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Certain Developments of Laguerre Equation and Laguerre Polynomials via Fractional Calculus

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Abstract: Recently, much interests have been paid in studying fractional calculus due to its effectiveness in modeling many of the natural phenomena. Motivated essentially by the success of the applications of the orthogonal polynomials, this paper is mainly devoted to developing Laguerre equation and Laguerre polynomials in the fractional calculus setting. We provide some type of generalizations of the classical Laguerre polynomials, via conformable fractional calculus. We start by solving the fractional Laguerre equation in the sense of conformable calculus about the fractional regular singular point. Next, we write the conformable fractional Laguerre polynomials (CFLPs), through various generating functions. Subsequently, Rodrigues' type representation formula of fractional order is reported, besides certain types of recurrence relations are then developed. The conformable fractional integral and the fractional Laplace transform, and the orthogonal property of CFLPs, are established. As an application, we present a numerical technique to obtain solutions of interesting differential equations in the frame of conformable derivative. For this purpose, a new operational matrix of the fractional derivative of arbitrary order for CFLPs is derived. This operational matrix is applied together with the generalized Laguerre tau method for solving general linear multi-term fractional differential equations (FDEs). The method has the advantage of obtaining the solution in terms of the CFLPs' parameters. Finally, some examples are given to illustrate the applicability and efficiency of the proposed method.

Keywords: Conformable fractional calculus, Fractional differential equations, Laguerre polynomials.

1 Introduction

Orthogonal polynomials such as Laguerre, Hermite, Legendre, Chebyshev, and Gegenbauer can be obtained through the well-known linear algebra method via Sturm-Liouville theory. The Sturm-Liouville theory is covered in most advanced physics and engineering courses. Over the last decades, the interest in Laguerre polynomials is considerably increased among engineers and scientists due to their vast potential of applications in several applied problems. For a detailed account of various properties, generalizations, and applications, in the classical case and in fractional context, the reader may be referred to earlier works of [1,2,3]. Further applications in various fields of mathematical physics, we mention the solving of delay differential equations [4,5], pantograph type Volterra integro-differential equations [6] and fractional differential equations [7,8,9,10,11,12,13, 14,15,16,17].

The Laguerre differential equation can be derived from the following hypergeometric differential equation which may be written as

$$p(x)\, f''(x) + q(x)\, f'(x) + \lambda f(x) = 0 \qquad (1.1)$$

where $f(x)$ is a real function of a real variable $f : I \to \mathbb{R}$, where $I \subset \mathbb{R}$ is an open interval of the real line, and $\lambda \in \mathbb{R}$ a corresponding eigenvalue, and the functions $p(x)$ and $q(x)$ are real polynomials of at most second order and first order, respectively. When $p(x)$ is a polynomial of the first degree, Eq. (1.1) takes the form

$$x f''(x) + (-ax + b + 1)\, f'(x) + \lambda f(x) = 0,$$

and when $a = 1$ and $b = 0$ one obtains the Laguerre equation.

The Laguerre differential equation and its solutions, that is, Laguerre polynomials, are found in many important physical problems, such as in the description of the transversal profile of Laguerre-Gaussian laser beams [18].

The practical importance of Laguerre polynomials was enhanced by Schrödinger's wave mechanics, where

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they occur in the radial wave functions of the hydrogen atom [19,1]. These polynomials are also used in problems involving the integration of Helmholtz's equation in parabolic coordinates, in the theory of propagation of electromagnetic waves along transmission lines, in describing the static Wigner functions of oscillator systems in quantum mechanics and in phase space [20], etc.

Nowadays, fractional calculus is regarded as a powerful and effective tool for modeling many physical phenomena. It appears in many fields of science and engineering, such as signal processing, finance and plasma physics, aerodynamics, control systems, viscoelasticity, bioengineering and biomedical [21,22, 23].

Many researchers tried to define fractional derivatives. Most of them used an integral form of fractional derivatives, such as Riemann-Liouville, Caputo, Grunwald-Letnikov, Riesz and Weyl, etc. Most of them are defined via fractional integrals, thus they inherit non-local properties from integral. Heredity and nonlocality are typical properties of these definitions [24], which are important in many application fields and are different from classical Newton-Leibniz calculus.

Although non-local fractional derivatives give natural memory and genetic effects in the physical system, the fractional derivatives obtained in this kind of calculus seem very complicated and lose some basic properties of general derivatives, such as product rule, quotient rule, and chain rule. Accordingly in 2014, the authors in [25] proposed another type of local fractional derivative which is considered as a well-defined fractional derivative named "'conformable fractional derivative"' (CFD) depending just on the fundamental definition of the limit of the usual derivative. This new proposal of CFD enjoys most properties which coincide with Newton derivative and can be used to solve fractional differential equations more easily (see[26,27,28]).

The CFD runs as follows

Definition 11*[25] Let $\mathscr{F}(x)$ be a real function such that $\mathscr{F}:[0,\infty)\to\mathbb{R}$. Then the conformable fractional derivative of order $\alpha\in(0,1]$ of $\mathscr{F}(x)$ is defined by*

$$\mathscr{D}_x^{\alpha}\mathscr{F}(x)=\lim_{\delta\to 0}\frac{\mathscr{F}\left(x+\delta x^{1-\alpha}\right)-\mathscr{F}(x)}{\delta},\quad x>0. \tag{1.2}$$

If $\mathscr{F}(x)$ is $\alpha-$differentiable in some $(0,a)$, $a>0$ and $\lim\limits_{x\to 0^+}\mathscr{D}_x^{\alpha}\mathscr{F}(x)$ exists, then define $\mathscr{D}_x^{\alpha}\mathscr{F}(0)=\lim\limits_{x\to 0^+}\mathscr{D}_x^{\alpha}\mathscr{F}(x)$.

Note that if $\mathscr{F}(x)$ is differentiable, then $\mathscr{D}_x^{\alpha}\mathscr{F}(x)=x^{1-\alpha}\mathscr{F}'(x)$, where $\mathscr{F}'(x)=\lim\limits_{\delta\to 0}\frac{\mathscr{F}(x+\delta)-\mathscr{F}(x)}{\delta}$.

Also, the conformable fractional integral was suggested in [25] in the following way

Definition 12*Let $0<\alpha\leq 1$, and $\mathscr{F}:(0,\infty)\to\mathbb{R}$ be α-differentiable, then the conformable fractional integral denoted by I_{α}^{a} is defined by*

$$I_{\alpha}^{a}\mathscr{F}(t)=I_1^{a}\left(t^{\alpha-1}\mathscr{F}\right)=\int_a^t\frac{\mathscr{F}(x)}{x^{1-\alpha}}dx,\ t\geq 0. \tag{1.3}$$

Remark 11

(1)Unlike the classical fractional calculus, the authors in [25] showed that the CFD satisfies the product rule and the chain rule.

(2)As an amazing fact, the derivative of a constant in conformable sense vanishes whereas the case for Riemann-Liouville FD is not.

(3)In the case of $\alpha=1$ in (1.2), it is easy to get the first-order derivative in the classical case. Further, note that a function can be α-differentiable at a point even though it is not differentiable, for instance, take $\mathscr{F}(x)=2\sqrt{x}$, then $\mathscr{D}^{\frac{1}{2}}\mathscr{F}(x)=1$. Thus $\mathscr{D}^{\frac{1}{2}}\mathscr{F}(0)=1$. However, $\mathscr{D}^{1}\mathscr{F}(0)$ does not exist, which is different from the classical derivatives.

(4)It is worth noting that the solution of the fractional equation $\mathscr{D}^{\frac{1}{2}}\phi+\phi=0$, by using Caputo or Riemann-Liouville definitions, is required to apply either the fractional power series technique or the Laplace transform. However, the use of conformable definition with $\mathscr{D}^{\alpha}(e^{\frac{1}{\alpha}x^{\alpha}})=e^{\frac{1}{\alpha}x^{\alpha}}$, one can easily see that $\phi=ce^{-2\sqrt{x}}$ is the general solution.

For more details about conformable derivatives, we refer to [29,30,31,32,33,34,35].

Motivated by the above-mentioned discussion, the goal of this work is to present further investigations on the above mentioned Laguerre equation and the associated Laguerre polynomials in the context of conformable fractional calculus. The distinct results obtained through the current work will be useful for investigators in various disciplines of applied sciences and engineering.

