] *and B. S. El-Desouky*[1]

[1]Mathematics Department, Faculty of Science, Mansoura University, Mansoura 35516, Egypt
[2]Mathematics Department, Faculty of Science, Al-Azhar University, Nasr city 11884, Cairo, Egypt
[3]High Institute of Computer and Management Information System, First Statement, New Cairo 11865, Cairo, Egypt
[4]Department of Basic Science, Faculty of Engineering, The British University in Egypt, El Sherouk City, Cairo, Egypt
[5]Mathematics Department, Faculty of Science, Al-Azhar University (Girls Branch), Nasr City, Cairo, Egypt

Abstract: In this article, we introduce a simple algorithm to generating a new type-II progressive censoring scheme for two samples. It is observed that the proposed algorithm can be applied for any continues probability distribution. Moreover, the description model and necessary assumptions are discussed. In addition, the steps of simple generation algorithm along with programming steps are also constructed on real example. The inference of two Weibull Frechet populations are discussed under the proposed algorithm. Both classical and Bayesian inferential approaches of the distribution parameters are discussed. Furthermore, approximate confidence intervals are constructed based on the asymptotic distribution of the maximum likelihood estimators. Also, we apply four methods of Bootstrap to construct confidence intervals. From Bayesian aspect, the Bayes estimates of the unknown parameters are evaluated by applying the Markov chain Monte Carlo technique and credible intervals are also carried out. The Bayes inference based on the squared error and LINEX loss functions is obtained. Simulation results have been implemented to obtain the accuracy of the estimators. Finally, one real data set has been analyzed for illustrative purposes both the proposed algorithm and methods of estimation.

Keywords: New type-II progressive censoring, Programming steps, Scheme censoring, Simple algorithm, Weibull Fréchet model

1 Introduction

The progressive censoring methods, in general, can not be implemented without withdrawal schemes prefixed in advance. For joint progressive censoring, the withdrawal scheme R is the composition of two random vectors S and T, where S and T are the numbers of units withdrawn at the time of the ith failure that belongs to the first and second sample, respectively.

S and T vectors are also prefixed before starting the experiment and hence the experimenter or the researcher can not modify or adapt the S or T or both during the experiment. Also, in the new joint progressive censoring, the experimenter can not modify or adapt the withdrawal schemes for both two samples, but he has a fair plan about what he is going to do whether the failure comes from the first or second sample, as the number of removable units (involved failure units from both samples) is still equal for $k = 1, 2, \cdots, k-1$.

In joint censoring, the researcher or experimenter should assign two withdrawal schemes S and T with k elements, while in the new joint censoring, he should only assign one withdrawal scheme R with $k-1$ elements. Hence the new joint censoring scheme reduces the biases for choosing the withdrawal schemes and then we expect to get more accurate inferential results.

In new joint progressive censoring, the experimenter can not determine exactly the number of removal units at the termination of the experiment R_k, but he can expect how many it will be depending on whether the k failure comes from the first or second sample.

The joint progressive censoring (**JPC**) scheme is quite useful to compare the lifetime distribution of products from different units which are being manufactured by two different lines in the same facility, see excellent reference Rasouli and Balakrishnan[1]. Recently, a new joint progressive type-II censoring (**NJPC**) scheme introduced by Mondal and Kundu[2] can be briefly stated as follows.

* Corresponding author e-mail: ethar.shokr@gmail.com

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It is assumed that two random samples of products, the first size is m and the second is n are selected from two different populations $Pop1$ and $Pop2$, respectively. These two independent samples are placed on a life testing experiment simultaneously.

Let $k < min\{m,n\}$ be the total number of failures to be observed and $R_1, \cdots, R_{k-1}$ are such that $\sum_{i=1}^{k-1} R_i + 1 < min\{m,n\}$. Suppose the first failure takes place at the time point W_1 and it comes from $Pop1$, then R_1 units are randomly chosen from the remaining $m-1$ surviving units of $Pop1$ and they are removed. At the same time $(R_1 + 1)$ units are randomly chosen from n surviving units of $Pop2$ and they are removed. Suppose the next failure takes place at the time point W_2 and it comes from $Pop2$, then $R_2 + 1$ units are chosen at random from the remaining $m - 1 - R_1$ surviving units of $Pop1$, and they are removed. At the same time R_2 units are chosen at random from the remaining $n - 2 - R_1$ surviving units of $Pop2$, and they are removed, and soon. Finally, at the time of the kth failure, it may be either from $Pop1$ or from $Pop2$, all the remaining items from both the populations are removed and the experiment stops.

This description is schematically illustrated through the figures in Mondal and Kundu [2]. Further, they also define a new set of random variables $Z_1, \cdots, Z_k$ where

$$Z_i = \begin{cases} 1, \text{ if the } j^{th} \text{ failure takes place from } Pop1 \\ 0, \text{ if the } j^{th} \text{ failure takes place from } Pop2 \end{cases}$$
$$, j = 1, 2, \cdots, min\{m,n\}.$$

Hence, for a NJPC scheme, the data will be of the form (W, Z), where $W = W_1, \cdots, W_k$, $W_1 \leq \cdots \leq W_k$ and $Z = (Z_1 \cdots, Z_k)$. In addition, they introduced a new algorithm to generate sample from a given NJPC based on lemma for the distribution (exponential distribution) under study, see Section 4 Page 2608 [2]. In this article, we introduce a simple algorithm to generate sample from a given NJPC for any continues probability distribution without any lemma related to the distribution. Moreover, we discuss the maximum likelihood, Bayes, and four parametric bootstrap methods for estimating the unknown parameters of the Weibull Fréchet distribution.

The layout of this article is organized as follows. In Section 2 we introduce the simple algorithm to generate sample from a given NJPC. The analysis of a data set mainly for illustrative purposes is provided in Section 3. NJPC is applied on two Weibull– Fréchet populations in Section 5. Different confidence intervals methods are constructed in Section 6. Section 7, provides the Bayesian analysis using square error and LINEX loss functions. Simulation study and two examples one of them used simulated data and the other used a real data sets have been analyzed in Section 8. Finally, some concluding remarks are given in Section 9.

2 Generation of NJPC

In this section, we introduce the proposed simple algorithm to generate sample from NJPC scheme. The steps of this algorithm can be described as follows:

Step 1 Specify m, n, k and scheme R.

Step 2 Generate two random sample from any continues probability distribution, such that $X \sim Pop1$ and $Y \sim Pop2$ in simulated data case. Or input the data as two vectors or as two lists in real data case.

Step 3 Loop implementation is required over the two data lists x and y to compare between the values based on an indicator list, say z_0, as follows

$$z_0 = \begin{cases} 1, Z_i = x_i \ (i^{th} \text{observation is coming from } x) \\ 0, Z_i = y_i \ (i^{th} \text{observation is coming from } y) \end{cases}$$
$$, i = 1, 2, \cdots, min\{m,n\}.$$

Step 4 Define a new withdrawal scheme R_i, $i = 1, 2, \cdots, k$ as follows
i
$$R_i = \begin{cases} R_i, & Z_i = x_i \\ (i^{th} \text{observation is coming from } x) \\ R_i + 1, & Z_i = y_i \\ (i^{th} \text{observation is coming from } y) \end{cases}$$
$$, i = 1, 2, \cdots, k-1.$$

ii Now, define the final withdrawn, R_k, according to Kundu's paper [2], as follows
$$R_k = \begin{cases} m - 1 - \sum_{i=1}^{k-1}(R_i + 1), & Z_i = x_i \\ , (i^{th} \text{observation is coming from } x) \\ n - \sum_{i=1}^{k-1}(R_i + 1), & Z_i = y_i \\ , (i^{th} \text{observation is coming from } y) \end{cases}$$

iii Then, the modified withdrawal scheme, R, is $R_i \cup R_k$, $i = 1, \cdots, k-1$.

Step 5 Join the two data sets as one data set, say, $J = x \cup y$ and sort these set.

Step 6 Using the algorithm proposed by Balakrishnan and Sandhu [3], the new joint progressive censored sample can be obtained.

3 Data Analysis

This section discusses the analysis of a data set specifically for the purpose of clarifying the steps of the new algorithm. These data sets were taken from Proschan [4] and they were used by Rasouli and Balakrishnan [5]. Recently, they were used by Mondal and Kundu [2]. The data represent the intervals between failures (in hours) of the air conditioning system of a leet of 13 Boeing 720 jet airplanes.

Mondal and Kundu [2] Page 2614, considered the planes "7913 and 7914" during their study of NJPC from the exponential distribution. In this paper, we considered the same planes to illustrative the simple algorithm proposed in generating NJPC.

The following steps herein are applied to real data that is stated in Mondal and Kundu [2], each steps of the algorithm is implemented using Mathematica program Version 9. The steps are adapted to be implemented throughout other software like Matlab, R and Python programs.

m=24; n=27; k=8;
R=0, 0, 0, 0, 0, 0, 0;
x=3, 5, 5, 13, 14, 15, 22, 22, 23, 30, 36, 39, 44, 46, 50, 72, 79, 88, 97, 102,139, 188, 197, 210;
y=1, 4, 11, 16, 18, 18, 18, 24, 31, 39, 46, 51, 54, 63, 68, 77, 80, 82, 97, 106, 111, 141, 142, 163, 191, 206, 216;
z_0=Table[If[x[[i]]<y[[i]], 1, 0],i, 1, Min[m, n]];
R=Table[If[z_0[[i]]==1, R[[i]]=R[[i]], R[[i]]=R[[i]]+1], i, 1, k-1];
If[z_0[[k]]==1,   R_k=m-1-$\sum_{i=1}^{k-1}$(R[[i]]+1),
R_k=n-k-$\sum_{i=1}^{k-1}$(R[[i]]+1)];
R=Join[R, R_k];
J_0=Join[x, y];
J=Sort[J_0];
ij=0;
Do[
Jprog[j]=J[[1]];
J=Table[J[[i]], i, 2, m+n-ij];
ij=ij+R[[j]]+1;
J=Sort[RandomSample[J, m+n-ij]];
, j, 1, k];
J=Sort[Table[Jprog[i], i, 1, k]];
z=Table[If[MemberQ[x, J[[h]]], z=1, z=0], h, 1, k, 1];

1, 3, 4, 5, 5, 11, 13, 15
0, 1, 0, 1, 1, 0, 1, 1

4 Weibull Fréchet Model

The Weibull Fréchet model had been discussed in Afify et al. [6] as a mixture of Weibull and the Fréchet distributions. The probability density function (pdf) and cumulative distribution function (cdf) of a random variable X has $WFr(\alpha, \beta, \lambda, \theta)$ are given, respectively, by

$$
\begin{aligned}
f(x) = {}& \lambda\theta\beta\alpha^{\beta}x^{-\beta-1}\exp\left[-\theta\left(\frac{\alpha}{x}\right)^{\beta}\right] \\
& \times \left\{1 - \exp\left[-\left(\frac{\alpha}{x}\right)^{\beta}\right]\right\}^{-\theta-1} \\
& \times \exp\left(-\lambda\left\{\exp\left[\left(\frac{\alpha}{x}\right)^{\beta}\right] - 1\right\}^{-\theta}\right),
\end{aligned} \tag{1}
$$

and

$$
F(x) = 1 - \exp\left(-\lambda\left\{\exp\left[\left(\frac{\alpha}{x}\right)^{\beta}\right] - 1\right\}^{-\theta}\right), \tag{2}
$$

where $\alpha, \beta, \lambda, \theta > 0$, $x > 0$, α is a scale parameter and β, λ, θ are shape parameters, moreover, Figures 1 and 2 display some plots of the pdf and cdf of WFr distribution for selected values of α, β, λ and θ. Also, Figures 3 and 4 display the reliability and hazard rate functions of
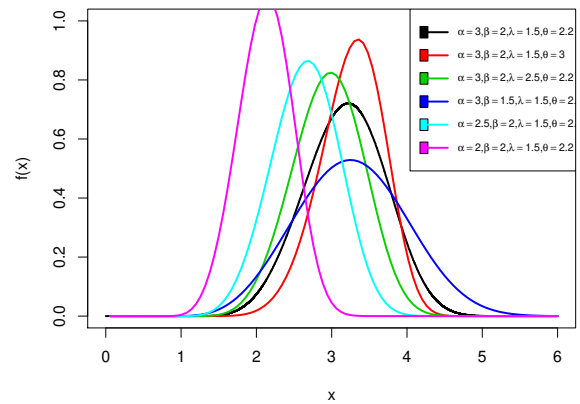
Fig. 1: pdf for WFr model
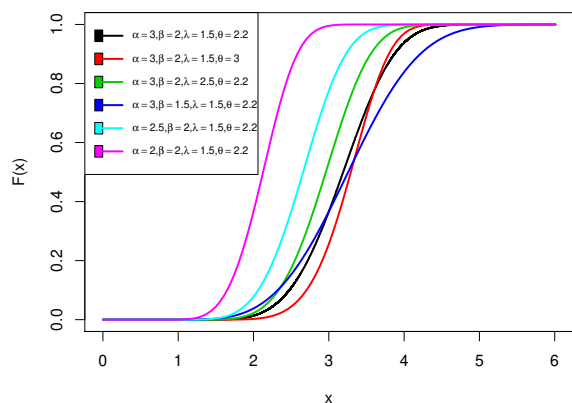
Fig. 2: cdf for WFr model

