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M. Seham
Department of Mathematics, Faculty of Science, Aswan University, Aswan, Egypt\\ Academy of Scientific Research and Techonolgy(ASRT), Egypt, seham__mohamed@sci.aswu.edu.eg

M. Amira Younis
Department of Mathematics, Faculty of Science, Aswan University, Aswan, Egypt\\ Academy of Scientific Research and Techonolgy(ASRT), Egypt, seham__mohamed@sci.aswu.edu.eg

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Statistical Inference of Weibull Extension Distribution under Imprecise Data

M. Seham[1,2,*] *and M. Amira Younis*[1,2]

[1]Department of Mathematics, Faculty of Science, Aswan University, Aswan, Egypt
[2]Academy of Scientific Research and Techonolgy(ASRT), Egypt

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Abstract: In this paper, we present the maximum likelihood (ML) and Bayes estimation of the unknown parameters, the reliability and hazard functions of the Weibull extension distribution based on progressively Type-II censoring scheme from fuzzy lifetime data. For the computation of Bayes estimates, we proposed using Tierney-Kadane's approximation under square error and LINEX loss functions. The performance of the maximum likelihood and Bayes estimators compared in terms of their mean squared errors (MSEs) through the simulation study. The results indicated that MSEs based on Tierney-Kadane's approximation are less than based on the ML method. Finally, to demonstrate the efficiency of the proposed methods, two real data sets are analyzed.

Keywords: Bayes estimates; Fuzzy lifetime data; Tierney-Kadane's approximation;Maximum likelihood estimates; Progressive Type-II censoring; squared error and linear exponential loss functions.

1 Introduction

The Weibull extension distribution (WED) was proposed by [1]. A random variable X is said to follow the WED if its probability density function (PDF) is given by

$$f(x) = \alpha\beta x^{\alpha-1} exp\left[x^{\alpha} - \beta\left(e^{x^{\alpha}} - 1\right)\right] ,x > 0 \ , \quad (1)$$

and its cumulative distribution function (CDF) is given by

$$F(x) = 1 - exp\left[-\beta\left(e^{x^{\alpha}} - 1\right)\right] \quad \alpha, \beta, \ x > 0, \quad (2)$$

the corresponding survival function (Reliability) is

$$R(x) = 1 - F(X) = exp\left[-\beta\left(e^{x^{\alpha}} - 1\right)\right] \quad ,x > 0 \quad (3)$$

the corresponding failure rate function of this distribution is

$$h(x) = \frac{f(x)}{R(x)} = \alpha\beta x^{\alpha-1} e^{x^{\alpha}} \quad ,x > 0\,. \quad (4)$$

The WED is the most useful extensions of the Weibull distribution. This distribution can have a bathtub-shaped hazard rate or an increasing hazard rate function. Statistical developments in WED have been numerous. We mention tail shapes classifications, distribution of extreme values, statistical inference, hypothesis tests, and goodness of fit tests [2]. Changing points of mean residual life and failure rate functions [3]. The exact confidence intervals for the shape parameter based on censored samples were derived by [4,5] and Explicit expressions for moments [6]. Quasi-likelihood estimation [7]. A generalization is referred to as the increasing, decreasing, unimodal, and bathtub-shaped distribution[8]. Bayesian analysis using Markov chain Monte Carlo simulation [9]. Estimation of the reliability function under type I and type-II censorings [10]. Regression models and their estimation, sensitivity, and residual analysis [11,12]. Estimation based on generalized order statistics [13]. A generalization is referred to as the complementary exponential power lifetime distribution [14]. A generalization based on a new family of distributions [15]. Estimation under progressive type-II censoring [16]. A method for determining acceleration factor [17]. A generalization is referred to as the exponentiated modified Weibull extension [18]. A discrete analog of the model [19]. a Bayes comparison with a competing model[20]. The statistical analysis of this model as a bathtub-shaped hazard rate function was discussed by [21]. The confidence intervals of the parameters of the Weibull