The outline of the paper is as follows. In section 2, we present some basic concepts which will be used in the sequel. Through section 3, a solution of the fractional Laguerre equation in the sense of conformable calculus about the fractional regular singular point is obtained. Various generating functions of CFLPs are established in section 4. Rodrigues' type representation formula of fractional order in sense of conformable derivative is reported in section 5. The pure recurrence relations and differential recurrence relations are the subject of section 6. The conformable fractional integral and the fractional Laplace transform of CFLPs is derived in section 7. In section 8, we introduce a detailed study on orthogonality property and an overview of approximation theory. In section 9, a new operational matrix of fractional derivative of arbitrary order for CFLPs is derived. In view of tau method, numerical solutions of some linear multi-term fractional differential equations are established. The obtained results showed the efficiency and applicability of the provided method. Finally, concluding remarks are appended in section 10.

2 Certain basic tools

In the sequel, the following formula of Gauss hypergeometric function ${}_2F_1\left(a,b;c;x\right)$, is in need.

$$ {}_2F_1\left(a_1,a_2;a_3;\zeta\right)=\sum_{n=0}^{\infty}\frac{\left(a_1\right)_n\left(a_2\right)_n}{\left(a_3\right)_n}\frac{\zeta^n}{n!}\qquad\left(\left|\zeta\right|<1\right)$$

where $\left(a\right)_n$ denotes the Pochhammer symbol defined, in terms of Gamma functions, by

$$\left(a\right)_n=\frac{\Gamma\left(a+n\right)}{\Gamma\left(a\right)}=a\left(a+1\right)\left(a+2\right)\ldots\left(a+n-1\right),$$

$n\in\mathbb{N}$ and $\left(a\right)_0=1$

In many places of the work here the rearrangement of terms in iterated series, is commonly used. To simplify many proofs of the presented results the following useful lemma is needed (see [36])

Lemma 21

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\mathscr{A}_{k,n}=\sum_{m=0}^{\infty}\sum_{j=0}^{m}\mathscr{A}_{j,m-j}=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\mathscr{A}_{k,n-k}\tag{2.1}$$

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n}\mathscr{B}_{k,n}=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\mathscr{B}_{k,n+k}\tag{2.2}$$

3 Solutions of the conformable fractional Laguerre differential equation

In our current study, we are interested to consider a generalization of fractional Laguerre differential equation, where the involving derivative is CFD. More precisely, we study the equation in the form

$$x^{\alpha}\mathscr{D}^{\alpha}\mathscr{D}^{\alpha}y+\alpha\left(1-x^{\alpha}\right)\mathscr{D}^{\alpha}y+\alpha^2ny=0\tag{3.1}$$

It is clear that $x=0$ is a $\alpha-$regular singular point, Hence, to find the solution of (3.1), we proceed as follows.

Let $y=\sum\limits_{k=0}^{\infty}a_kx^{\alpha(k+c)}$, $a_0\neq0$ be the series solution of equation (3.1) about $x=0$. Then from the basic properties of the CFD we get

$$\mathscr{D}^{\alpha}y=\sum_{k=0}^{\infty}\alpha\left(k+c\right)a_kx^{\alpha(k+c-1)}$$

and

$$\mathscr{D}^{\alpha}\mathscr{D}^{\alpha}y=\sum_{k=0}^{\infty}\alpha^2\left(k+c\right)\left(k+c-1\right)a_kx^{\alpha(k+c-2)}$$

Thus, owing to (1.2), we have

$$\begin{aligned}x^{\alpha}\sum_{k=0}^{\infty}\alpha^2\left(k+c\right)\left(k+c-1\right)a_kx^{\alpha(k+c-2)}\\+\alpha\left(1-x^{\alpha}\right)\sum_{k=0}^{\infty}\alpha\left(k+c\right)a_kx^{\alpha(k+c-1)}\\+\alpha^2n\sum_{k=0}^{\infty}a_kx^{\alpha(k+c)}=0\end{aligned}\tag{3.2}$$

Therefore,

$$\begin{aligned}\sum_{k=0}^{\infty}\left(k+c\right)\left(k+c-1\right)a_kx^{\alpha(k+c-1)}+\sum_{k=0}^{\infty}\left(k+c\right)a_kx^{\alpha(k+c-1)}\\-\sum_{k=0}^{\infty}\left(k+c\right)a_kx^{\alpha(k+c)}+\sum_{k=0}^{\infty}na_kx^{\alpha(k+c)}=0\end{aligned}$$

$$\sum_{k=0}^{\infty}\left(k+c\right)^2a_kx^{\alpha(k+c-1)}-\sum_{k=0}^{\infty}\left(k+c-n\right)a_kx^{\alpha(k+c)}=0$$

Equaling the coefficient of $x^{\alpha(c-1)}$ we have $c=0$

Again, equaling the coefficient of $x^{\alpha(k+c)}$ we have the recursion formula

$$a_{k+1}=\frac{k+c-n}{\left(k+c+1\right)^2}a_k$$

Substituting by $c=0$, we get

$$a_{k+1}=\frac{\left(-1\right)n-k}{\left(k+1\right)^2}a_k$$

at $k=0$, we have

$$a_1=\frac{\left(-1\right)n}{1^2}a_0$$

at $k=1$, we have

$$a_2=\frac{\left(-1\right)^2n\left(n-1\right)}{1^2.2^2}a_0$$

at $k=2$, we have

$$a_3=\frac{\left(-1\right)^3n\left(n-1\right)\left(n-2\right)}{1^2.2^2.3^3}a_0$$

$$\ldots$$

In general, we have

$$\begin{aligned}a_k&=\frac{\left(-1\right)^kn\left(n-1\right)\left(n-2\right)\ldots\left(n-k+1\right)}{\left(k!\right)^2}a_0\\&=\frac{\left(-1\right)^kn!}{\left(k!\right)^2\left(n-k\right)!}a_0\end{aligned}$$

The constant a_0 is usually chosen so that the polynomial solution at $x=1$ equal 1. So, the value to be given to a_0 is

$a_0 = 1$. Since (3.1) has $x = 0$, as a regular singular point, its solution is written as

$$y = \sum_{k=0}^{\infty} a_k x^{\alpha(k+c)} = \sum_{k=0}^{\infty} \frac{(-1)^k n!}{(k!)^2 (n-k)!} x^{\alpha k} \tag{3.3}$$

Keeping in mind that the factorial function is always chosen to be non-negative, thus $n-k \geq 0$ and hence $k \leq n$.
Therefore (3.3) becomes

$$y =: L_n^{\alpha}(x) = \sum_{k=0}^{n} \frac{(-1)^k n!}{(k!)^2 (n-k)!} x^{\alpha k}; \quad \alpha \in (0,1]. \tag{3.4}$$

which is the αk^{th} conformable fractional Laguerre polynomials. Clearly, the first four terms of $L_n^{\alpha}(x)$ are

$$L_0^{\alpha}(x) = 1,\ L_1^{\alpha}(x) = 1 - t^{\alpha},\ L_2^{\alpha}(x) = \frac{1}{2!}\left(t^{2\alpha} - 4t^{\alpha} + 2\right), \tag{3.5}$$
$$L_3^{\alpha}(x) = \frac{1}{3!}\left(-t^{3\alpha} + 9t^{2\alpha} - 18t^{\alpha} + 6\right)$$

Figs.1 and 2 show graphs of CFLPs (3.4) for various values of n and α.

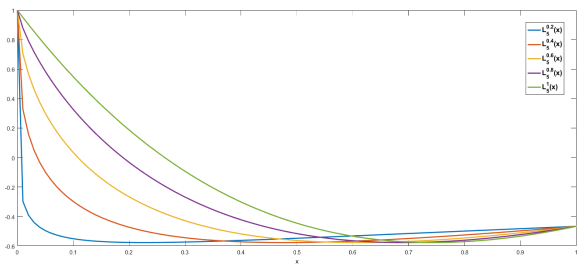

Fig. 1: Graph of the CFLPs with $n = 5$ and various values of $\alpha = 0.2, 0.4, 0.6, 0.8, 1$.

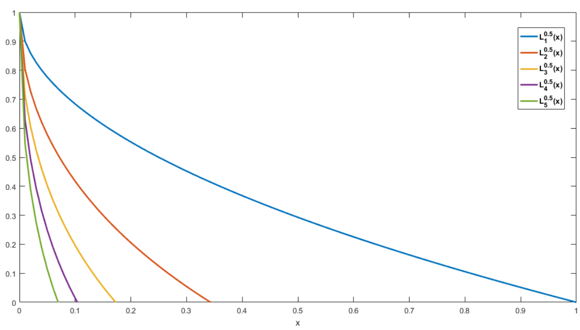

Fig. 2: Graph of CFLPs with $\alpha = 0.5$ and various values of $n = 1, 2, ..., 5$.