$WFr(\alpha, \beta, \lambda, \theta)$ which are given, respectively, by Eqs 3, 4 as

$$
r(t) = \exp\left(-\lambda\left\{\exp\left[\left(\frac{\alpha}{t}\right)^{\beta}\right] - 1\right\}^{-\theta}\right), \tag{3}
$$

and

$$
\begin{aligned}
h(t) = {}& \lambda\theta\beta\alpha^{\beta}t^{-\beta-1}\exp\left(-\theta\left(\frac{\alpha}{t}\right)^{\beta}\right) \\
& \times \left(1 - \exp\left[-\left(\frac{\alpha}{t}\right)^{\beta}\right]\right)^{(-\theta-1)}. \tag{4}
\end{aligned}
$$

Fig. 3: Reliability function for WFr model
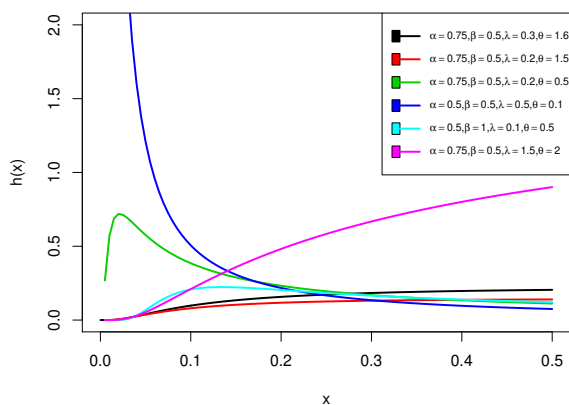
Fig. 4: Hazard rate function for WFr model

The WFr distribution has been shown to be useful for modeling and analyzing the life time data in medical and biological sciences, engineering, etc. So, it has been received the greatest attention from theoretical and applied statisticians primarily due to its use in reliability and life testing studies see, [7,8]. This distribution is a very flexible model that approaches to different distributions when its parameters are changed. It contains the following new special models:

1. $WFr(\alpha,1,\lambda,\theta)$ refers to the Weibull inverse exponential model.
2. $WFr(\alpha,\beta,\lambda,1)$ reduces to the exponential Fréchet model.
1. $WFr(\alpha,2,\lambda,\theta)$ follows the Weibull inverse Rayleigh model.

3. $WFr(\alpha,2,\lambda,1)$ is the exponential inverse Rayleigh model.
4. $WFr(\alpha,1,\lambda,1)$ follows the exponential inverse exponential model.

5 Maximum Likelihood Estimation

In this section, maximum likelihood estimation is used to estimate the unknown parameters on the basis of the new type-II progressive censoring scheme for two samples. Assume, the lifetime of m element of product A are i.i.d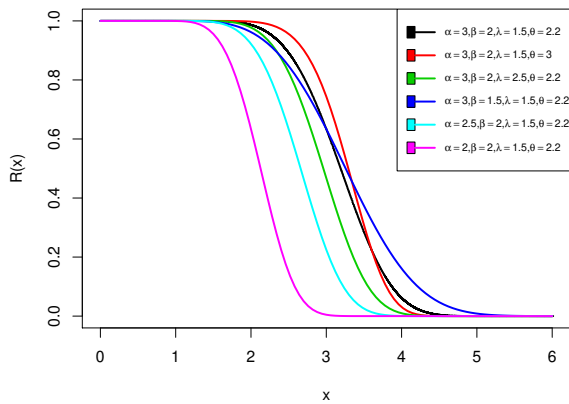. r.v. from Weibull Fréchet $WFr(\alpha_1,\beta_1,\lambda_1,\theta_1)$ population, also the lifetime of n element of product B are i.i.d. r. v. from $WFr(\alpha_2,\beta_2,\lambda_2,\theta_2)$. Hence, the log-likelihood function according to Mondal and Kundu [2] is given by:

$$
\begin{aligned}
\ell = {}& \ln c + m_r \ln \lambda_1 + m_r \ln \theta_1 + m_r \ln \beta_1 + m_r \beta_1 \ln \alpha_1 \\
& - (\beta_1 + 1) \sum_{i=1}^{r} z_i \ln w_i + n_r \ln \lambda_2 + n_r \ln \theta_2 + n_r \ln \beta_2 \\
& + n_r \beta_2 \ln \alpha_2 - (\beta_2 + 1) \sum_{i=1}^{r} (1 - z_i) \ln w_i \\
& - \theta_1 \sum_{i=1}^{r} z_i \left(\frac{\alpha_1}{w_i}\right)^{\beta_1} - \theta_2 \sum_{i=1}^{r} (1 - z_i) \left(\frac{\alpha_2}{w_i}\right)^{\beta_2} \\
& - \lambda_1 \sum_{i=1}^{r} (z_i + R_i) \left(\exp\left[\left(\frac{\alpha_1}{w_i}\right)^{\beta_1}\right] - 1 \right)^{-\theta_1} \\
& - \lambda_2 \sum_{i=1}^{r} (1 - z_i + R_i) \left(\exp\left[\left(\frac{\alpha_2}{w_i}\right)^{\beta_2}\right] - 1 \right)^{-\theta_2} \\
& - (\theta_1 + 1) \sum_{i=1}^{r} z_i \ln \left[1 - \exp\left[-\left(\frac{\alpha_1}{w_i}\right)^{\beta_1}\right] \right] \\
& - (\theta_2 + 1) \sum_{i=1}^{r} (1 - z_i) \ln \left[1 - \exp\left[-\left(\frac{\alpha_2}{w_i}\right)^{\beta_2}\right] \right]. \quad (5)
\end{aligned}
$$

Then, the normal equations to obtain estimates of the unknown parameters are given by:

$$
\begin{aligned}
& \frac{m_r \beta_1}{\alpha_1} - \theta_1 \sum_{i=1}^{r} \left(\frac{\beta_1 z_i}{w_i}\right) \left(\frac{\alpha_1}{w_i}\right)^{\beta_1 - 1} \\
& - (1 + \theta_1) \sum_{i=1}^{r} \frac{z_i \beta_1 \left(\frac{\alpha_1}{w_i}\right)^{\beta_1 - 1} \exp\left[-\left(\frac{\alpha_1}{w_i}\right)^{\beta_1}\right]}{\left(1 - \exp\left[-\left(\frac{\alpha_1}{w_i}\right)^{\beta_1}\right] \right) \times w_i} \\
& + \lambda_1 \theta_1 \beta_1 \sum_{i=1}^{r} \frac{(z_i + R_i)}{w_i} \left(\frac{\alpha_1}{w_i}\right)^{\beta_1 - 1} \exp\left[\left(\frac{\alpha_1}{w_i}\right)^{\beta_1}\right] \\
& \times \left(\exp\left[\left(\frac{\alpha_1}{w_i}\right)^{\beta_1}\right] - 1 \right)^{-\theta_1 - 1} = 0, \quad (6)
\end{aligned}
$$

$$\frac{n_r \beta_2}{\alpha_2} - \theta_2 \sum_{i=1}^{r} \left(\frac{\beta_2 (1-z_i)}{w_i} \right) \left(\frac{\alpha_2}{w_i} \right)^{\beta_2 - 1}$$

$$-(1+\theta_2) \sum_{i=1}^{r} \frac{(1-z_i)\beta_2 \left(\frac{\alpha_2}{w_i} \right)^{\beta_2 - 1} \exp\left[-\left(\frac{\alpha_2}{w_i} \right)^{\beta_2} \right]}{\left(1 - \exp\left[-\left(\frac{\alpha_2}{w_i} \right)^{\beta_2} \right] \right) \times w_i}$$

$$+\lambda_2 \theta_2 \beta_2 \sum_{i=1}^{r} \frac{(1-z_i+R_i)}{w_i} \left(\frac{\alpha_2}{w_i} \right)^{\beta_2 - 1} \exp\left[\left(\frac{\alpha_2}{w_i} \right)^{\beta_2} \right]$$

$$\times \left(\exp\left[\left(\frac{\alpha_2}{w_i} \right)^{\beta_2} \right] - 1 \right)^{-\theta_2 - 1} = 0, \tag{7}$$

$$\frac{m_r}{\beta_1} + m_r \ln(\alpha_1) - \sum_{i=1}^{r} z_i \ln(w_i) - \theta_1 \sum_{i=1}^{r} z_i \left(\frac{\alpha_1}{w_i} \right)^{\beta_1} \ln\left(\frac{\alpha_1}{w_i} \right)$$

$$-(1+\theta_1) \sum_{i=1}^{r} \frac{z_i \left(\frac{\alpha_1}{w_i} \right)^{\beta_1} \exp\left[-\left(\frac{\alpha_1}{w_i} \right)^{\beta_1} \right] \ln\left(\frac{\alpha_1}{w_i} \right)}{1 - \exp\left[-\left(\frac{\alpha_1}{w_i} \right)^{\beta_1} \right]}$$

$$+\lambda_1 \theta_1 \sum_{i=1}^{r} (z_i+R_i) \ln\left(\frac{\alpha_1}{w_i} \right) \exp\left[\left(\frac{\alpha_1}{w_i} \right)^{\beta_1} \right]$$

$$\times \left(\exp\left[\left(\frac{\alpha_1}{w_i} \right)^{\beta_1} \right] - 1 \right)^{-\theta_1 - 1} \left(\frac{\alpha_1}{w_i} \right)^{\beta_1} = 0, \tag{8}$$

$$\frac{n_r}{\beta_2} - \sum_{i=1}^{r} (1-z_i) \ln(w_i) - \theta_2 \sum_{i=1}^{r} (1-z_i) \left(\frac{\alpha_2}{w_i} \right)^{\beta_2} \ln\left(\frac{\alpha_2}{w_i} \right)$$

$$-(1+\theta_2) \sum_{i=1}^{r} \frac{(1-z_i) \left(\frac{\alpha_2}{w_i} \right)^{\beta_2} \exp\left[-\left(\frac{\alpha_2}{w_i} \right)^{\beta_2} \right] \ln\left(\frac{\alpha_2}{w_i} \right)}{1 - \exp\left[-\left(\frac{\alpha_2}{w_i} \right)^{\beta_2} \right]}$$

$$+n_r \ln(\alpha_2) + \lambda_2 \theta_2 \sum_{i=1}^{r} (1-z_i+R_i) \ln\left(\frac{\alpha_2}{w_i} \right)$$