* Corresponding author e-mail: seham_mohamed@sci.aswu.edu.eg

extension model using the conditional inference based on the generalized order statistics were derived by [22].
Various censoring schemes are available in the literature to deal with the investigator's need and the limitations of the experiment. The progressively Type-II censoring schemes have drawn attention by many authors during recent years. Since The most common censoring Type-I, Type-II censoring schemes do not permit the experimenter to eliminate units before ending the experiment. A situation that is often met in some of the life testing experiment, particularly in those experiments where costly items are put to test, is that units are removed from the experiment before the final termination of the experiment. i.e., removal of items is allowed prior to completion of the experiment and such type of scheme is known as progressive censoring scheme. It was presented by [23]. From that point forward, numerous authors including, see, [24,25,26,27,28,29] have examined the estimation methods and applications of progressively censored samples for different lifetime models. A broad audit of the literature on progressive censoring can be seen from [30]. Furthermore, [31] these censoring permits removals within Type-II censoring scheme in the following manner. Suppose that n units are placed on a life test and the experimenter decides beforehand a quantity m, the number of units to be failed. Now at the time of the first failure, $\mathbb{R}_1$ of the remaining $\mathrm{n}-1$ surviving units are randomly removed from the experiment. Then after the second observed failure, $\mathbb{R}_2$ of the remaining $\mathrm{n}-\mathbb{R}_1-2$ units are randomly removed from the experiment. Finally, at the time of the mth failure, all the remaining $(n-\mathrm{m}-\sum_{\mathrm{i}=1}^{\mathrm{m}-1}\mathbb{R}_{\mathrm{i}}=\mathbb{R}_{\mathrm{m}})$ of surviving units are removed from the experiment. The above research results for estimating parameters of various lifetime distributions under general progressive Type-II censoring are restricted to precise data. However, in numerous fields of application, it is sometimes difficult to obtain exact observations of lifetime. The obtained lifetime data may be imprecise most of the time. For example, consider a life-testing experiment in which n identical batteries are placed on a test, and we are interested in the lifetime of these batteries. A tested battery may be considered a failure, or-strictly speaking-as nonconforming, when at least one value of its parameters falls beyond specification limits.
However, in practice, the observer does not have the possibility to measure all parameters and cannot precisely determine the moment of failures, but rather he/she can only approximate it by means of the following imprecise quantities: "approximately less than 1000 hour ", "approximately 1500 to 2000 hours ", "approximately 2500 hours ", "approximately 3000 hours ", "approximately 3500 to 4000 hours ", "approximately higher than 4500 hours ", and so on. Classical statistical procedures and Bayesian inference are not suitable for dealing with such imprecise cases. In order to model imprecise lifetimes, it is necessary to generalize the real numbers. These lifetimes can be represented by fuzzy numbers. A fuzzy number is a subset, denoted by $\tilde{\mathrm{x}}$, of the set of real numbers (denoted by $\mathbb{R}$) and characterized by the so-called membership function $\mu_{\tilde{x}}(.)$. Fuzzy numbers satisfy the following constraints, see, [32,33,34,35,36]. The probability measure for fuzzy events was discussed by [37]. Also,[38] studied membership functions and probability measures of fuzzy sets. [39] have considered generalized parametric procedures for reliability characteristic including fuzzy point estimators and generalized Bayesian procedures. [40,41,42,43,44] have also discussed the estimation of the survival function and Hazard rate. [45] discussed two types of estimators for cumulative distribution function (CDF) based on fuzzy data. Recently [46] discussed various classical and Bayesian methods of estimation for Weibull distribution when data is available in the form of fuzzy numbers. Also, [47] proposed a new method to determine the maximum likelihood estimate of the scale parameter of a Rayleigh distribution under doubly Type-II censored sample from fuzzy data, and [48] discussed inferences for the Rayleigh distribution based on progressive Type-II fuzzy censored data. [49] Derived Bayesian estimation procedures for the parameter and reliability function of Rayleigh distribution based on fuzzy lifetime data. [50] Proposed a new method to determine the maximum likelihood estimate for exponential mean parameter under progressive Type-II censoring from imprecise lifetime. [51] Studied inferences for Lindley distribution based on fuzzy data. [52] Discussed various classical and methods of estimation for Rayleigh Distribution under Type-II progressive hybrid censored fuzzy Data.
In the present paper, we obtain the ML and Bayes estimates for the unknown parameters, reliability, and hazard functions of WED when the lifetime observations are reported under progressively Type-II censoring scheme in the form of fuzzy numbers. The present paper is organized as follows. In section 2, we obtain the ML estimate of the parameters of WED. Next, in Section 3, Bayes estimators under squared error and linear exponential (LINEX) loss functions by using Tierney-Kadane's approximation techniques with informative priors are obtained. Simulation studies are carried out in Section 4 to assess the performance of the proposed methods. In section 5. Also, two real data set are analyzed for illustrative purposes. Finally, the conclusions are given in section 6.