4 Generating functions

Generating functions are important way to transform formal power series into functions and to analyze asymptotic properties of sequences. In what follows we characterize the CFLPs by means of various generating functions.

Theorem 41 *For $\alpha \in (0,1]$, the generating relation of conformable fractional Laguerre polynomials $L_n^{\alpha}(x)$, can be written as*

$$g(x,t) = e^{t^{\alpha}} J_0^{\alpha}\left(2\sqrt{xt}\right) = \sum_{n=0}^{\infty} \frac{L_n^{\alpha}(x)\, t^{\alpha n}}{n!} \tag{4.1}$$

where $J_0^{\alpha}(x)$ is the conformable fractional Bessel function [37].

Proof. Directly from (3.4), we have

$$\sum_{n=0}^{\infty} \frac{L_n^{\alpha}(x)\, t^{\alpha n}}{n!} = \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{(-1)^k x^{\alpha k}}{(n-k)!\,(k!)^2} t^{\alpha n}$$

Using lemma 21, one obtains

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{L_n^{\alpha}(x)\, t^{\alpha n}}{n!} &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-1)^k x^{\alpha k}}{n!\,(k!)^2} t^{\alpha(n+k)} \\
&= \sum_{n=0}^{\infty} \frac{t^{\alpha n}}{n!} . \sum_{k=0}^{\infty} \frac{(-1)^k (xt)^{\alpha k}}{(k!)^2} \\
&= e^{t^{\alpha}} J_0^{\alpha}\left(2\sqrt{xt}\right)
\end{aligned}$$

as required.

Theorem 42 *Let $\alpha \in (0,1]$ and c an arbitrary real numbers, then the generating function of CFLPs $L_n^{\alpha}(x)$, can be given by the following formula:*

$$\sum_{n=0}^{\infty} \frac{(c)_n L_n^{\alpha}(x)\, t^{\alpha n}}{n!} = \frac{1}{(1-t^{\alpha})^c}\, {}_1F_1\left(c; 1; \frac{-x^{\alpha} t^{\alpha}}{1-t^{\alpha}}\right) \tag{4.2}$$

Proof. In view of (3.4), we get

$$\sum_{n=0}^{\infty} \frac{(c)_n L_n^{\alpha}(x)\, t^{\alpha n}}{n!} = \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{(-1)^k (c)_n x^{\alpha k}}{(n-k)!\,(k!)^2} t^{\alpha n}$$

Using lemma 21 and the identity $(c)_n = (c+k)_n (c)_k$, we have

$$\sum_{n=0}^{\infty} \frac{(c)_n L_n^{\alpha}(x)\, t^{\alpha n}}{n!} = \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} \frac{(c+k)_n}{n!} t^{\alpha n} . \frac{(-1)^k (c)_k x^{\alpha k} t^{\alpha k}}{(k!)^2}$$

Owing to the binomial series we obtain

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{(c)_n L_n^{\alpha}(x)\, t^{\alpha n}}{n!} &= \sum_{k=0}^{\infty} \frac{1}{(1-t^{\alpha})^{c+k}} \frac{(-1)^k (c)_k x^{\alpha k} t^{\alpha k}}{(k!)^2} \\
&= \frac{1}{(1-t^{\alpha})^c} \sum_{k=0}^{\infty} \frac{(c)_k}{(k!)^2} \left(\frac{-x^{\alpha} t^{\alpha}}{1-t^{\alpha}}\right)^k \\
&= \frac{1}{(1-t^{\alpha})^c}\, {}_1F_1\left(c; 1; \frac{-x^{\alpha} t^{\alpha}}{1-t^{\alpha}}\right)
\end{aligned}$$

as desired.

Corollary 41*For $c=1$ in the previous theorem, we get*

$$\sum_{n=0}^{\infty} L_n^{\alpha}(x)\,t^{\alpha n} = \frac{1}{(1-t^{\alpha})}\,{}_1F_1\left(1;1;\frac{-x^{\alpha}t^{\alpha}}{1-t^{\alpha}}\right) = \frac{1}{(1-t^{\alpha})}\exp\left(\frac{-x^{\alpha}t^{\alpha}}{1-t^{\alpha}}\right). \tag{4.3}$$

5 Rodrigues Formula for CFLPs

Rodrigues Formula is one of the main tools to define a sequence of orthogonal polynomials [38]. Once we have this formula, a lot of interesting properties of the polynomials can be characterized. For this reason, generalizations of these formulas paid much attention to mathematicians in the last two decades, both to include fractional order differentiation and to define new classes of special functions.
Owing to the notation of CFD we have $\mathscr{D}^{\alpha n}=\mathscr{D}^{\alpha}D^{\alpha}\mathscr{D}^{\alpha}...\mathscr{D}^{\alpha};n-$ times, and relying on the fact $\mathscr{D}^{\alpha}x^{k}=kx^{k-\alpha}$, one can provide the Rodrigues formula for CFLPs $L_n^{\alpha}(x)$ through the following result

Theorem 51*The Laguerre polynomials $L_n^{\alpha}(x)$, can be formulated in the sense of conformable derivative as:*

$$L_n^{\alpha}(x) = \frac{e^{x^{\alpha}}}{\alpha^n n!}\mathscr{D}^{\alpha n}\left[x^{\alpha n}e^{-x^{\alpha}}\right] \tag{5.1}$$

*Proof.*In virtue of the conformable derivative, we have

$$\mathscr{D}^{\alpha(n-k)}x^{\alpha n} = \frac{\alpha^{n-k}n!}{k!}x^{\alpha k}, \text{ and } \mathscr{D}^{\alpha k}e^{-x^{\alpha}} = (-1)^k\alpha^k e^{-x^{\alpha}} \tag{5.2}$$

Using (3.4) and (5.2), it follows that

$$\begin{aligned} L_n^{\alpha}(x) &= \sum_{k=0}^{n}\frac{e^{x^{\alpha}}\mathscr{D}^{\alpha(n-k)}x^{\alpha n}.\mathscr{D}^{\alpha k}e^{-x^{\alpha}}}{(n-k)!k!\alpha^n} \\ &= \frac{e^{x^{\alpha}}}{n!\alpha^n}\sum_{k=0}^{n}\frac{n!\mathscr{D}^{\alpha(n-k)}x^{\alpha n}.\mathscr{D}^{\alpha k}e^{-x^{\alpha}}}{(n-k)!k!} \\ &= \frac{e^{x^{\alpha}}}{n!\alpha^n}\sum_{k=0}^{n}\binom{n}{k}\mathscr{D}^{\alpha(n-k)}x^{\alpha n}.\mathscr{D}^{\alpha k}e^{-x^{\alpha}} \end{aligned} \tag{5.3}$$

Noting that, the CFD achieves well both the product and Leibniz rules unlike the case of the old fractional calculus. Hence, in view of Leibniz rule, (5.3) becomes

$$L_n^{\alpha}(x) = \frac{e^{x^{\alpha}}}{\alpha^n n!}\mathscr{D}^{\alpha n}\left[x^{\alpha n}e^{-x^{\alpha}}\right]$$

as required.