$$\times \exp\left[\left(\frac{\alpha_2}{w_i} \right)^{\beta_2} \right] \left(\exp\left[\left(\frac{\alpha_2}{w_i} \right)^{\beta_2} \right] - 1 \right)^{-\theta_2 - 1} \left(\frac{\alpha_2}{w_i} \right)^{\beta_2}$$

$$= 0, \tag{9}$$

$$\frac{m_r}{\lambda_1} - \sum_{i=1}^{r} (z_i+R_i) \left(\exp\left[\left(\frac{\alpha_1}{w_i} \right)^{\beta_1} \right] - 1 \right)^{-\theta_1} = 0, \tag{10}$$

$$\frac{n_r}{\lambda_2} - \sum_{i=1}^{r} (1-z_i+R_i) \left(\exp\left[\left(\frac{\alpha_2}{w_i} \right)^{\beta_2} \right] - 1 \right)^{-\theta_2} = 0, \tag{11}$$

$$\frac{m_r}{\theta_1} - \sum_{i=1}^{r} z_i \left(\frac{\alpha_1}{w_i} \right)^{\beta_1} + \lambda_1 \sum_{i=1}^{r} (z_i+R_i)$$

$$\times \left(\exp\left[\left(\frac{\alpha_1}{w_i} \right)^{\beta_1} \right] - 1 \right)^{-\theta_1} \ln\left(\exp\left[\left(\frac{\alpha_1}{w_i} \right)^{\beta_1} \right] - 1 \right)$$

$$-\sum_{i=1}^{r} z_i \ln\left(1 - \exp\left[-\left(\frac{\alpha_1}{w_i} \right)^{\beta_1} \right] \right) = 0, \tag{12}$$

and

$$\frac{n_r}{\theta_2} - \sum_{i=1}^{r} (1-z_i) \left(\frac{\alpha_2}{w_i} \right)^{\beta_2} + \lambda_2 \sum_{i=1}^{r} (1-z_i+R_i)$$

$$\times \left(\exp\left[\left(\frac{\alpha_2}{w_i} \right)^{\beta_2} \right] - 1 \right)^{-\theta_2} \ln\left(\exp\left[\left(\frac{\alpha_2}{w_i} \right)^{\beta_2} \right] - 1 \right)$$

$$-\sum_{i=1}^{r} (1-z_i) \ln\left(1 - \exp\left[-\left(\frac{\alpha_2}{w_i} \right)^{\beta_2} \right] \right) = 0. \tag{13}$$

Thus, from Eqs. 10 and 11 the MLEs of λ_i, $i = 1, 2$ can be given, respectively, by

$$\hat{\lambda}_1 = m_r \left[\sum_{i=1}^{r} (z_i+R_i) \left(\exp\left[\left(\frac{\alpha_1}{w_i} \right)^{\beta_1} \right] - 1 \right)^{-\theta_1} \right]^{-1}, \tag{14}$$

and

$$\hat{\lambda}_2 = n_r \left[\sum_{i=1}^{r} (1-z_i+R_i) \left(\exp\left[\left(\frac{\alpha_2}{w_i} \right)^{\beta_2} \right] - 1 \right)^{-\theta_2} \right]^{-1}. \tag{15}$$

The solutions of nonlinear equations are the maximum likelihood estimators of the parameters α_i, β_i, λ_i and θ_i, $i = 1, 2$. Since it is difficult to find explicit solutions for the parameters due to the above system of nonlinear equations, so we can obtain the required solutions based on some numerical approximation methods such as the Newton-Raphson. For more details about the iteration algorithm used in this method see EL-Sagheer [9]. To get some better accuracy in estimation of the parameters this requires information or prior knowledge of the estimation process. This knowledge or information is incorporated into the estimation process through the prior distributions. We will discuss these technique in Section 7, such technique is known as Bayesian technique.

6 Construction of Confidence Intervals

In this section, we derive the approximate confidence intervals based on the normality property of maximum likelihood estimations and also the bootstrap confidence intervals are derived for the parameters α_i, β_i, λ_i and θ_i, $i = 1, 2$.

6.1 Approximate Confidence Intervals

The asymptotic confidence intervals for the parameters are established by employing the variance of the parameters. We acquire the asymptotic confidence intervals for the parameters from the variance-covariance matrix, which is also known as the inverse Fisher information matrix. The Fisher information matrix is a generalization of the Fisher information amount. The Fisher information amount represents the average amount of information about the state parameters in a certain

sense that a sample of random variables can provide. Let $I(\Omega_i) = I_{i,j}(\Omega_i)$ where $i, j = 1, 2, \cdots, 8$ and $(\Omega_1, \Omega_2, \Omega_3, \Omega_4, \Omega_5, \Omega_6, \Omega_7, \Omega_8) = (\alpha_1, \alpha_2, \beta_1, \beta_2, \lambda_1, \lambda_2, \theta_1, \theta_2)$ be the Fisher information matrix of the parameter Ω_i, where

$$\hat{I}(\Omega_i) = \left(-E\left(\frac{\partial^2 \ell}{\partial \Omega_i \partial \Omega_j} \right) \right).$$

So, the asymptotic variance-covariance matrix is given by

$$\hat{V} = V_{i,j} = (cov(\hat{\Omega}_i, \hat{\Omega}_j)) = \hat{I}^{-1}(\hat{\Omega}_i). \quad (16)$$

Therefore, the $(1 - \delta)100\%$ approximate confidence intervals (ACIs) for Ω_i, $i = 1, \cdots, 8$ are given by

$$[\hat{\Omega}_i \pm Z_{\delta/2} \sqrt{\widehat{var(\Omega_i)}}]. \quad (17)$$

6.2 Bootstrap Confidence Intervals

A parametric bootstrap interval provides much more information about the population value of the quantity of interest than does a point estimation . Also, it is evident that the confidence intervals based on the asymptotic results don't perform very well for small sample size. For this, we propose to use confidence intervals based on the parametric bootstrap methods: (i) percentile bootstrap method (Boot-p), see Efron and Tibshirani [10], (ii) bootstrap-t (Boot-t), see Hall [11], (iii) Bias-Corrected bootstrap CI (Boot-BC) and (iv) bias-corrected accelerated bootstrap CI (Boot-BCa), see DiCiccio and Efron [12] and Efron and Tibshirani [13]. Moreover the algorithms for estimating the confidence intervals using these methods are given bellow:

6.2.1 Algorithm 1: Boot-p CI

1. From the joint original sample W, compute the MLEs of the unknown parameters $\Omega = (\alpha_1, \alpha_2, \beta_1, \beta_2, \lambda_1, \lambda_2, \theta_1, \theta_2)$ and denoted them by $\hat{\Omega} = (\hat{\alpha}_1, \hat{\alpha}_2, \hat{\beta}_1, \hat{\beta}_2, \hat{\lambda}_1, \hat{\lambda}_2, \hat{\theta}_1, \hat{\theta}_2)$ by solving Eqs. (6–13).
2. Use the values of $\hat{\Omega}$ to generate a bootstrap NJPC sample $w^* = (w_1^*, \cdots, w_r^*)$ with different values of R_i, $i = 1, \cdots, r$ by using the algorithm presented in this paper.
3. Compute the bootstrap estimate of Ω based on this bootstrap sample and denoted them by $\hat{\Omega}_1^*$.
4. Repeat step 2 and 3 $Nboot$ times to get $\hat{\Omega}_1^*, \hat{\Omega}_2^*, \cdots, \hat{\Omega}_{Nboot}^*$ where $\hat{\Omega}_i^* = (\alpha_{1i}^*, \alpha_{2i}^*, \beta_{1i}^*, \beta_{2i}^*, \lambda_{1i}^*, \lambda_{2i}^*, \theta_{1i}^*, \theta_{2i}^*)$, $i = 1, 2, \cdots, Nboot$.
5. Arrange $\hat{\Omega}_i^*$ in ascending order and denote them by $\hat{\Omega}_{(1)}^*, \hat{\Omega}_{(2)}^*, \cdots, \hat{\Omega}_{(Nboot)}^*$.
6. The $(1 - \delta)100\%$ approximate bootstrap-p CI for Ω is obtained as (L, U), where $L = \hat{\Omega}_{Nboot(\frac{\delta}{2})}^*$ and $U = \hat{\Omega}_{Nboot(1-\frac{\delta}{2})}^*$.

6.2.2 Algorithm 2: Boot-t CI

1-3 The same as Algorithm 1: Boot-p CI.

4. Compute the variance-covariance matrix for $\hat{\Omega}^*$ by Eq. 16, then, calculate the value of the statistic $T_\Omega^* = \frac{\hat{\Omega}^* - \hat{\Omega}}{\sqrt{\widehat{var(\Omega^*)}}}$.
5. Repeat step 1 – 4 for $Nboot$ times to get $T_{\Omega_1}^*, \cdots, T_{\Omega_{Nboot}}^*$.
6. Arrange them in ascending order and get $T_{\Omega_{(1)}}^*, T_{\Omega_{(2)}}^*, \cdots, T_{\Omega_{(Nboot)}}^*$.
7. The $(1 - \delta)100\%$ approximate bootstrap-t confidence interval for Ω is obtained as (L, U), where

$$L = \hat{\Omega} + \sqrt{var(\widehat{\Omega_{(Nboot(\frac{\delta}{2}))}^*})} * T_{\Omega_{(Nboot(\frac{\delta}{2}))}}^*,$$

and

$$U = \hat{\Omega} + \sqrt{var(\widehat{\Omega_{(Nboot(1-\frac{\delta}{2}))}^*})} * T_{\Omega_{(Nboot(1-\frac{\delta}{2}))}}^*.$$

6.2.3 Algorithm 3: Boot-BC CI

1-4 The same as Algorithm 1: Boot-p CI.

5. Calculate the probability p_0 based on the ordered bootstrap distribution of $\hat{\Omega}_i^*$

$$P_0 = p(\hat{\Omega}_i^* < \hat{\Omega}) = \frac{\hat{\Omega}_i^* < \hat{\Omega}}{N}, \; where \; i = 1, \cdots, Nboot. \quad (18)$$

5. Put χ and χ^{-1} denote to the CDF and inverse CDF of standard normal variable z, and then, the bias-correction constant z_0 is defined as follow:

$$z_0 = \chi^{-1}(P_0) = \chi^{-1}\left(\frac{N(\hat{\Omega}^* < \hat{\Omega})}{N} \right), \quad (19)$$

where $p(\hat{\Omega}^* < \hat{\Omega}) = G(z_0)$ and $G(.)$ is the CDF of bootstrap distribution.

6. The percentiles of the ordered bootstrap distribution of $\hat{\Omega}$ are given as

$$L = \chi(2z_0 + z_{\delta/2}), \quad U = \chi(2z_0 + z_{1-\delta/2}). \quad (20)$$

7. The approximate Boot-BC $100(1 - \delta)\%$ CI of $\hat{\Omega} = (\alpha_1, \alpha_2, \beta_1, \beta_2, \lambda_1, \lambda_2, \theta_1, \theta_2)$ is given by

$$\left[\hat{\Omega}_{Boot-BC(\frac{\delta}{2})}, \hat{\Omega}_{Boot-BC(1-\frac{\delta}{2})} \right].$$

6.2.4 Algorithm 4: Boot-BCa CI

1-3 The same as Algorithm 1: Boot-p CI.

4. Put $\chi(z) = \eta$ be the standard normal CDF where $z_\eta = \chi^{-1}(\eta)$ and the bias-correction constant z_0 is defined in Eq. 19, hence

$$\hat{\Omega}^*_{Boot-BCa} = G^{-1}\left[\chi\left(z_0 + \frac{z_0 + z_\eta}{1 - a(z_0 + z_\eta)}\right)\right], \quad (21)$$

where a is indicated to the acceleration factor and it can be calculated by using a jackknife approach and can be expressed as

$$a = \frac{\sum_{i=1}^{N}(\hat{\Omega} - \hat{\Omega}_i)^3}{6\left(\sum_{i=1}^{N}(\hat{\Omega} - \hat{\Omega}_i)^2\right)^{3/2}}, \quad i = 1, 2, \cdots, Nboot. \quad (22)$$

5. The approximate Boot-BCa $100(1 - \delta)\%$ CI of $\hat{\Omega}$ is given by

$$\left[\hat{\Omega}_{Boot-BCa(\frac{\delta}{2})}, \hat{\Omega}_{Boot-BCa(1-\frac{\delta}{2})}\right].$$

7 Bayesian Estimation

The Bayesian approach allows one to summarize prior subjective knowledge or technical information concerning the lifetime parameters into a prior density. In this section, we discuss how to obtain the Bayes estimates and the corresponding credible intervals for the unknown parameters $\alpha_j, \beta_j, \lambda_j$ and θ_j, $j = 1, 2$. Let us consider the parameters η_i are independent and follow the gamma prior distributions, the prior density functions of η_i are given by

$$\pi(\eta_i) \propto \eta_i^{c_i-1} e^{-k_i\eta_i}$$

where $c_i, k_i > 0$, $i = 1, \cdots, 8$ and $(\eta_1, \eta_2, \eta_3, \eta_4, \eta_5, \eta_6, \eta_7, \eta_8) = (\alpha_1, \alpha_2, \beta_1, \beta_2, \lambda_1, \lambda_2, \theta_1, \theta_2)$. Then, using the likelihood function, the joint posterior density functions of η_i in given (W, Z) is

$$\begin{aligned}
\pi^*(\eta_i \mid W, Z) \propto{}& \alpha_1^{m_r\beta_1+c_1-1} e^{-k_1\alpha_1} \beta_1^{m_r+c_3-1} e^{-k_3\beta_1} \\
&\times \lambda_1^{m_r+c_5-1} e^{-k_5\lambda_1} \theta_1^{m_r+c_7-1} e^{-k_7\theta_1} \alpha_2^{n_r\beta_2+c_2-1} e^{-k_2\alpha_2} \\
&\times \beta_2^{n_r+c_4-1} e^{-k_4\beta_2} \lambda_2^{n_r+c_6-1} e^{-k_6\lambda_2} \theta_2^{n_r+c_8-1} \\
&\times e^{-k_8\theta_2} \prod_{i=1}^{r}(w_i^{z_i(-\beta_1-1)}) \prod_{i=1}^{r}(w_i^{(1-z_i)(-\beta_2-1)}) \\
&\times \exp\left[-\theta_1 \sum_{i=1}^{r}\left(\frac{\alpha_1}{w_i}\right)^{\beta_1} z_i\right] \exp\left[-\theta_2 \sum_{i=1}^{r}\left(\frac{\alpha_2}{w_i}\right)^{\beta_2}\right. \\
&\times (1-z_i)\bigg] \prod_{i=1}^{r}\left\{1 - \exp\left[-\left(\frac{\alpha_1}{w_i}\right)^{\beta_1}\right]\right\}^{-z_i(\theta_1+1)} \\
&\times \prod_{i=1}^{r}\left\{1 - \exp\left[-\left(\frac{\alpha_2}{w_i}\right)^{\beta_2}\right]\right\}^{-(1-z_i)(\theta_2+1)} \\
&\times \exp\left(-\lambda_1 \sum_{i=1}^{r}(z_i + R_i)\left(\left\{\exp\left[\left(\frac{\alpha_1}{w_i}\right)^{\beta_1}\right]\right.\right.\right.
\end{aligned}$$

$$\begin{aligned}
&\times \bigg] - 1\bigg\}^{-\theta_1}\bigg)\bigg) \exp\left(-\lambda_2 \sum_{i=1}^{r}(1-z_i+R_i)\right. \\
&\times \left(\left\{\exp\left[\left(\frac{\alpha_2}{w_i}\right)^{\beta_2}\right] - 1\right\}^{-\theta_2}\right)\bigg). \quad (23)
\end{aligned}$$

It is impossible to compute (23) analytically. The Markov chain Monte Carlo (MCMC) method provides an alternative method for the parameters estimation. In the following subsection, we propose using the MCMC technique to obtain the bayes estimates of the unknown parameters and construct the corresponding credible intervals.

7.1 MCMC method

The MCMC techniques are a general simulation method for sampling from posterior distributions and computing posterior quantities of interest, we need to sample successively from a target distribution. The Gibbs algorithm requires to decompose the joint posterior distribution into full conditional distributions for each parameter in the model, then sample from each one of these conditional distributions and compute the Bayes estimates of unknown parameters and the corresponding credible intervals of them. The conditional densities of $\alpha_j, \beta_j, \lambda_j$ and θ_j are obtained as follows:

$$\begin{aligned}
\pi_1^*(\alpha_1|\alpha_2, \beta_j, \lambda_j, \theta_j) \propto{}& \alpha_1^{m_r\beta_1+c_1-1} \prod_{i=1}^{r}\bigg\{1 - \exp\bigg[- \\
&\left(\frac{\alpha_1}{w_i}\right)^{\beta_1}\bigg]\bigg\}^{-z_i(\theta_1+1)} \exp\left[-\alpha_1 k_1 - \theta_1 \sum_{i=1}^{r}\left(\frac{\alpha_1}{w_i}\right)^{\beta_1} z_i\right] \\
&\times \exp\left(-\lambda_1 \sum_{i=1}^{r}(z_i + R_i)\left(\left\{\exp\left[\left(\frac{\alpha_1}{w_i}\right)^{\beta_1}\right] - 1\right\}^{-\theta_1}\right)\right), \\
& \quad (24)
\end{aligned}$$

$$\begin{aligned}
\pi_2^*(\alpha_2|\alpha_1, \beta_j, \lambda_j, \theta_j) \propto{}& \alpha_2^{n_r\beta_2+c_2-1} \prod_{i=1}^{r}\bigg\{1 - \exp\bigg[- \\
&\left(\frac{\alpha_2}{w_i}\right)^{\beta_2}\bigg]\bigg\}^{-(1-z_i)(\theta_2+1)} \exp\left[-\alpha_2 k_2 - \theta_2 \sum_{i=1}^{r}\left(\frac{\alpha_2}{w_i}\right)^{\beta_2}\right. \\
&(1-z_i)\bigg] \exp\left(-\lambda_2 \sum_{i=1}^{r}(1-z_i+R_i)\left(\left\{\exp\left[\left(\frac{\alpha_2}{w_i}\right)^{\beta_2}\right]\right.\right.\right. \\
&\left.\left.\left. -1\right\}^{-\theta_2}\right)\right), \quad (25)
\end{aligned}$$

$$\begin{aligned}
\pi_3^*(\beta_1|\alpha_j, \beta_2, \lambda_j, \theta_j) \propto{}& \beta_1^{m_r+c_3-1} \prod_{i=1}^{r}(w_i^{z_i(-\beta_1-1)}) \\
&\exp\left[-\beta_1 k_3 - \theta_1 \sum_{i=1}^{r}\left(\frac{\alpha_1}{w_i}\right)^{\beta_1} z_i\right] \\
&\times \prod_{i=1}^{r}\left\{1 - \exp\left[-\left(\frac{\alpha_1}{w_i}\right)^{\beta_1}\right]\right\}^{-z_i(\theta_1+1)} \\
&\exp\left(-\lambda_1 \sum_{i=1}^{r}(z_i + R_i)\left(\left\{\exp\left[\left(\frac{\alpha_1}{w_i}\right)^{\beta_1}\right] - 1\right\}^{-\theta_1}\right)\right), \\
& \quad (26)
\end{aligned}$$

$$\pi_4^*(\beta_2|\alpha_j,\beta_1,\lambda_j,\theta_j) \propto \beta_2^{n_r+c_4-1} \prod_{i=1}^{r} (w_i^{(1-z_i)(-\beta_2-1)})$$

$$\exp\left[-\beta_2 k_4 - \theta_2 \sum_{i=1}^{r} \left(\frac{\alpha_2}{w_i}\right)^{\beta_2}(1-z_i)\right]$$

$$\prod_{i=1}^{r}\left\{1-\exp\left[-\left(\frac{\alpha_2}{w_i}\right)^{\beta_2}\right]\right\}^{-(1-z_i)(\theta_2+1)}$$

$$\exp\left(-\lambda_2 \sum_{i=1}^{r}(1-z_i+R_i)\left(\left\{\exp\left[\left(\frac{\alpha_2}{w_i}\right)^{\beta_2}\right]-1\right\}^{-\theta_2}\right.\right.$$

$$\left.\left.\vphantom{\sum}\right)\right), \tag{27}$$

$$\pi_5^*(\lambda_1|\alpha_j,\beta_j,\lambda_2,\theta_j) \propto \lambda_1^{m_r+c_5-1}\exp\left\{-\lambda_1\right.$$

$$\times\left(k_5 + \sum_{i=1}^{r}(z_i+R_i)\left(\left\{\exp\left[\left(\frac{\alpha_1}{w_i}\right)^{\beta_1}\right]-1\right\}^{-\theta_1}\right)\right)\right\}, \tag{28}$$

$$\pi_6^*(\lambda_2|\alpha_j,\beta_j,\lambda_2,\theta_j) \propto \lambda_2^{n_r+c_6-1}\exp\left\{-\lambda_2\left(k_6\right.\right.$$

$$\left.\left.+\sum_{i=1}^{r}(1-z_i+R_i)\left(\left\{\exp\left[\left(\frac{\alpha_2}{w_i}\right)^{\beta_2}\right]-1\right\}^{-\theta_2}\right)\right)\right\}, \tag{29}$$

$$\pi_7^*(\theta_1|\alpha_j,\beta_j,\lambda_j,\theta_2) \propto \theta_1^{m_r+c_7-1}\prod_{i=1}^{r}\left\{1-\exp\left[-\right.\right.$$

$$\left.\left.\left(\frac{\alpha_1}{w_i}\right)^{\beta_1}\right]\right\}^{-z_i(\theta_1+1)}\exp\left[-\theta_1\left(k_7+\sum_{i=1}^{r}\left(\frac{\alpha_1}{w_i}\right)^{\beta_1}z_i\right)\right]$$

$$\times\exp\left(-\lambda_1\sum_{i=1}^{r}(z_i+R_i)\left(\left\{\exp\left[\left(\frac{\alpha_1}{w_i}\right)^{\beta_1}\right]-1\right\}^{-\theta_1}\right)\right), \tag{30}$$

and

$$\pi_8^*(\theta_2|\alpha_j,\beta_j,\lambda_j,\theta_1) \propto \theta_2^{n_r+c_8-1}\prod_{i=1}^{r}\left\{1-\exp\left[-\right.\right.$$

$$\left.\left.\left(\frac{\alpha_2}{w_i}\right)^{\beta_2}\right]\right\}^{-(1-z_i)(\theta_2+1)}\exp\left[-\theta_2\left(k_8+\sum_{i=1}^{r}\left(\frac{\alpha_2}{w_i}\right)^{\beta_2}\right.\right.$$

$$\left.\left.\times(1-z_i)\right)\right]\exp\left(-\lambda_2\sum_{i=1}^{r}(1-z_i+R_i)\left(\left\{\exp\left[\left(\frac{\alpha_2}{w_i}\right)^{\beta_2}\right]\right.\right.\right.$$

$$\left.\left.\left.-1\right\}^{-\theta_2}\right)\right). \tag{31}$$

The full conditional posterior density of λ_1 and λ_2 are, respectively, $Gamma(m_r + c_5, k_5 + \sum_{i=1}^{r}(z_i + R_i)\left(\left\{\exp\left[\left(\frac{\alpha_1}{w_i}\right)^{\beta_1}\right]-1\right\}^{-\theta_1}\right))$ and $Gamma(n_r+c_6,k_6+\sum_{i=1}^{r}(1-z_i+R_i)\left(\left\{\exp\left[\left(\frac{\alpha_2}{w_i}\right)^{\beta_2}\right]-1\right\}^{-\theta_2}\right))$, which come directly from Eqs. 28 and 29. Thus, using any gamma routine, we can generate the samples of λ_1 and λ_2. While the conditional posterior distributions of α_j, β_j and θ_j cannot be reduced analytically to well known distribution, and therefore it is not possible to sample directly by standard methods, but the plots of them show that they are similar to normal distribution (see Figure 5).