6 Certain types of recurrence relations

This section is devoted to establishing pure recurrence relation and then characterizes some differential formulas. Our first result is:

Theorem 61*The Laguerre polynomials $L_n^{\alpha}(x)$ in the conformable sense, satisfy the following pure recurrence formula*

$$(n+1)L_{n+1}^{\alpha}(x) = (2n+1-x^{\alpha})L_n^{\alpha}(x) - nL_{n-1}^{\alpha}(x), \tag{6.1}$$

$\alpha\in(0,1]$

*Proof.*Acting by the conformable fractional operator $\mathscr{D}^{\alpha}$ on the generating function (4.3) with respect to t, we get

$$\sum_{n=0}^{\infty} L_n^{\alpha}(x)\,n\alpha t^{\alpha(n-1)} = \frac{\alpha}{(1-t^{\alpha})^2}e^{\frac{-x^{\alpha}t^{\alpha}}{1-t^{\alpha}}} - \frac{\alpha x^{\alpha}}{(1-t^{\alpha})^2}\frac{1}{1-t^{\alpha}}e^{\frac{-x^{\alpha}t^{\alpha}}{1-t^{\alpha}}} \tag{6.2}$$

Using (4.3), we have

$$\sum_{n=0}^{\infty} L_n^{\alpha}(x)\,nt^{\alpha(n-1)} = \frac{1}{1-t^{\alpha}}\sum_{n=0}^{\infty}L_n^{\alpha}(x)\,t^{\alpha n} - \frac{x^{\alpha}}{(1-t^{\alpha})^2}\sum_{n=0}^{\infty}L_n^{\alpha}(x)\,t^{\alpha n} \tag{6.3}$$

Multiplying throughout by $(1-t^{\alpha})^2$, we have

$$(1-t^{\alpha})^2\sum_{n=0}^{\infty}nL_n^{\alpha}(x)\,t^{\alpha(n-1)} = (1-t^{\alpha})\sum_{n=0}^{\infty}L_n^{\alpha}(x)\,t^{\alpha n}$$

$$-x^{\alpha}\sum_{n=0}^{\infty}L_n^{\alpha}(x)\,t^{\alpha n}$$

Therefore,

$$\sum_{n=0}^{\infty}nL_n^{\alpha}(x)\,t^{\alpha(n-1)} = \sum_{n=0}^{\infty}(2n+1-x^{\alpha})L_n^{\alpha}(x)\,t^{\alpha n}$$

$$-\sum_{n=0}^{\infty}(n+1)L_n^{\alpha}(x)\,t^{\alpha(n+1)}$$

Relabeling the summation so that the general power appear as $t^{\alpha n}$ in each gives

$$\sum_{n=-1}^{\infty}(n+1)L_{n+1}^{\alpha}(x)\,t^{\alpha n} = \sum_{n=0}^{\infty}(2n+1-x^{\alpha})L_n^{\alpha}(x)\,t^{\alpha n} - \sum_{n=0}^{\infty}nL_{n-1}^{\alpha}(x)\,t^{\alpha n} \tag{6.4}$$

Equating the coefficient of $t^{\alpha n}$ on both sides of (6.4), we obtain

$$(n+1)L_{n+1}^{\alpha}(x) = (2n+1-x^{\alpha})L_n^{\alpha}(x) - nL_{n-1}^{\alpha}(x)$$

and the proof is therefore established.

For fractional differential formulas, we give the next result.

Theorem 62*For $\alpha\in(0,1]$, the $\alpha-$Laguerre polynomials $L_n^{\alpha}(x)$, are characterized through the following differential formulas*

$$\mathscr{D}^{\alpha}L_n^{\alpha}(x) = \mathscr{D}^{\alpha}L_{n-1}^{\alpha}(x) - \alpha L_{n-1}^{\alpha}(x) \tag{6.5}$$

$$\mathscr{D}^{\alpha}L_n^{\alpha}(x) = -\alpha\sum_{s=0}^{n-1}L_s^{\alpha}(x) \tag{6.6}$$

$$x^{\alpha}\mathscr{D}^{\alpha}L_n^{\alpha}(x) = n\alpha\left[L_n^{\alpha}(x) - L_{n-1}^{\alpha}(x)\right] \tag{6.7}$$

*Proof.*According to the conformable fractional differential operator and in view of (4.3), it follows that

$$\mathscr{D}^{\alpha}\left\{\frac{1}{1-t^{\alpha}}e^{\frac{-x^{\alpha}t^{\alpha}}{1-t^{\alpha}}}\right\}=\mathscr{D}^{\alpha}\left[\sum_{n=0}^{\infty}L_{n}^{\alpha}(x)t^{\alpha n}\right]$$

$$\frac{-\alpha t^{\alpha}}{1-t^{\alpha}}\cdot\frac{1}{1-t^{\alpha}}e^{\frac{-x^{\alpha}t^{\alpha}}{1-t^{\alpha}}}=\sum_{n=0}^{\infty}\mathscr{D}^{\alpha}\left[L_{n}^{\alpha}(x)\right]t^{\alpha n}$$

Also, using (4.3), it is easily verify

$$\frac{-\alpha t^{\alpha}}{1-t^{\alpha}}\sum_{n=0}^{\infty}L_{n}^{\alpha}(x)t^{\alpha n}=\sum_{n=0}^{\infty}\mathscr{D}^{\alpha}\left[L_{n}^{\alpha}(x)\right]t^{\alpha n}$$

Therefore,

$$-\alpha\sum_{n=0}^{\infty}L_{n}^{\alpha}(x)t^{\alpha(n+1)}=\sum_{n=0}^{\infty}\mathscr{D}^{\alpha}\left[L_{n}^{\alpha}(x)\right]t^{\alpha n}-\qquad(6.8)$$

$$\sum_{n=0}^{\infty}\mathscr{D}^{\alpha}\left[L_{n}^{\alpha}(x)\right]t^{\alpha(n+1)}$$

Equating the coefficient of $t^{\alpha n}$ in both sides of (6.8), we get

$$-\alpha L_{n-1}^{\alpha}(x)=\mathscr{D}^{\alpha}\left[L_{n}^{\alpha}(x)\right]-\mathscr{D}^{\alpha}\left[L_{n-1}^{\alpha}(x)\right]$$

Thus, the result (6.5) is therefore performed.
Applying relation (6.5) recursively, we obtain

$$\mathscr{D}^{\alpha}L_{n}^{\alpha}(x)=-\alpha\left[L_{n-2}^{\alpha}(x)+L_{n-1}^{\alpha}(x)\right]$$

Repeating the relation (6.5), $n-$times and noting that $D^{\alpha}L_{0}^{\alpha}(x)=0$, we obtain (6.6).
To deduce relation (6.7), applying the conformable derivative by (6.1), we get

$$(n+1)\mathscr{D}^{\alpha}L_{n+1}^{\alpha}(x)+(x^{\alpha}-1-2n)\mathscr{D}^{\alpha}L_{n}^{\alpha}(x)\qquad(6.9)$$

$$+L_{n}^{\alpha}(x)+n\mathscr{D}^{\alpha}L_{n-1}^{\alpha}(x)=0$$

and by writing (6.5), in the equivalent useful forms:

$$\mathscr{D}^{\alpha}L_{n+1}^{\alpha}(x)=D^{\alpha}L_{n}^{\alpha}(x)-\alpha L_{n}^{\alpha}(x)\qquad(6.10)$$

$$\mathscr{D}^{\alpha}L_{n-1}^{\alpha}(x)=\mathscr{D}^{\alpha}L_{n}^{\alpha}(x)+\alpha L_{n-1}^{\alpha}(x)\qquad(6.11)$$

substituting from (6.10) and (6.11) in (6.9), we obtain (6.7).

7 Conformable fractional integral and fractional Laplace transform of CFLPs

7.1 Conformable fractional integral of CFLPs

Taking into account the α-integral given in Definition 12, we provide the integral of CFLPs.
Thus according to Definition 12, it follows that

$$I_{\alpha}f(x)=\int_{0}^{x}t^{\alpha-1}f(t)\,dt.\qquad(7.1)$$

In this regard, we state the following important result given in [25].

Lemma 71*Let $f:[0,\infty)\to\mathbb{R}$ be α-differentiable for $\alpha\in(0,1]$, then for all $x>0$ one can write:*

$$I_{\alpha}\mathscr{D}^{\alpha}(f(x))=f(x)-f(0)\qquad(7.2)$$

With the use of (7.1) and (7.2), we give the following result

Theorem 71*For $\gamma\in(0,1]$, then the conformable fractional integral of order γ, of CFLPs, $L_{n}^{\alpha}(x)$ can be written as*

$$I_{\gamma}L_{n}^{\alpha}(x)=\sum_{k=0}^{n}\frac{(-1)^{k}n!}{(n-k)!(k!)^{2}(\alpha k+\gamma)}x^{\alpha k+\gamma}\qquad(7.3)$$

*Proof.*In view of (7.1) and (3.4), we obtain

$$\begin{aligned}I_{\gamma}L_{n}^{\alpha}(x)&=\int_{0}^{x}t^{\gamma-1}\sum_{k=0}^{n}\frac{(-1)^{k}n!}{(n-k)!(k!)^{2}}t^{\alpha k}dt\\&=\sum_{k=0}^{n}\frac{(-1)^{k}n!}{(n-k)!(k!)^{2}}\int_{0}^{x}t^{\alpha k+\gamma-1}dt\\&=\sum_{k=0}^{n}\frac{(-1)^{k}n!}{(n-k)!(k!)^{2}(\alpha k+\gamma)}x^{\alpha k+\gamma}\end{aligned}$$

and the result follows.