So, to generate random numbers from these distribution, we use the Metropolis–Hastings (M–H) algorithm with normal proposal distribution. Now, the following steps illustrate the process of M–H algorithm within Gibbs sampling to simulate the posterior samples as suggested by Tierney [14].
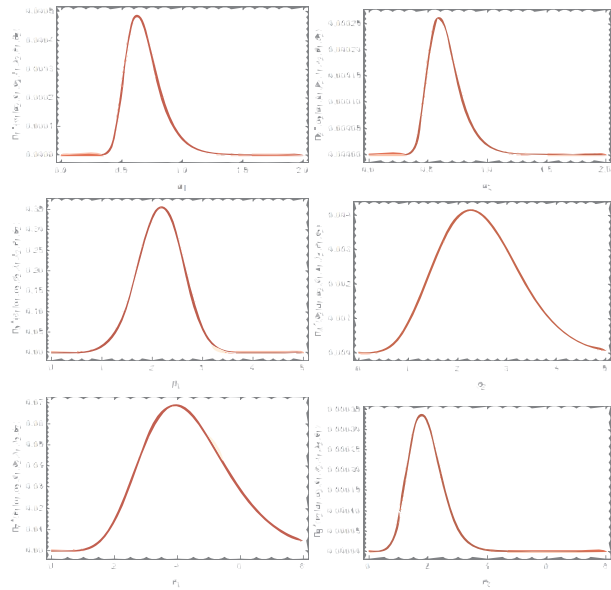
Fig. 5: Posterior density functions for $\alpha_1, \alpha_2, \beta_1, \beta_2, \theta_1$ and θ_2

1. Start with $(\alpha_1^{(0)}, \alpha_2^{(0)}, \beta_1^{(0)}, \beta_2^{(0)}, \lambda_1^{(0)}, \lambda_2^{(0)}, \theta_1^{(0)}, \theta_2^{(0)})$.
2. Set $k = 1$.
3. Generate $\lambda_1^{(k)}$ from
   $Gamma(m_r + c_5, k_5 + \sum_{j=1}^{r}(z_j + R_j)\left(\exp\left[\left(\frac{\alpha_1^{(k-1)}}{w_j}\right)^{\beta_1^{(k-1)}}\right]-1\right)^{-\theta_1^{(k-1)}})$.
4. Generate $\lambda_2^{(k)}$ from
   $Gamma(n_r + c_6, k_6 + \sum_{j=1}^{r}(1 - z_j + R_j)\left(\exp\left[\left(\frac{\alpha_2^{(k-1)}}{w_j}\right)^{\beta_2^{(k-1)}}\right]-1\right)^{-\theta_2^{(k-1)}})$.
5. Using M-H,
   i Generate α_1^* from $N(\alpha_1^{(k-1)}, var(\alpha_1))$, α_2^* from $N(\alpha_2^{(k-1)}, var(\alpha_2))$, β_1^* from $N(\beta_1^{(k-1)}, var(\beta_1))$, β_2^* from $N(\beta_2^{(k-1)}, var(\beta_2))$, θ_1^* from $N(\theta_1^{(k-1)}, var(\theta_1))$ and θ_2^* from $N(\theta_2^{(k-1)}, var(\theta_2))$.
   ii Evaluate the acceptance probabilities:
   $$\psi_1 = min[1, \frac{\pi_1^*(\alpha_1^*|\alpha_2^{(k-1)},\beta_1^{(k-1)},\beta_2^{(k-1)},\lambda_1^{(k)},\lambda_2^{(k)},\theta_1^{(k-1)},\theta_2^{(k-1)},\underline{w})}{\pi_1^*(\alpha_1^{(k-1)}|\alpha_2^{(k-1)},\beta_1^{(k-1)},\beta_2^{(k-1)},\lambda_1^{(k)},\lambda_2^{(k)},\theta_1^{(k-1)},\theta_2^{(k-1)},\underline{w})}].$$
   $$\psi_2 = min[1, \frac{\pi_2^*(\alpha_2^*|\alpha_1^{(k)},\beta_1^{(k-1)},\beta_2^{(k-1)},\lambda_1^{(k)},\lambda_2^{(k)},\theta_1^{(k-1)},\theta_2^{(k-1)},\underline{w})}{\pi_2^*(\alpha_2^{(k-1)}|\alpha_1^{(k)},\beta_1^{(k-1)},\beta_2^{(k-1)},\lambda_1^{(k)},\lambda_2^{(k)},\theta_1^{(k-1)},\theta_2^{(k-1)},\underline{w})}].$$

$$\psi_3 = min[1,$$
$$\frac{\pi_3^*(\beta_1^* | \alpha_1^{(k)}, \alpha_2^{(k)}, \beta_2^{(k-1)}, \lambda_1^{(k)}, \lambda_2^{(k)}, \theta_1^{(k-1)}, \theta_2^{(k-1)}, \underline{w})}{\pi_3^*(\beta_1^{(k-1)} | \alpha_1^{(k)}, \alpha_2^{(k)}, \beta_2^{(k-1)}, \lambda_1^{(k)}, \lambda_2^{(k)}, \theta_1^{(k-1)}, \theta_2^{(k-1)}, \underline{w})}].$$

$$\psi_4 = min[1,$$
$$\frac{\pi_4^*(\beta_2^* | \alpha_1^{(k)}, \alpha_2^{(k)}, \beta_1^{(k)}, \lambda_1^{(k)}, \lambda_2^{(k)}, \theta_1^{(k-1)}, \theta_2^{(k-1)}, \underline{w})}{\pi_4^*(\beta_2^{(k-1)} | \alpha_1^{(k)}, \alpha_2^{(k)}, \beta_1^{(k)}, \lambda_1^{(k)}, \lambda_2^{(k)}, \theta_1^{(k-1)}, \theta_2^{(k-1)}, \underline{w})}].$$

$$\psi_5 = min[1,$$
$$\frac{\pi_7^*(\theta_1^* | \alpha_1^{(k)}, \alpha_2^{(k)}, \beta_1^{(k)}, \beta_2^{(k)}, \lambda_1^{(k)}, \lambda_2^{(k)}, \theta_2^{(k-1)}, \underline{w})}{\pi_7^*(\theta_1^{(k-1)} | \alpha_1^{(k)}, \alpha_2^{(k)}, \beta_1^{(k)}, \beta_2^{(k)}, \lambda_1^{(k)}, \lambda_2^{(k)}, \theta_2^{(k-1)}, \underline{w})}].$$

$$\psi_6 = min[1,$$
$$\frac{\pi_8^*(\theta_2^* | \alpha_1^{(k)}, \alpha_2^{(k)}, \beta_1^{(k)}, \beta_2^{(k)}, \lambda_1^{(k)}, \lambda_2^{(k)}, \theta_1^{(k)}, \underline{w})}{\pi_8^*(\theta_2^{(k-1)} | \alpha_1^{(k)}, \alpha_2^{(k)}, \beta_1^{(k)}, \beta_2^{(k)}, \lambda_1^{(k)}, \lambda_2^{(k)}, \theta_1^{(k)}, \underline{w})}].$$

iii Generate r_1, r_2, r_3, r_4, r_5 and r_6 from a Uniform distribution $(0,1)$.

If $r_1 < \psi_1$, set $\alpha_1^{(k)} = \alpha_1^*$, otherwise $\alpha_1^{(k)} = \alpha_1^{(k-1)}$.

If $r_2 < \psi_2$, set $\alpha_2^{(k)} = \alpha_2^*$, otherwise $\alpha_2^{(k)} = \alpha_2^{(k-1)}$.

If $r_3 < \psi_3$, set $\beta_1^{(k)} = \beta_1^*$, otherwise $\beta_1^{(k)} = \beta_1^{(k-1)}$.

If $r_4 < \psi_4$, set $\beta_2^{(k)} = \beta_2^*$, otherwise $\beta_2^{(k)} = \beta_2^{(k-1)}$.

If $r_5 < \psi_5$, set $\theta_1^{(k)} = \theta_1^*$, otherwise $\theta_1^{(k)} = \theta_1^{(k-1)}$.

If $r_6 < \psi_6$, set $\theta_2^{(k)} = \theta_2^*$, otherwise $\theta_2^{(k)} = \theta_2^{(k-1)}$.

6. Set $k = k + 1$.

7. Repeat steps 3–6 N times.

Hence, to evaluate the convergence and to remove the affection of the selection of initial value, we remove the first M simulated values. Then the selected sample is $\alpha_j^{(k)}, \beta_j^{(k)}, \lambda_j^{(k)}$ and $\theta_j^{(k)}$, $j = 1, 2$ and $k = M + 1, ..., N$. So, for large N, the Bayes estimates of $\nu = \alpha_j, \beta_j, \lambda_j$ or θ_j is given by

$$\hat{\nu}_{MC} = \frac{1}{N-M} \sum_{k=M+1}^{N} \nu^{(k)}. \qquad (32)$$

To compute the credible intervals of $\alpha_j, \beta_j, \lambda_j$ and θ_j, make $\alpha_j^{(k)}, \beta_j^{(k)}, \lambda_j^{(k)}$ and $\theta_j^{(k)}$, $k = 1, 2, ..., N$, in ascending order, hence $(1 - \delta)100\%$ CI of them is given by

$$[\nu_{(N(\delta/2))}, \nu_{(N(1-\delta/2))}].$$

8 Simulation Results and Real Data Analysis

The main object of this section is to illustrate numerically the results discussed in the previous sections.

8.1 Simulation Study

To illustrate the use of the estimation method proposed in this article, a **NJPC** samples are generated from two WFr populations with various samples sizes as $m, n = 10, 20, 30, 40, 50, 60, 70, 80, 90$ and various values of joint progressive type II censoring schemes (JP-II-CS) $r = 5, 10, 15, 20, 25, 30, 35, 40, 45, 50$. Also, set the parameters $(\alpha_1, \alpha_2, \beta_1, \beta_2, \lambda_1, \lambda_2, \theta_1, \theta_2) = (0.5, 0.6, 2.5, 2.69, 0.69, 0.8, 1.57, 1.8)$. The MSEs,

lentghs of 95% CIs and the corresponding coverage probability (CP) for the parameters $\alpha_1, \alpha_2, \beta_1, \beta_2, \lambda_1, \lambda_2, \theta_1$ and θ_2 have been evaluated using MLEs and MCMC with 10000 observations under SE and LINEX loss functions. This process is repeated 1000 times and the results of the mean values of mean square error (MSE), lengths and CP, are displayed in Tables (1–13).

8.2 Real data analysis

In this case, we analyze the data stated in Section 3. Based on the Kolmogorov-Smirnov (K-S) test, we checked if the two data sets fit the WFr model or not. For set 1 (x), the calculated value of the K-S test is 0.108225 and the associated P-value equal 0.912838 and for set 2 (y), the calculated value of the K-S test is 0.123457 and the associated P-value equal 0.759875. So, the two sets fit the WFr model very well. Also, we have just plotted the data in Figure 6 for the first sample and in Figure 7 for the second sample. These figures show also that WFr can be a good model fitted this data.

From the above two data sets, a **NJPC** samples with the censoring scheme have been generated with $m = 24$ for the first sample and $n = 27$ for the second sample, hence, $N = m + n$ denoted the total sample size and when $r = 8$ and $R = (2_{(7)})$, the generated data sets are $w = (1, 3, 4, 5, 11, 13, 14, 15)$ with $Z = (0, 1, 0, 1, 0, 1, 1, 1)$. For Bayesian estimation, MCMC method is used based on 10000 MCMC samples and discard the first 2000 values as 'burn-in', also the Bayesian estimates for $\alpha_1, \alpha_2, \beta_1, \beta_2, \lambda_1, \lambda_2, \theta_1$ and θ_2 are obtained under SE loss and LINEX loss functions. Based on the above information, we get:

1. Table 14 displays the point estimates based on MLEs, Bootstrap and MCMC methods for $\alpha_1, \alpha_2, \beta_1, \beta_2, \lambda_1, \lambda_2, \theta_1$ and θ_2.
2. Tables (15–18) display the results of 95% ACIs, using MLE, four Bootstrap CIs and MCMC for the unknown parameters.

9 Conclusion

The proposed algorithm has been applied to the same two data sets used by Mondal and Kundu and we obtained the same results. The results can easily be obtained through the proposed algorithm for any two continuous probability distributions without establishing complicated theoretical results as these theorems that were proved in case of applying the new joint progressive censoring to the exponential distribution with one parameter, see Mondal and Kundu [2]. The proposed algorithm is extremely flexible to be implemented using one of the familiar software programs, like Mathematica, Matlab, R and Python.

Also, in this article, the MLEs, Bootstrap CIs and Bayesian estimation based on squared error loss and LINEX lose functions for the unknown parameters of two WFr distributions has been discussed based on **NJPC**. A simulation study is used to compare the performance of the proposed methods for different sample sizes m, n and different schemes.
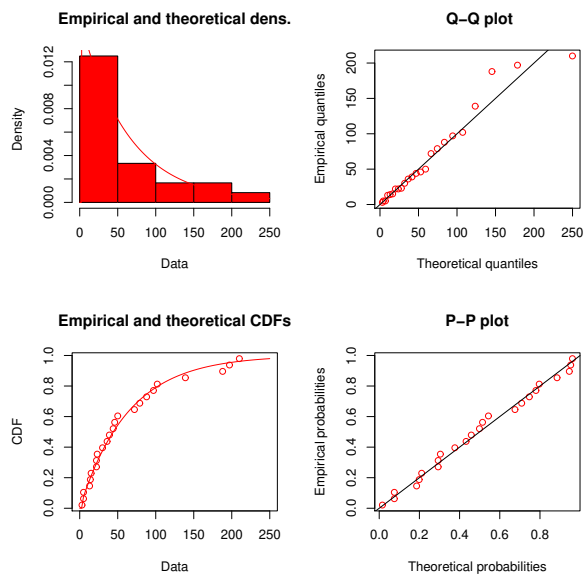
Fig. 6: Fitted first sample

Fig. 7: Fitted second sample

1. It is evident that from Tables (15–18), the Boot-BC and Boot-BCa are better than Boot-t and Boot-p for parameters α_i, β_i and λ_1 and Booot-t is the best for parameters λ_2 and θ_i in the sense of having smallest lengths.
2. It is observed that from Tables (1–13), the MSEs of MLE is smaller than the MSEs of MCMC. Also CP of MCMC is smaller than CP of MLE. Then, the performance of the MLE estimates for the parameter α_i, β_i, λ_i and θ_i, $i = 1,2$ are better than the Bayes estimates.
3. The Bayes estimates under LINEX with $q = 0.5$ provides better estimates in the sense of having smaller MSEs as shown in Tables (1–13).

Conflicts of Interests
The authors declare that they have no conflicts of interests

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Table 1: MSE, length and coverage probability (CP) of estimates for the parameter α_1.

r	(m,n)	MLE			MCMC				
		MSE	Length	CP	SE	LINEX		Length	CP
						q=-0.5	q=0.5		
5	(10,10)	0.0152	2.3575	0.994	0.0155	0.0154	0.0154	1.4811	0.994
	(10,20)	0.0107	2.3337	0.995	0.0113	0.0113	0.0112	1.4108	0.993
	(20,10)	0.0100	2.3142	0.996	0.0104	0.0105	0.0104	1.4517	0.995
10	(20,20)	0.0100	2.2993	0.993	0.0102	0.0104	0.0101	1.5323	0.994
	(20,30)	0.0098	2.2519	0.994	0.0100	0.0101	0.0100	1.4029	0.995
	(30,20)	0.0098	2.2716	0.995	0.0099	0.0099	0.0098	1.4012	0.993
15	(30,30)	0.0097	2.3310	0.994	0.0098	0.0098	0.0098	1.5022	0.995
	(30,40)	0.0084	2.2657	0.995	0.0104	0.0104	0.0102	1.4266	0.995
	(40,30)	0.0084	2.2517	0.995	0.0088	0.0088	0.0086	1.4039	0.994
20	(40,40)	0.0079	2.2952	0.994	0.0088	0.0089	0.0088	1.4056	0.993
	(40,50)	0.0081	2.2104	0.996	0.0088	0.0088	0.0086	1.4003	0.995
	(50,40)	0.0079	2.2280	0.996	0.0082	0.0083	0.0080	1.3832	0.995
25	(50,50)	0.0070	2.2439	0.995	0.0072	0.0073	0.0072	1.4116	0.997
	(50,60)	0.0063	2.1673	0.997	0.0070	0.0071	0.0070	1.4046	0.996
	(60,50)	0.0060	2.1427	0.996	0.0063	0.0062	0.0062	1.3734	0.997
30	(60,60)	0.0056	2.1185	0.995	0.0058	0.0058	0.0057	1.3842	0.995
	(60,70)	0.0055	2.1039	0.996	0.0057	0.0057	0.0055	1.3844	0.993
	(70,60)	0.0054	2.1434	0.996	0.0055	0.0055	0.0054	1.3896	0.994
35	(70,70)	0.0053	2.1731	0.995	0.0064	0.0065	0.0064	1.4073	0.995
	(70,80)	0.0050	3.1444	0.998	0.0052	0.0052	0.0052	1.3543	0.998
	(80,70)	0.0043	2.0579	0.997	0.0045	0.0044	0.0043	1.3530	0.995
40	(80,80)	0.0039	2.1008	0.997	0.0040	0.0041	0.0040	1.3031	0.995
	(80,90)	0.0035	2.0920	0.996	0.0037	0.0037	0.0036	1.5302	0.996
	(90,80)	0.0035	2.0489	0.997	0.0035	0.0036	0.0035	1.2938	0.997
45	(90,90)	0.0030	2.0050	0.997	0.0032	0.0033	0.0031	1.2415	0.994
	(90,100)	0.0027	2.0309	0.996	0.0030	0.0030	0.0028	1.2773	0.994
	(100,90)	0.0027	1.9563	0.998	0.0028	0.0029	0.0027	1.2095	0.996
50	(100,100)	0.0020	1.8422	0.995	0.0020	0.0020	0.0020	1.2024	0.995
	(100,110)	0.0018	1.9259	0.997	0.0020	0.0021	0.0019	1.2074	0.997
	(110,100)	0.0011	1.9094	0.998	0.0010	0.0012	0.0010	1.2053	0.998

Table 2: MSE, length and coverage probability (CP) of estimates for the parameter α_2.

r	(m,n)	MLE			MCMC				
		MSE	Length	CP	SE	LINEX		Length	CP
						q=-0.5	q=0.5		
5	(10,10)	0.0171	2.0578	0.996	0.0173	0.0173	0.0172	1.6333	0.995
	(10,20)	0.0154	1.8109	0.995	0.0156	0.0157	0.0155	1.6023	0.995
	(20,10)	0.0121	1.9183	0.997	0.0122	0.0122	0.0121	1.5342	0.997
10	(20,20)	0.0118	1.8763	0.998	0.0123	0.0124	0.0120	1.6191	0.995
	(20,30)	0.0115	1.8752	0.997	0.0119	0.0120	0.0118	1.6147	0.995
	(30,20)	0.0114	1.8143	0.995	0.0119	0.0119	0.0119	1.5058	0.997

Table 3: continued Table 2.

r	(m,n)	MSE	Length	CP	SE	q=-0.5	q=0.5	Length	CP
15	(30,30)	0.0114	1.8436	0.995	0.0118	0.0118	0.0117	1.5043	0.996
	(30,40)	0.0098	1.7845	0.996	0.0099	0.0098	0.0098	2.5022	0.997
	(40,30)	0.0096	1.8886	0.994	0.0096	0.0097	0.0096	1.5032	0.995
20	(40,40)	0.0088	1.7999	0.995	0.0089	0.0089	0.0088	1.5122	0.995
	(40,50)	0.0087	1.7771	0.995	0.0090	0.0090	0.0089	1.5073	0.995
	(50,40)	0.0074	1.7628	0.996	0.0075	0.0076	0.0075	1.5003	0.997
25	(50,50)	0.0071	1.7681	0.997	0.0074	0.0076	0.0074	1.4074	0.995
	(50,60)	0.0069	1.7124	0.996	0.0072	0.0072	0.0072	1.4861	0.996
	(60,50)	0.0061	1.7269	0.995	0.0063	0.0063	0.0062	1.4054	0.996
30	(60,60)	0.0057	1.6908	0.994	0.0062	0.0063	0.0060	1.4056	0.995
	(60,70)	0.0047	1.5836	0.994	0.0059	0.0060	0.0058	1.4269	0.995
	(70,60)	0.0045	1.6709	0.996	0.0046	0.0046	0.0046	1.4083	0.997
35	(70,70)	0.0044	1.6028	0.995	0.0045	0.0046	0.0045	1.5167	0.998
	(70,80)	0.0037	1.6003	0.995	0.0043	0.0044	0.0043	1.4093	0.995
	(80,70)	0.0037	1.5847	0.994	0.0038	0.0039	0.0038	1.4056	0.994
40	(80,80)	0.0032	1.6376	0.995	0.0033	0.0033	0.0032	1.3970	0.995
	(80,90)	0.0029	1.6247	0.995	0.0032	0.0033	0.0030	1.3788	0.993
	(90,80)	0.0030	1.5921	0.996	0.0031	0.0031	0.0030	1.3980	0.995
45	(90,90)	0.0028	1.6974	0.996	0.0035	0.0034	0.0035	1.3975	0.996
	(90,100)	0.0025	1.6830	0.994	0.0028	0.0029	0.0028	1.3736	0.997
	(100,90)	0.0023	1.6119	0.995	0.0027	0.0026	0.0026	1.3352	0.994
50	(100,100)	0.0024	1.5617	0.994	0.0026	0.0027	0.0025	1.2804	0.996
	(100,110)	0.0023	1.5415	0.995	0.0023	0.0023	0.0023	1.2043	0.995
	(110,100)	0.0016	1.5359	0.997	0.0018	0.0018	0.0017	1.1955	0.996

Table 4: MSE, length and coverage probability (CP) of estimates for the parameter β_1.

r	(m,n)	MLE			MCMC				
		MSE	Length	CP	SE	LINEX		Length	CP
						q=-0.5	q=0.5		
5	(10,10)	0.0980	2.5105	0.995	0.0983	0.0983	0.0981	1.9006	0.994
	(10,20)	0.0910	2.4377	0.994	0.0911	0.0912	0.0911	1.9028	0.995
	(20,10)	0.0873	2.4188	0.995	0.0875	0.0875	0.0873	1.6004	0.995
10	(20,20)	0.0857	2.5313	0.996	0.0856	0.0856	0.0856	1.8016	0.998
	(20,30)	0.0832	2.4415	0.995	0.0841	0.0841	0.0840	1.7042	0.994
	(30,20)	0.0830	2.3794	0.994	0.0832	0.0833	0.0832	1.7387	0.997
15	(30,30)	0.0827	2.3446	0.994	0.0830	0.0830	0.0829	1.8113	0.994
	(30,40)	0.0801	2.2321	0.993	0.0801	0.0802	0.0801	1.7012	0.995
	(40,30)	0.0788	2.1875	0.995	0.0789	0.0789	0.0788	1.8027	0.995
20	(40,40)	0.0759	2.2046	0.993	0.0761	0.0761	0.0760	1.7829	0.990
	(40,50)	0.0745	2.2423	0.994	0.0747	0.0747	0.0747	1.7849	0.992
	(50,40)	0.0733	2.1934	0.996	0.0735	0.0735	0.0734	1.6415	0.995
25	(50,50)	0.0721	2.1896	0.997	0.0724	0.0724	0.0722	1.7509	0.996
	(50,60)	0.0719	2.1701	0.995	0.0721	0.0721	0.0720	1.6353	0.995
	(60,50)	0.0691	2.1678	0.996	0.0693	0.0693	0.0692	1.6213	0.995
30	(60,60)	0.0673	2.1597	0.995	0.0676	0.0677	0.0676	1.6908	0.993
	(60,70)	0.0661	2.0734	0.994	0.0664	0.0665	0.0664	1.7503	0.995
	(70,60)	0.0653	2.0486	0.996	0.0654	0.0654	0.0654	1.7413	0.996
35	(70,70)	0.0649	2.1033	0.995	0.0649	0.0650	0.0649	1.7914	0.997
	(70,80)	0.0639	2.0291	0.995	0.0638	0.0638	0.0638	1.5487	0.997
	(80,70)	0.0605	2.0141	0.994	0.0606	0.0606	0.0606	1.4921	0.996
40	(80,80)	0.0555	1.8301	0.996	0.0555	0.0555	0.0555	1.4406	0.996
	(80,90)	0.0546	1.9807	0.996	0.0550	0.0551	0.0549	1.3542	0.995
	(90,80)	0.0517	1.9020	0.997	0.0522	0.0522	0.0520	1.5830	0.997
45	(90,90)	0.0433	1.9359	0.996	0.0435	0.0436	0.0434	1.4119	0.996
	(90,100)	0.0405	1.9829	0.997	0.0406	0.0406	0.0405	1.4380	0.995
	(100,90)	0.0358	1.9243	0.997	0.0361	0.0363	0.0360	1.4753	0.994

Table 5: continued Table 4.