Remark 71*If $\alpha=\gamma$ in (7.3), we have*

$$I_{\alpha}L_{n}^{\alpha}(x)=\frac{1}{\alpha}\sum_{k=0}^{n}\frac{(-1)^{k}n!}{(n-k)!(k!)^{2}(k+1)}x^{\alpha(k+1)}\qquad(7.4)$$

Now, the following result allows us to express the conformable fractional integral of Laguerre polynomials in terms of CFLPs.

Theorem 72*Let $\alpha\in(0,1]$, the conformable fractional integral I_{α} of CFLPs, satisfies the relation*

$$I_{\alpha}=\frac{1}{\alpha}\left[L_{n}^{\alpha}(x)-L_{n+1}^{\alpha}(x)\right]\qquad(7.5)$$

*Proof.*Relation (6.5) gives

$$\alpha L_{n}^{\alpha}(x)=\mathscr{D}^{\alpha}L_{n}^{\alpha}(x)-\mathscr{D}^{\alpha}L_{n+1}^{\alpha}(x)$$

Acting by the conformable fractional integral on both sides, we obtain

$$\alpha I_{\alpha}L_{n}^{\alpha}(x)=I_{\alpha}\mathscr{D}^{\alpha}L_{n}^{\alpha}(x)-I_{\alpha}\mathscr{D}^{\alpha}L_{n+1}^{\alpha}(x)$$

Using (7.2) and noting that $L_{n}^{\alpha}(0)=1$, we have

$$\alpha I_{\alpha}L_{n}^{\alpha}(x)=L_{n}^{\alpha}(x)-L_{n+1}^{\alpha}(x)$$

just as required in theorem 72.

7.2 Fractional Laplace transform of the CFLPs, $L_n^\alpha(x)$

In [26], Abdeljawad defined the fractional Laplace transform in the conformable sense as follows:

Definition 71[26] *Let $\alpha \in (0,1]$ and $f:[0,\infty) \to \mathbb{R}$ be real valued function. Then the fractional Laplace transform of order α is defined by*

$$\mathscr{L}_\alpha[f(t)] = F_\alpha(s) = \int_0^\infty e^{-s\left(\frac{t^\alpha}{\alpha}\right)} f(t)\, d_\alpha t \tag{7.6}$$

$$= \int_0^\infty e^{-s\left(\frac{t^\alpha}{\alpha}\right)} f(t)\, t^{\alpha-1} dt.$$

Remark 72*If $\alpha = 1$, then (7.6) is the classical definition of the Laplace transform of integer order.*

Also, the author in [26] gave the following interesting results.

Lemma 72[26] *Let $\alpha \in (0,1]$ and $f:[0,\infty) \to \mathbb{R}$ be real valued function such that*
$\mathscr{L}_\alpha[f(t)] = F_\alpha(s)$ exist. Then $F_\alpha(s) = \mathscr{L}\left[f(\alpha t)^{\frac{1}{\alpha}}\right]$, where $\mathscr{L}[f(t)] = \int_0^\infty e^{-st} f(t)\, dt$.

Lemma 73[26] *The following the conformable fractional Laplace transform of certain functions:*

(1)$\mathscr{L}_\alpha[1] = \frac{1}{s};\ \ s>0$

(2)$\mathscr{L}_\alpha[t^p] = \alpha^{\frac{p}{\alpha}} \frac{\Gamma\left(1+\frac{p}{\alpha}\right)}{s^{1+\frac{p}{\alpha}}};\ \ s>0$

(3)$\mathscr{L}_\alpha\left[e^{k\frac{t^\alpha}{\alpha}}\right] = \frac{1}{s-k}$

Owing to the definition of CFLPs and applying the conformable fractional Laplace transform operator of an arbitrary order $\gamma \in (0,1]$, we have

$$\mathscr{L}_\gamma[L_n^\alpha(x)] = \mathscr{L}_\gamma\left[\sum_{k=0}^{n} \frac{(-1)^k n!}{(n-k)!\,(k!)^2} x^{\alpha k}\right] \tag{7.7}$$

$$= \sum_{k=0}^{n} \frac{(-1)^k n!}{(n-k)!\,(k!)^2} \mathscr{L}_\gamma\left\{x^{\alpha k}\right\}$$

Using (2) of lemma 73, we obtain

$$\mathscr{L}_\gamma[L_n^\alpha(x)] = \sum_{k=0}^{n} \frac{(-1)^k n!}{(n-k)!\,(k!)^2} \gamma^{\frac{k\alpha}{\gamma}} \frac{\Gamma\left(1+\frac{k\alpha}{\gamma}\right)}{s^{1+\frac{k\alpha}{\gamma}}} \tag{7.8}$$

Remark 73*If $\gamma = \alpha$ in (7.8) we have*

$$\mathscr{L}_\alpha[L_n^\alpha(x)] = \sum_{k=0}^{n} \frac{(-1)^k n!}{(n-k)!\,(k!)^2} \frac{\alpha^k\, \Gamma(1+k)}{s^{1+k}}$$

$$= \frac{1}{s}\left(1-\frac{\alpha}{s}\right)^n$$

8 Orthogonality

Along the same lines in scalar case, one can use the conformable derivative to introduce the orthogonality relation as follows

Theorem 81*The conformable fractional Laguerre polynomials $L_n^\alpha(x)$, are orthogonal with respect to the weight function $w(x) = x^{\alpha-1}\ e^{-x^\alpha}$, over the interval $[0,\infty)$ as follows*

$$\int_0^\infty x^{\alpha-1}\ e^{-x^\alpha} L_n^\alpha(x) L_m^\alpha(x)\, dx = \frac{1}{\alpha}\delta_{nm},\quad \alpha \in (0,1]. \tag{8.1}$$

where δ_{nm} is the familiar Kronker delta.

*Proof.*Since $L_n^\alpha(x)$ is a solution of the conformable fractional Laguerre equation (3.1), then it satisfies the equation

$$x^\alpha e^{-x^\alpha} \mathscr{D}^\alpha \mathscr{D}^\alpha L_n^\alpha(x) + \alpha e^{-x^\alpha}(1-x^\alpha)\mathscr{D}^\alpha L_n^\alpha(x) \tag{8.2}$$

$+\alpha^2 n e^{-x^\alpha} L_n^\alpha(x) = 0.$
For our propose we rewrite (8.2) in the more useful form:

$$\mathscr{D}^\alpha\left[x^\alpha e^{-x^\alpha} \mathscr{D}^\alpha L_n^\alpha(x)\right] + \alpha^2 n e^{-x^\alpha} L_n^\alpha(x) = 0, \tag{8.3}$$

as is easily verified. Eq. (8.3) together with

$$\mathscr{D}^\alpha\left[x^\alpha e^{-x^\alpha} \mathscr{D}^\alpha L_m^\alpha(x)\right] + \alpha^2 m e^{-x^\alpha} L_m^\alpha(x) = 0, \tag{8.4}$$

Multiplying (8.3) by $L_m^\alpha(x)$ and (8.4) by $L_n^\alpha(x)$ and subtracting the resulting equations, we have

$$\alpha^2(n-m)e^{-x^\alpha} L_n^\alpha(x) L_m^\alpha(x) = L_n^\alpha(x)\mathscr{D}^\alpha\left[x^\alpha e^{-x^\alpha}\mathscr{D}^\alpha L_m^\alpha(x)\right]$$
$$-L_m^\alpha(x)\mathscr{D}^\alpha\left[x^\alpha e^{-x^\alpha}\mathscr{D}^\alpha L_n^\alpha(x)\right]$$

Applying the conformable fractional integral (1.3) over the interval $[0,\infty)$, we obtain

$$\alpha^2(n-m)\int_0^\infty e^{-x^\alpha} L_n^\alpha(x) L_m^\alpha(x)\, d_\alpha x$$

$$= \int_0^\infty L_n^\alpha(x)\mathscr{D}^\alpha\left[x^\alpha e^{-x^\alpha}\mathscr{D}^\alpha L_m^\alpha(x)\right] d_\alpha x$$

$$- \int_0^\infty L_m^\alpha(x)\mathscr{D}^\alpha\left[x^\alpha e^{-x^\alpha}\mathscr{D}^\alpha L_n^\alpha(x)\right] d_\alpha x$$

Performing integration by parts [26] on the right-hand side, we get

$$\alpha^2(n-m)\int_0^\infty e^{-x^\alpha} L_n^\alpha(x) L_m^\alpha(x)\, d_\alpha x = 0$$