50	(100,100)	0.0285	1.8655	0.998	0.0287	0.0288	0.0287	1.3952	0.997
	(100,110)	0.0253	1.8810	0.998	0.0259	0.0261	0.0258	1.3477	0.997
	(110,100)	0.0207	1.7732	0.998	0.0215	0.0218	0.0214	1.3423	0.998

Table 6: MSE, length and coverage probability (CP) of estimates for the parameter β_2.

r	(m,n)	MLE			MCMC				
		MSE	Length	CP	SE	LINEX		Length	CP
						q=-0.5	q=0.5		
5	(10,10)	0.0818	2.1838	0.993	0.0819	0.0820	0.0818	1.6345	0.994
	(10,20)	0.0793	2.0131	0.993	0.0795	0.0796	0.0795	1.5830	0.995
	(20,10)	0.0785	2.1951	0.995	0.0787	0.0786	0.0786	1.5308	0.992
10	(20,20)	0.0754	2.1985	0.994	0.0755	0.0755	0.0754	1.5211	0.990
	(20,30)	0.0743	2.1924	0.993	0.0744	0.0744	0.0745	1.6372	0.993
	(30,20)	0.0727	2.1655	0.995	0.0728	0.0729	0.0728	1.5827	0.995
15	(30,30)	0.0714	2.1619	0.996	0.0714	0.0714	0.0714	1.6108	0.996
	(30,40)	0.0707	2.1547	0.995	0.0710	0.0710	0.0709	1.5019	0.995
	(40,30)	0.0704	2.0117	0.995	0.0712	0.0713	0.0712	1.3947	0.994
20	(40,40)	0.0596	2.0965	0.994	0.0598	0.0597	0.0597	1.5112	0.995
	(40,50)	0.0701	1.5495	0.997	0.0702	0.0703	0.0702	1.6133	0.998
	(50,40)	0.0698	1.1164	0.994	0.0700	0.0701	0.0699	1.5144	0.995
25	(50,50)	0.0691	1.2029	0.996	0.0696	0.0696	0.0695	1.4239	0.995
	(50,60)	0.0681	1.6038	0.996	0.0683	0.0683	0.0683	1.4221	0.994
	(60,50)	0.0678	1.3863	0.994	0.0679	0.0679	0.0678	1.4012	0.994
30	(60,60)	0.0631	1.5025	0.997	0.0635	0.0636	0.0634	1.4315	0.995
	(60,70)	0.0571	1.6485	0.995	0.0573	0.0572	0.0572	1.4062	0.997
	(70,60)	0.0555	1.6790	0.995	0.0565	0.0566	0.0565	1.4949	0.995
35	(70,70)	0.0547	1.7601	0.997	0.0551	0.0550	0.0550	1.5108	0.997
	(70,80)	0.0532	1.8933	0.995	0.0534	0.0534	0.0533	1.6016	0.995
	(80,70)	0.0539	1.8829	0.996	0.0542	0.0543	0.0540	1.6014	0.994
40	(80,80)	0.0509	1.6340	0.997	0.0512	0.0513	0.0510	1.5135	0.996
	(80,90)	0.0458	1.5507	0.996	0.0461	0.0460	0.0459	1.5118	0.995
	(90,80)	0.0423	1.9351	0.993	0.0425	0.0425	0.0424	1.5547	0.995
45	(90,90)	0.0411	1.9380	0.995	0.0414	0.0413	0.0413	1.5300	0.997
	(90,100)	0.0384	1.8901	0.997	0.0386	0.0386	0.0385	1.5387	0.997
	(100,90)	0.0309	1.8493	0.996	0.0312	0.0312	0.0310	1.5031	0.996
50	(100,100)	0.0288	1.8946	0.995	0.0289	0.0289	0.0288	1.4137	0.998
	(100,110)	0.0223	1.8748	0.998	0.0227	0.0228	0.0226	1.4730	0.997
	(110,100)	0.0198	1.8738	0.998	0.0201	0.0201	0.0200	1.4325	0.997

Table 7: MSE, length and coverage probability (CP) of estimates for the parameter λ_1.

r	(m,n)	MLE			MCMC				
		MSE	Length	CP	SE	LINEX		Length	CP
						q=-0.5	q=0.5		
5	(10,10)	0.0694	2.6872	0.984	0.0726	0.0727	0.0724	2.1217	0.985
	(10,20)	0.0611	2.5627	0.986	0.0624	0.0625	0.0622	2.1337	0.986
	(20,10)	0.0537	2.6398	0.985	0.0557	0.0555	0.0549	2.2366	0.984
10	(20,20)	0.0485	2.6004	0.987	0.0531	0.0533	0.0527	2.1546	0.985
	(20,30)	0.0433	2.6032	0.985	0.0528	0.0535	0.0521	2.0864	0.985
	(30,20)	0.0421	2.5875	0.987	0.0469	0.0467	0.0471	1.8582	0.985
15	(30,30)	0.0369	2.5579	0.990	0.0374	0.0387	0.0378	1.9879	0.987
	(30,40)	0.0368	2.5212	0.985	0.0383	0.0383	0.0384	1.9375	0.993
	(40,30)	0.0362	2.5707	0.989	0.0412	0.0418	0.0408	2.0718	0.991
20	(40,40)	0.0340	2.5696	0.990	0.0386	0.0395	0.0383	2.1458	0.990
	(40,50)	0.0312	2.5368	0.990	0.0329	0.0326	0.0333	2.0201	0.991
	(50,40)	0.0309	2.4801	0.995	0.0373	0.0376	0.0368	2.0196	0.987

Table 8: continued Table 7.

25	(50,50)	0.0345	2.4773	0.965	0.0355	0.0362	0.0350	1.8821	0.989
	(50,60)	0.0319	2.4228	0.990	0.0314	0.0308	0.0308	1.8885	0.995
	(60,50)	0.0286	2.4178	0.985	0.0319	0.0323	0.0315	1.8729	0.988
30	(60,60)	0.0276	2.4314	0.997	0.0307	0.0309	0.0302	1.9806	0.987
	(60,70)	0.0259	2.4487	0.993	0.0268	0.0265	0.0261	1.9773	0.995
	(70,60)	0.0240	2.4107	0.997	0.0247	0.0245	0.0242	2.0741	0.990
35	(70,70)	0.0210	2.4210	0.995	0.0246	0.0243	0.0240	1.9814	0.991
	(70,80)	0.0133	2.4595	0.996	0.0232	0.0340	0.0325	1.8556	0.989
	(80,70)	0.0123	2.4557	0.994	0.0211	0.0201	0.0221	1.9377	0.993
40	(80,80)	0.0125	2.3741	0.993	0.0198	0.0188	0.0188	1.9492	0.990
	(80,90)	0.0111	2.3948	0.998	0.0146	0.0145	0.0142	1.9977	0.991
	(90,80)	0.0102	2.3723	0.996	0.0133	0.0123	0.0144	1.9074	0.993
45	(90,90)	0.0100	2.3707	0.995	0.0121	0.0125	0.0119	1.8390	0.994
	(90,100)	0.0101	2.3038	0.997	0.0118	0.0122	0.0115	1.8924	0.995
	(100,90)	0.0098	2.2816	0.996	0.0109	0.0115	0.0109	1.8001	0.995
50	(100,100)	0.0098	2.3000	0.996	0.0112	0.0115	0.0111	1.7830	0.995
	(100,110)	0.0098	2.2593	0.995	0.0105	0.0110	0.0105	1.7046	0.996
	(110,100)	0.0096	2.2287	0.998	0.0105	0.0107	0.0114	1.7080	0.997

Table 9: MSE, length and coverage probability (CP) of estimates for the parameter λ_2.

r	(m,n)	MLE			MCMC				
		MSE	Length	CP	SE	LINEX		Length	CP
						q=-0.5	q=0.5		
5	(10,10)	0.0887	2.3889	0.985	0.1481	0.1477	0.1476	1.8500	0.989
	(10,20)	0.0763	2.3749	0.987	0.0780	0.0776	0.0768	1.8015	0.991
	(20,10)	0.0749	2.3519	0.988	0.0753	0.0747	0.0744	1.9028	0.988
10	(20,20)	0.0709	2.3008	0.988	0.0751	0.0744	0.0748	1.9893	0.987
	(20,30)	0.0686	2.2345	0.995	0.0744	0.0739	0.0735	1.9846	0.985
	(30,20)	0.0684	2.2075	0.985	0.0723	0.0724	0.0723	1.9581	0.986
15	(30,30)	0.0649	2.2903	0.990	0.0709	0.0701	0.0701	1.8996	0.987
	(30,40)	0.0639	2.2400	0.985	0.0687	0.0689	0.0686	1.9032	0.990
	(40,30)	0.0610	2.2096	0.988	0.0656	0.0646	0.0646	1.8591	0.990
20	(40,40)	0.0574	2.1326	0.991	0.0625	0.0624	0.0624	1.8296	0.988
	(40,50)	0.0589	2.1627	0.985	0.0598	0.0600	0.0599	1.7275	0.989
	(50,40)	0.0539	2.1802	0.995	0.0587	0.0576	0.0569	1.7054	0.993

Table 10: continued Table 9.