For $n \neq m$, we have

$$\int_0^\infty e^{-x^\alpha} L_n^\alpha(x) L_m^\alpha(x)\, d_\alpha x = \int_0^\infty x^{\alpha-1}\, e^{-x^\alpha} L_n^\alpha(x) L_m^\alpha(x)\, dx = 0. \tag{8.5}$$

Now, for $n = m$. In the view of the generating function (4.3), we can easily write

$$\sum_{n=0}^\infty e^{-x^\alpha} \left[L_n^\alpha(x)\right]^2 t^{2\alpha n} = e^{-x^\alpha} . \frac{1}{(1-t^\alpha)^2} \exp\left(\frac{-2x^\alpha t^\alpha}{1-t^\alpha}\right)$$

Applying the conformable fractional integral (1.3) over the interval $[0,\infty)$, we get

$$\begin{aligned}
&\sum_{n=0}^\infty \int_0^\infty e^{-x^\alpha} \left[L_n^\alpha(x)\right]^2 t^{2\alpha n} d_\alpha x \\
&= \frac{1}{(1-t^\alpha)^2} \int_0^\infty \exp\left[-\left(\frac{1+t^\alpha}{1-t^\alpha}\right) x^\alpha\right] d_\alpha x \\
&= \left[\frac{1}{(1-t^\alpha)^2} \frac{1}{-\alpha\left(\frac{1+t^\alpha}{1-t^\alpha}\right)} \exp\left[-\left(\frac{1+t^\alpha}{1-t^\alpha}\right) x^\alpha\right]\right]_0^\infty \\
&= \frac{1}{\alpha\left(1-t^{2\alpha}\right)} = \frac{1}{\alpha} \sum_{n=0}^\infty t^{2\alpha n}
\end{aligned}$$

Equating the coefficient of $t^{2\alpha n}$, we have

$$\int_0^\infty e^{-x^\alpha} \left[L_n^\alpha(x)\right]^2 t^{2\alpha n} d_\alpha x = \int_0^\infty x^{\alpha-1}\, e^{-x^\alpha} \left[L_n^\alpha(x)\right]^2 dx = \frac{1}{\alpha}. \tag{8.6}$$

Hence, the result is established.

8.1 Overview of approximation theory

Analogously to the scalar case and due to the orthogonality property we say that a set of conformable fractional polynomials forms a simple set of fractional polynomials [1]. Consequently, the following results can be determined.

Proposition 81*Suppose that $\{\psi_{\alpha n}(x)\}$ is a simple set of the conformable fractional polynomials. If $P(x)$ is a polynomial of degree αn, then for certain constants we have the expansion*

$$P(x) = \sum_{k=0}^n a_k \psi_{\alpha k}(x), \tag{8.7}$$

where a_k are functions of k and of any parameters involved in $P(x)$.

[1] The definition of the simple set of polynomials can be found in [36].

8.1.1 Approximation in terms of $L_n^\alpha(x)$

It is useful to seek an expansion in terms of CFLPs, $L_n^\alpha(x)$ of the form

$$f(x) = \sum_{n=0}^\infty a_n L_n^\alpha(x),\ x \in (0,\infty). \tag{8.8}$$

Noting that convergence of (8.8) is actually guaranteed provided that $f(x)$ is sufficiently well behaved. The formula (8.8) can be determined in terms of $x^{\alpha n}$, $\alpha \in (0,1]$ which can be characterized by means of $L_n^\alpha(x)$. Proposition 81 concludes that any polynomial can be expanded in a series of CFLPs merely due to the fact that $L_n^\alpha(x)$ forms a simple set. In fact the orthogonality property of $\{L_n^\alpha(x)\}$ has an important role in determined the coefficients. So that it is necessary to have the expansion of $x^{\alpha n}$ in a series of CFLPs. For such application in function theory we refer to the work [39, 40, 41, 42, 43, 44].

8.1.2 The expansion of $x^{\alpha n}, \alpha \in (0,1]$

Theorem 82*The expansion of $x^{\alpha n}, \alpha \in (0,1], n$ is non-negative integer, can be characterized in terms of $L_n^\alpha(x)$ as follows*

$$x^{\alpha n} = \sum_{k=0}^n \frac{(-1)^k (n!)^2}{(n-k)!k!} L_k^\alpha(x) \tag{8.9}$$

*Proof.*In view of (4.1), we obtain

$$J_0^\alpha\left(2\sqrt{xt}\right) = e^{-t^\alpha} \sum_{k=0}^\infty \frac{L_k^\alpha(x)\ t^{\alpha k}}{k!}$$

Using the definition of conformable fractional Bessel function and lemma 21, we get

$$\begin{aligned}
\sum_{n=0}^\infty \frac{(-1)^n x^{\alpha n} t^{\alpha n}}{(n!)^2} &= \sum_{n=0}^\infty \sum_{k=0}^\infty \frac{(-1)^n L_k^\alpha(x)\ t^{\alpha(n+k)}}{n!k!} \\
&= \sum_{n=0}^\infty \sum_{k=0}^n \frac{(-1)^{n-k} L_k^\alpha(x)\ t^{\alpha n}}{(n-k)!k!}
\end{aligned}$$

Equating the coefficient of $t^{\alpha n}$, we obtain

$$\frac{(-1)^n x^{\alpha n}}{(n!)^2} = \sum_{k=0}^n \frac{(-1)^{n-k} L_k^\alpha(x)}{(n-k)!k!}$$

Hence, the result (8.9), is established.

Example 81*Using theorem 82, we can write for example*

$$\begin{aligned}
x^{\frac{1}{2}} &= L_0^{\frac{1}{2}}(x) - L_1^{\frac{1}{2}}(x), \\
\textit{and}\ \ x^{\frac{3}{2}} &= 6L_0^{\frac{1}{2}}(x) - 18L_1^{\frac{1}{2}}(x) + 18L_2^{\frac{1}{2}}(x) - 6L_3^{\frac{1}{2}}(x).
\end{aligned}$$

8.1.3 The expansion of analytic functions

Classically, the expansion theory was treated through several approaches, see for example [45,46]. For approximation theory of analytic functions by means of a sequence of polynomials, we refer to the classical works of Whittaker [40], Boas [41] and recently in higher dimensions [42,43,44]. The previous theorem 82 is then useful to determinate an explicit representation of analytic function by means of CFLPs series. Beforehand let us recall the following amazing fact.

In classical analysis, Taylor's series expansion of a function f near certain points does not always exist, unlike the situation in the conformable fractional analysis. This fact was illustrated by Abdeljawad [26] who showed the existence of the conformable fractional power series representation for an infinity $\alpha-$differentiable function, for $\alpha \in (0,1]$. In fact, the following interesting result was proved in [26].

Theorem 83_Given x_0 be fixed and assume that f is an infinity conformable fractional differentiable function for $\alpha \in (0,1]$. Then there exist a Taylor conformable fractional representation in the form._

$$f(x) = \sum_{k=0}^{\infty} \frac{(\mathscr{D}^{\alpha} f)^{(k)}(x_0)}{\alpha^k k!} (x - x_0)^{\alpha k}, \tag{8.10}$$

$x_0 < x < x_0 + R^{1/\alpha}, \quad R > 0$

where $(\mathscr{D}^{\alpha} f)^{(k)}(x_0)$ means acting by the CFD repeatedly k times at x_0.

Consequently when $x_0 = 0$, we immediately get the α-Maclaurin expansion.

$$f(x) = \sum_{k=0}^{\infty} \frac{(\mathscr{D}^{\alpha} f)^{(k)}(0)}{\alpha^k k!} x^{\alpha k}, \quad 0 < x < R^{1/\alpha}, \quad R > 0, \tag{8.11}$$

where also $(\mathscr{D}^{\alpha} f)^{(k)}(0)$ means as we pointed above at $x = 0$.
With the aid of the formula (8.11) and applying theorem (82) attain the following.

$$f(x) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{a_n}{n!} \frac{(-1)^k (n!)^2}{(n-k)!k!} L_k^{\alpha}(x).$$

Hence,

$$f(x) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^k (n+k)! a_{n+k}}{n!k!} L_k^{\alpha}(x). \tag{8.12}$$

Using the orthogonality relation and theorem 82. it can be easy to establish the following result

Theorem 84_For $\alpha \in (0,1]$, the conformable fractional Laguerre polynomials $L_n^{\alpha}(x)$, satisfy the following integral relations_

$$\int_0^{\infty} e^{-x^{\alpha}} x^{\alpha k} L_n^{\alpha}(x)\, d_{\alpha}x = 0, \quad k = 0,1,2,\ldots,(n-1), \tag{8.13}$$

$$\int_0^{\infty} e^{-x^{\alpha}} x^{\alpha n} L_n^{\alpha}(x)\, d_{\alpha}x = \frac{(-1)^n n!}{\alpha}. \tag{8.14}$$

where $d_{\alpha}x = x^{\alpha-1}dx$.

9 Applications

9.1 Fundamental Results

Let $u(x)$ is a square integrable function over $(0,\infty)$ i.e $\int_0^{\infty} u^2(x)dx < \infty$, then $u(x)$ may be expressed in terms of CFLPs $L_n^{\alpha}(x)$, as

$$u(x) = \sum_{j=0}^{\infty} a_j L_j^{\alpha}(x), \tag{9.1}$$

where

$$a_j = \alpha \int_0^{\infty} x^{\alpha-1} e^{-x^{\alpha}} u(x) L_j^{\alpha}(x)\, dx, \quad j = 0,1,2,\ldots \tag{9.2}$$

In practice, only the first $(M+1)-$terms of CFLPs, are considered. Then, we have

$$u_M(x) = \sum_{j=0}^{M} a_j L_j^{\alpha}(x) = A^T \Psi(x) \tag{9.3}$$

where the conformable fractional Laguerre coefficient vector A and the conformable fractional Laguerre vector $\Psi(x)$ are given by

$$A^T = [a_0, a_1, a_2, \ldots, a_M] \text{ and } \Psi(x) = [L_0^{\alpha}(x), L_1^{\alpha}(x), L_2^{\alpha}(x), \ldots, L_M^{\alpha}(x)]^T \tag{9.4}$$

Now, the main objective of this subsection is to drive a new operational matrix of conformable fractional derivative for the conformable fractional Laguerre vector as:

Theorem 91_Let $\alpha \in (0,1]$, $\Psi(x)$ be the conformable fractional Laguerre vector defined in (9.4), and $\lambda > 0$, then_

$$\mathscr{D}^{\lambda} \Psi(x) \simeq \Upsilon^{\lambda} \Psi(x), \tag{9.5}$$

where Υ^{λ} is $(M+1) \times (M+1)$ operational matrix of fractional derivative of order λ in the conformable sense and is defined as follows:

$$\Upsilon^{\lambda} = \begin{pmatrix} \eta(0,0) & \eta(0,0) & \ldots & \eta(0,0) \\ \eta(1,0) & \eta(1,1) & \ldots & \eta(1,M) \\ \eta(2,0) & \eta(2,1) & \ldots & \eta(2,M) \\ \ldots & \ldots & \ldots & \ldots \\ \eta(M,0) & \eta(M,1) & \ldots & \eta(M,M) \end{pmatrix} \tag{9.6}$$

where $\eta(n,j)=0$ when $\alpha k\in\mathbb{N}_0$ and $\alpha k<\lambda$, in the other wise

$$\eta(n,j)=\sum_{k=0}^{n}\sum_{r=0}^{j}\frac{(-1)^{k+r}n!j!}{(n-k)!(k!)^2(j-r)!(r!)^2}\cdot\frac{\Gamma(\alpha k+1)\Gamma\left(k+r-\frac{\lambda}{\alpha}+1\right)}{\Gamma(\alpha k-\lceil\lambda\rceil+1)} \tag{9.7}$$

where $\lceil\lambda\rceil$ is the usual ceil function i.e (the least integer greater than or equal to λ).

*Proof.*Using the analytic form of the CFLPs $L_n^\alpha(x)$ of degree αn (3.4) and the conformable fractional operator (1.2), we have

$$\begin{aligned}\mathscr{D}^\lambda L_n^\alpha(x)&=\sum_{k=0}^{n}\frac{(-1)^k n!}{(n-k)!(k!)^2}\mathscr{D}^\lambda x^{\alpha k}\\&=\sum_{k=0}^{n}b_{k,n}\cdot\frac{\Gamma(\alpha k+1)}{\Gamma(\alpha k-\lceil\lambda\rceil+1)}x^{\alpha k-\lambda},\end{aligned} \tag{9.8}$$

where, $b_{k,n}=0$ when $\alpha k\in\mathbb{N}_0$ and $\alpha k<\lambda$, in the other case $b_{k,n}=\frac{(-1)^k n!}{(n-k)!(k!)^2}$.
Now, approximate $x^{\alpha k-\lambda}$ by $(M+1)-$terms of CFLPs, we have

$$x^{\alpha k-\lambda}\simeq\sum_{j=0}^{M}a_j\,L_j^\alpha(x) \tag{9.9}$$

where a_j is given from (9.2) with $u(x)=x^{\alpha k-\lambda}$ that is

$$a_j=\sum_{r=0}^{j}\frac{(-1)^r j!}{(j-r)!(r!)^2}.\Gamma\left(k+r-\frac{\lambda}{\alpha}+1\right),\quad j=0,1,2,...,M. \tag{9.10}$$

In virtue of (9.8) and (9.9), we get

$$D^\lambda L_n^\alpha(x)=\sum_{j=0}^{M}\eta(n,j)L_j^\alpha(x),\quad n=0,1,2,...,M. \tag{9.11}$$

where $\eta(n,j)=0$ when $\alpha k\in\mathbb{N}_0$ and $\alpha k<\lambda$, in the other wise

$$\eta(n,j)=\sum_{k=0}^{n}\sum_{r=0}^{j}\frac{(-1)^{k+r}n!j!}{(n-k)!(k!)^2(j-r)!(r!)^2}\cdot\frac{\Gamma(\alpha k+1)\Gamma\left(k+r-\frac{\lambda}{\alpha}+1\right)}{\Gamma(\alpha k-\lceil\lambda\rceil+1)}$$

Accordingly, (9.11) can be written in a vector form as follows:

$$D^\lambda L_n^\alpha(x)=[\eta(n,0),\eta(n,1),\eta(n,1),...,\eta(n,M)]\,\Psi(x),\quad n=0,1,2,...,M. \tag{9.12}$$

9.2 Proposed scheme

Now, we consider the generalized linear multi-order conformable fractional differential equation in the form:

$$\mathscr{D}^\lambda u(x)+\sum_{s=1}^{m}c_s\mathscr{D}^{\lambda_s}u(x)+c_{m+1}u(x)=c_{m+2}g(x) \tag{9.13}$$

with the initial conditions

$$u^{(i)}(0)=d_i,\quad i=0,1,2,...,\lceil\gamma\rceil-1, \tag{9.14}$$

where $\mathscr{D}^\lambda$, $0<\lambda_1<\lambda_2,...,\lambda_m<\lambda$ referes to the conformable fractional derivative of order λ and d_i, $i=0,1,2,...,\lceil\gamma\rceil-1$ are given constant.
To solve the problem in equations (9.13) and (9.14), it is required to approximate $u(x),\mathscr{D}^\lambda u(x)$, $\mathscr{D}^{\lambda_s}u(x)$ and $g(x)$ by the conformable fractional Laguerre polynomials as follows:

$$u(x)\simeq\sum_{j=0}^{M}a_j\,L_j^\alpha(x)=A^T\,\Psi(x) \tag{9.15}$$

$$\mathscr{D}^\lambda u(x)\simeq\sum_{j=0}^{M}a_j\,\mathscr{D}^\lambda L_j^\alpha(x)=A^T\,\mathscr{D}^\lambda\Psi(x)=A^T\,\Upsilon^\lambda\Psi(x) \tag{9.16}$$

$$\mathscr{D}^{\lambda_s}u(x)\simeq\sum_{j=0}^{M}a_j\,\mathscr{D}^{\lambda_s}L_j^\alpha(x)=A^T\,\mathscr{D}^{\lambda_s}\Psi(x)=A^T\,\Upsilon^{\lambda_s}\Psi(x) \tag{9.17}$$

and

$$g(x)\simeq\sum_{j=0}^{M}g_jL_j^\alpha(x)=G^T\Psi(x), \tag{9.18}$$

where the vector $G=[g_0,g_1,...,g_M]^T$ is known but the vector $A=[a_0,a_1,...,a_M]^T$ is unknown
Inserting equations (9.15)-(9.18) into (9.13), we can find the residual for equation (9.13) as

$$R_M(x)\simeq[A^T\,\Upsilon^\lambda\Psi(x)+A^T\sum_{s=1}^{m}c_s\Upsilon^{\lambda_s}\Psi(x)+c_{k+1}A^T\Psi(x)-c_{k+2}G^T\Psi(x)]$$

i.e

$$R_M(x)\simeq\left[A^T\,\Upsilon^\lambda+A^T\sum_{s=1}^{m}c_s\Upsilon^{\lambda_s}+c_{k+1}A^T-c_{k+2}G^T\right]\Psi(x) \tag{9.19}$$