25	(50,50)	0.0444	2.1119	0.995	0.0481	0.0487	0.0481	1.6807	0.991
	(50,60)	0.0404	2.0898	0.995	0.0454	0.0456	0.0450	2.6446	0.992
	(60,50)	0.0401	2.0032	0.995	0.0411	0.0411	0.0411	1.6053	0.995
30	(60,60)	0.0343	2.0266	0.996	0.0349	0.0354	0.0348	1.6136	0.990
	(60,70)	0.0312	1.9601	0.989	0.0346	0.0349	0.0343	1.6244	0.991
	(70,60)	0.0304	1.9782	0.991	0.0339	0.0344	0.0315	1.5374	0.993
35	(70,70)	0.0324	1.9901	0.992	0.0358	0.0357	0.0357	1.5701	0.993
	(70,80)	0.0282	1.9811	0.987	0.0314	0.0308	0.0307	1.5787	0.994
	(80,70)	0.0204	1.9705	0.990	0.0278	0.0281	0.0275	1.4717	0.990
40	(80,80)	0.0225	1.9927	0.995	0.0257	0.0252	0.0252	1.4229	0.991
	(80,90)	0.0172	1.9672	0.993	0.0189	0.0176	0.0173	1.4174	0.989
	(90,80)	0.0149	1.8873	0.995	0.0158	0.0153	0.0153	1.4107	0.994
45	(90,90)	0.0142	1.8690	0.995	0.0149	0.0148	0.0143	1.3803	0.993
	(90,100)	0.0135	1.8064	0.994	0.0143	0.0145	0.0139	1.3200	0.995
	(100,90)	0.0132	1.8801	0.996	0.0135	0.0135	0.0133	1.3013	0.995
50	(100,100)	0.0112	1.8152	0.994	0.0115	0.0115	0.0112	1.2980	0.994
	(100,110)	0.0110	1.7811	0.997	0.0115	0.0118	0.0111	1.2371	0.996
	(110,100)	0.0098	1.7340	0.996	0.0107	0.0108	0.0107	1.2291	0.997

Table 11: MSE, length and coverage probability (CP) of estimates for the parameter θ_1.

r	(m,n)	MLE			MCMC				
		MSE	Length	CP	SE	LINEX		Length	CP
						q=-0.5	q=0.5		
5	(10,10)	0.0788	2.5176	0.987	0.0791	0.0791	0.0791	2.0024	0.987
	(10,20)	0.0765	2.4364	0.989	0.0765	0.0765	0.0765	1.9190	0.989
	(20,10)	0.0752	2.3780	0.991	0.0754	0.0755	0.0754	1.8734	0.990
10	(20,20)	0.0757	2.3373	0.993	0.0759	0.0759	0.0759	1.8315	0.991
	(20,30)	0.0707	2.3626	0.995	0.0722	0.0723	0.0722	1.9829	0.990
	(30,20)	0.0694	2.3157	0.997	0.0694	0.0694	0.0694	1.9006	0.991
15	(30,30)	0.0692	2.3260	0.990	0.0697	0.0697	0.0697	1.9043	0.992
	(30,40)	0.0652	2.2985	0.989	0.0652	0.0652	0.0652	1.9008	0.992
	(40,30)	0.0621	2.3460	0.993	0.0624	0.0624	0.0624	1.8842	0.989
20	(40,40)	0.0586	2.2936	0.988	0.0594	0.0598	0.0594	1.9721	0.990
	(40,50)	0.0562	2.2720	0.991	0.0561	0.0561	0.0561	1.8537	0.996
	(50,40)	0.0552	2.2363	0.990	0.0551	0.0551	0.0551	1.8602	0.995
25	(50,50)	0.0537	2.2630	0.991	0.0572	0.0572	0.0572	1.8757	0.987
	(50,60)	0.0506	2.1932	0.991	0.0507	0.0506	0.0507	1.8190	0.993
	(60,50)	0.0508	2.1473	0.993	0.0515	0.0513	0.0513	1.8025	0.994
30	(60,60)	0.0417	2.1750	0.995	0.0420	0.0417	0.0417	1.8114	0.995
	(60,70)	0.0393	2.1165	0.993	0.0395	0.0395	0.0395	1.8015	0.995
	(70,60)	0.0394	2.0806	0.991	0.0395	0.0395	0.0395	1.7684	0.985
35	(70,70)	0.0341	2.0776	0.995	0.0341	0.0341	0.0341	1.7408	0.988
	(70,80)	0.0285	2.0980	0.995	0.0289	0.0289	0.0288	1.7537	0.986
	(80,70)	0.0277	2.0077	0.996	0.0278	0.0279	0.0278	1.7415	0.996
40	(80,80)	0.0249	1.9441	0.996	0.0247	0.0247	0.0247	1.7313	0.990
	(80,90)	0.0235	1.9479	0.998	0.0265	0.0265	0.0265	1.7005	0.991
	(90,80)	0.0229	1.9023	0.998	0.0239	0.0240	0.0239	1.7007	0.995
45	(90,90)	0.0247	1.9789	0.996	0.0284	0.0284	0.0284	1.7730	0.994
	(90,100)	0.0245	1.9529	0.995	0.0268	0.0268	0.0255	1.7574	0.994
	(100,90)	0.0251	1.9851	0.996	0.0260	0.0260	0.0260	1.6904	0.994
50	(100,100)	0.0189	1.9324	0.996	0.0238	0.0240	0.0238	1.6253	0.994
	(100,110)	0.0152	1.9838	0.997	0.0181	0.0185	0.0180	1.6573	0.994
	(110,100)	0.0137	1.9740	0.999	0.0145	0.0153	0.0139	1.6270	0.994

Table 12: MSE, length and coverage probability (CP) of estimates for the parameter θ_2.

r	(m,n)	MLE			MCMC				
		MSE	Length	CP	SE	LINEX		Length	CP
						q=-0.5	q=0.5		
5	(10,10)	0.0995	2.6313	0.990	0.0998	0.0999	0.0997	1.8101	0.990
	(10,20)	0.0983	2.5571	0.991	0.0984	0.0984	0.0984	1.8729	0.989
	(20,10)	0.0816	2.5743	0.993	0.0812	0.0812	0.0812	1.8439	0.986
10	(20,20)	0.0751	2.5854	0.991	0.0771	0.0771	0.0771	1.9819	0.990
	(20,30)	0.0647	2.5115	0.995	0.0648	0.0648	0.0648	1.8233	0.993
	(30,20)	0.0583	2.4330	0.994	0.0585	0.0586	0.0585	1.8045	0.992
15	(30,30)	0.0392	2.3843	0.995	0.0409	0.0409	0.0409	1.8191	0.993
	(30,40)	0.0353	2.3570	0.995	0.0356	0.0356	0.0355	1.7071	0.991
	(40,30)	0.0335	2.3780	0.995	0.0339	0.0339	0.0338	1.7053	0.994
20	(40,40)	0.0272	2.3576	0.995	0.0273	0.0273	0.0273	1.6302	0.993
	(40,50)	0.0289	2.3392	0.995	0.0286	0.0286	0.0286	1.7027	0.995
	(50,40)	0.0268	2.3900	0.994	0.0271	0.0271	0.0271	1.6742	0.995

Table 13: continued Table 12.

25	(50,50)	0.0226	2.2230	0.993	0.0225	0.0225	0.0225	1.6129	0.992
	(50,60)	0.0211	2.3369	0.996	0.0219	0.0219	0.0219	1.7818	0.994
	(60,50)	0.0206	1.2670	0.996	0.0247	0.0247	0.0244	1.6665	0.993
30	(60,60)	0.0201	2.2620	0.994	0.0210	0.0212	0.0207	1.6939	0.989
	(60,70)	0.0135	1.9982	0.990	0.0135	0.0135	0.0135	1.6031	0.996
	(70,60)	0.0109	1.9380	0.992	0.0113	0.0115	0.0110	1.6963	0.994
35	(70,70)	0.0105	2.1160	0.995	0.0120	0.0125	0.0125	1.6117	0.991
	(70,80)	0.0099	1.9707	0.996	0.0114	0.0115	0.0113	1.6037	0.989
	(80,70)	0.0094	1.9052	0.996	0.0115	0.0115	0.0115	1.6024	0.990
40	(80,80)	0.0097	1.9226	0.996	0.0114	0.0115	0.0099	1.6919	0.995
	(80,90)	0.0098	1.9124	0.994	0.0110	0.0109	0.0101	1.6623	0.995
	(90,80)	0.0092	1.9998	0.993	0.0103	0.0103	0.0100	1.6016	0.993
45	(90,90)	0.0099	2.0509	0.995	0.0105	0.0107	0.0104	1.5080	0.995
	(90,100)	0.0099	1.9840	0.995	0.0104	0.0107	0.0105	1.5309	0.997
	(100,90)	0.0098	1.9935	0.996	0.0114	0.0115	0.0110	1.5001	0.998
50	(100,100)	0.0097	2.0470	0.998	0.0124	0.0125	0.0124	1.4402	0.998
	(100,110)	0.0098	2.0350	0.997	0.0106	0.0110	0.0106	1.4890	0.999
	(110,100)	0.0090	1.9898	0.998	0.0100	0.0101	0.0100	1.4751	0.998

Table 14: Different point estimates of $\alpha_1, \alpha_2, \beta_1, \beta_2, \lambda_1, \lambda_2, \theta_1, \theta_2$.

Parameters	MLE	Boot-p	Boot-t	MCMC				
				SE		LINEX		
					q=-0.5	q=0.0001	q=0.5	
α_1	4.1700	4.0614	4.0806	4.1532	4.1575	4.1532	4.1519	
α_2	1.3500	1.3395	1.3385	1.3451	1.3473	1.3451	1.3430	
β_1	6.9000	7.0883	6.9829	6.9578	6.9584	6.9578	6.9548	
β_2	5.9000	6.1450	6.0164	6.1835	6.1366	6.1835	6.0952	
λ_1	0.0660	0.0610	0.0648	0.0645	0.0657	0.0645	0.0640	
λ_2	0.0100	0.0119	0.0102	0.0102	0.0102	0.0102	0.0101	
θ_1	0.1400	0.1643	0.1341	0.1429	0.1435	0.1429	0.1424	
θ_2	0.1800	0.1974	0.1790	0.1822	0.1821	0.1822	0.1825	

Table 15: 95% CI for α_1 and α_2.

Method	α_1			α_2		
	Lower	Upper	Length	Lower	Upper	Length
MLE	3.4329	4.9029	1.4700	0.7882	1.9118	1.1235
Boot-p	3.2393	4.1700	0.9307	1.2123	1.3651	0.1527
Boot-t	3.4101	4.1700	0.7599	1.2416	1.3500	0.1084
Boot-BC CI	3.4393	4.1700	0.7307	1.2585	1.3500	0.0915
Boot-BCa CI	3.4393	4.1700	0.7307	1.2593	1.3500	0.0907
MCMC	3.8471	4.6085	0.7614	0.9310	1.5055	0.5745

Table 16: 95% CI for β_1 and β_2.

Method	β_1			β_2		
	Lower	Upper	Length	Lower	Upper	Length
MLE	5.5381	8.2619	2.7238	4.8641	6.9359	2.0719
Boot-p	4.3558	7.6863	3.3305	4.8674	8.1489	3.2815
Boot-t	4.9000	7.2147	2.3147	5.4720	7.7937	2.3217
Boot-BC CI	5.6973	6.9000	1.2027	4.6018	5.9000	1.2982
Boot-BCa CI	5.6973	6.9000	1.2027	4.6018	5.9000	1.2982
MCMC	5.8031	7.4792	1.6761	5.5964	6.4732	0.8768

Table 17: 95% CI for λ_1 and λ_2.

Method	λ_1			λ_2		
	Lower	Upper	Length	Lower	Upper	Length
MLE	0.0001	0.1525	0.1732	0.0001	0.1241	0.1240
Boot-p	0.0141	0.0795	0.0654	0.0078	0.0189	0.0111
Boot-t	0.0454	0.0682	0.0228	0.0100	0.0126	0.0026
Boot-BC CI	0.0061	0.0660	0.0599	0.0049	0.0100	0.0051
Boot-BCa CI	0.0061	0.0660	0.0599	0.0049	0.0100	0.0051
MCMC	0.0275	0.0885	0.0610	0.0063	0.0157	0.0094

Table 18: 95% CI for θ_1 and θ_2.

Method	θ_1			θ_2		
	Lower	Upper	Length	Lower	Upper	Length
MLE	0.0122	0.2678	0.2557	0.0128	0.3472	0.3344
Boot-p	0.0734	0.3779	0.3044	0.1083	0.5874	0.4791
Boot-t	0.0636	0.1632	0.0995	0.1436	0.1914	0.0477
Boot-BC CI	0.0246	0.1400	0.1154	0.0602	0.1800	0.1198
Boot-BCa CI	0.0246	0.1400	0.1154	0.0602	0.1800	0.1198
MCMC	0.1157	0.1781	0.0624	0.1374	0.2389	0.1015