Owing to tau method (see [47,48]) we generate $(M-\lceil\gamma\rceil)$ linear equations by helpful of orthogonal property by using

$$\langle R_M(x),L_j^\alpha(x)\rangle_{w(x)}=\int_0^\infty x^{\alpha-1}e^{-x^\alpha}R_M(x).L_j^\alpha(x)\,dx=0. \tag{9.20}$$

From (9.14) we can obtain:

$$\begin{aligned} &u(0) = A^T \Psi(0) = d_0, \\ &u^{(1)}(0) = A^T \Psi^{(1)}(0) = d_1, \\ &u^{(2)}(0) = A^T \Psi^{(2)}(0) = d_2, \\ &. \\ &. \\ &. \\ &u^{(\lceil \gamma \rceil)}(0) = A^T\, \Psi^{(\lceil \gamma \rceil)}(0) = d_{\lceil \gamma \rceil}, \end{aligned} \tag{9.21}$$

The equations (9.20) and (9.21) generate $(M - \lceil \gamma \rceil)$ and $(\lceil \gamma \rceil + 1)$ set of linear equations respectively.
Solving these linear equations for unknown coefficient of the vector A, then $u(x)$ can be established.

9.3 Illustrative examples

Example 91*As the first example, we consider the inhomogeneous Bagley-Torvik fractional differential equation*

$$\mathscr{D}^2 u(x) + \mathscr{D}^{3/2} u(x) + u(x) = 1 + x, \tag{9.22}$$

subject to the initial conditions:

$$u(0) = 1, \;\; u'(0) = 1. \tag{9.23}$$

The exact solution of this problem is $u(x) = 1 + x$. (see [49,50,51]).
Applying the proposed scheme with $M = 2$, and $\alpha = 1$ we have the approximate solution

$$u(x) = a_0 L_0^{\alpha}(x) + a_1 L_1^{\alpha}(x) + a_2 L_2^{\alpha}(x) = A^T\, \Psi(x). \tag{9.24}$$

Hence, Eq. (9.22), becomes

$$A^T \left[\Upsilon^2 + \Upsilon^{3/2} + I \right] \Psi(x) = G^T \Psi(x) \tag{9.25}$$

Here, we have

$$\Upsilon^1 = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ -1 & -1 & 0 \end{bmatrix}, \;\; \Upsilon^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \tag{9.26}$$

$$\Upsilon^{3/2} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{\sqrt{\pi}}{2} & \frac{-\sqrt{\pi}}{4} & \frac{-\sqrt{\pi}}{16} \end{bmatrix} \text{ and } G = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$$

Therefore, using Eq. (9.20), we obtain

$$a_0 + \left(1 + \frac{\sqrt{\pi}}{2}\right) a_2 = 2 \tag{9.27}$$

Applying Eq. (9.21), we have

$$a_0 + a_1 + a_2 = 1 \tag{9.28}$$

$$a_0 + a_1 = 1 \tag{9.29}$$

Solving Eqs. (9.27), (9.28) and (9.29), we get

$$a_0 = 2, \;\; a_1 = -1, \text{ and } a_2 = 0. \tag{9.30}$$

Thus, we have

$$u(x) = (2, \; -1, \; 0) \begin{pmatrix} L_0^1(x) \\ L_1^1(x) \\ L_2^1(x) \end{pmatrix} = 1 + x,$$

which is the exact solution.

Example 92*Consider the following CFDE*

$$\mathscr{D}^{\lambda} u(x) + u(x) = 0, \quad \lambda \in (0,1], \;\; 0 \le x \le 1, \tag{9.31}$$

with

$$u(0) = 1. \tag{9.32}$$

The problem (9.31) has an exact solution in the form $u(x) = e^{\left(\frac{-x^{\lambda}}{\lambda}\right)}$, (see [50,51]).
This problem has been treated using different methods in Caputo sense, (see[50,51]).
We solve the problem (9.31), by applying the technique described in the previous subsection. The absolute error for $M = 10$, and various values of $\alpha = \lambda$ are shown in table 1, and we can see that we can achieve a good approximation with the exact solution using a few terms of conformable fractional Laguerre polynomials. Also, the numerical result for $u(x)$ at $M = 10$ with $\alpha = \lambda = 0.4, \, 0.6, \, 0.8, \, 0.9,$ and 1 are plotted in Fig. 3.

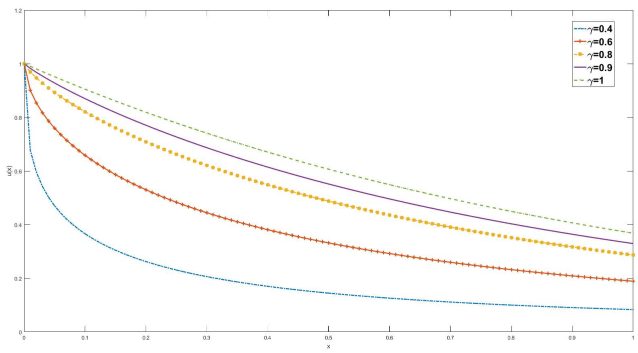

Fig. 3: Graph of the numerical solution $u_(x)$ of Example 92 for $M = 10$ with various values of $\alpha = \lambda = 0.4, \, 0.6, \, 0.8, \, 0.9,$ and 1.

Table 1: The absolute error for $M = 10$, and various values of $\alpha = \lambda = 0.4,\ 0.6,\ 0.8,\ 0.9,$ and 1, for Example 92

x	$\gamma = 0.4$	$\gamma = 0.6$	$\gamma = 0.8$	$\gamma = 1$
0.1	$4.0256833578.10^{-3}$	$3.053252644754.10^{-4}$	$1.80817274715639.10^{-4}$	$5.237431361688823.10^{-5}$
0.2	$6.9221049938.10^{-3}$	$5.553962049349.10^{-4}$	$4.61879235811313.10^{-5}$	$4.287895424511354.10^{-5}$
0.3	$7.4914126535.10^{-3}$	$1.125596510475.10^{-3}$	$1.20890231216475.10^{-4}$	$4.693164363294121.10^{-6}$
0.4	$7.0045089315.10^{-3}$	$1.407650550408.10^{-3}$	$2.54670049942785.10^{-4}$	$3.987324972045628.10^{-5}$
0.5	$5.9881878156.10^{-3}$	$1.462666989978.10^{-3}$	$3.34703362446343.10^{-4}$	$7.691936263802814.10^{-5}$
0.6	$4.7050760526.10^{-3}$	$1.350034699207.10^{-3}$	$3.58743647587700.10^{-4}$	$9.890731483230397.10^{-5}$
0.7	$3.2999503461.10^{-3}$	$1.119073935016.10^{-3}$	$3.32711340581993.10^{-4}$	$1.029759044721605.10^{-4}$
0.8	$1.8575168159.10^{-3}$	$8.090853661598.10^{-4}$	$2.66056667450276.10^{-4}$	$8.956447421627215.10^{-5}$
0.9	$4.2906650059.10^{-4}$	$4.509086007730.10^{-4}$	$1.69384005589968.10^{-4}$	$6.130357176535860.10^{-5}$
1	$9.5386883330.10^{-4}$	$6.854369738516.10^{-5}$	$5.31792314418302.10^{-5}$	$2.213388537324782.10^{-5}$

10 Concluding Remarks

The main purpose of the present paper is to develop a study on the fractional Laguerre equation and fractional Laguerre polynomial initiated recently in [3] in the sense of conformable derivative. We obtained solutions of the conformable fractional Laguerre equation about the fractional regular singular point. Several basic properties of the conformable fractional Laguerre polynomials are reported. Because of the orthogonality property of the conformable fractional Laguerre functions, it can be employed as a basic of operational matrix together with generalized tau method for solving general linear multi-term fractional differential equations. The main advantage of this method lies in its easiness since it relies on conformable fractional Laguerre polynomials. This method contrasts in simplicity with standard methods based on solving the fractional differential equations using other techniques found in the literature. Supported examples are given to ensure the applicability and efficiency of the introduced method. Therefore, the results of this work are variant, significant and so it is interesting and capable to develop its study in the future such as generalized Laguerre polynomials eg. [1,2].

Conflict of interest

The authors declare that there is no conflict regarding the publication of this paper.

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