2 Maximum Likelihood Estimation

Suppose that n independent units are put on a test and that the lifetime distribution of each unit is given by $f(x;\theta)$. Now consider the problem where under a progressive Type-II censoring scheme, failure times are not observed precisely and only partial information about them are available so we can represent them as fuzzy numbers $\tilde{\mathrm{x}}_{\mathrm{i}}=(\mathrm{a}_{\mathrm{i}},\mathrm{c}_{\mathrm{i}},\mathrm{b}_{\mathrm{i}}),\mathrm{i}=1,\ldots,\mathrm{m}$ and its corresponding membership function $(\mu_{\tilde{x}_1},\mu_{\tilde{x}_2},\ldots,\mu_{\tilde{x}_{\mathrm{m}}})$. Let

$c_{(1)} \le c_{(2)} \le \cdots \le c_{(m)}$ denote the ordered values of the means of these fuzzy numbers. The lifetime of $\mathbb{R}_i$surviving units, which are removed from the test after the ith failure, can be encoded as fuzzy numbers $(\tilde{z}_1, \tilde{z}_2, \ldots, \tilde{z}_m)$ where every $\tilde{z}_i$ is a $1 \times R_i$ vector with the membership functions

$$\mu_{\tilde{z}_{ij}} = \begin{cases} 0 & z \le C_i, \\ 1 & z > C_i, \end{cases} \qquad j, \ldots, \mathbb{R}_i$$

The fuzzy data $\tilde{w} = (\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_m, \tilde{z}_1, \tilde{z}_2, \ldots, \tilde{z}_m)$ where $\tilde{z}_i$ is a $1\times$ $\mathbb{R}_i$ vector with $\tilde{z}_i = \left(\tilde{z}_{i1}, \tilde{z}_{i2}, \ldots, \tilde{z}_{i\mathbb{R}i}\right)$ or $i = 1, \ldots, m$ is thus the set of observed life times.
The likelihood function for observed data $\widetilde{W}$ can be obtained by using Zadeh's definition of the probability of a fuzzy event; see [37], as

$$\begin{aligned} L_o(\widetilde{w}; \alpha, \beta) &\propto \prod_{i=1}^{m} \int f(x)\, \mu_{\tilde{x}_i}(x) dx \\ &\cdot \prod_{i=1}^{m} \prod_{j=1}^{R_i} \int f(z)\, \mu_{\tilde{z}_{ij}}(z) dz \\ &= K \prod_{i=1}^{m} \int \alpha\beta\, x^{\alpha-1} e^{\left(x^{\alpha} - \beta\left(e^{x^{\alpha}} - 1\right)\right)} \mu_{\tilde{x}_i}(x)\, dx \\ &\cdot \prod_{i=1}^{m} \prod_{j=1}^{R_i} \int \alpha\beta\, z^{\alpha-1} e^{\left(z^{\alpha} - \beta\left(e^{z^{\alpha}} - 1\right)\right)} \mu_{\tilde{z}_{ij}}(z)\, dz. \end{aligned} \tag{5}$$

Then the observed data log-likelihood function can be obtained by using the expression (5) as follows:

$$\begin{aligned} L(\widetilde{w}, \alpha, \beta) &= \sum_{i=1}^{m} \log \int \alpha\beta x^{\alpha-1} e^{\left(x^{\alpha} - \beta\left(e^{x^{\alpha}} - 1\right)\right)} \mu_{\tilde{x}_i}(x)\, dx \\ &+ \sum_{i=1}^{m} \sum_{j=1}^{R_i} \log \int \alpha\beta z^{\alpha-1} e^{\left(z^{\alpha} - \beta\left(e^{z^{\alpha}} - 1\right)\right)} \mu_{\tilde{z}_{ij}}(z)\, dz \\ &= m(\log\alpha + \log\beta) - \beta \sum_{i=1}^{m} R_i (e^{c_{(i)}^{\alpha}} - 1) \\ &+ \sum_{i=1}^{m} \log \int x^{\alpha-1} e^{\left(x^{\alpha} - \beta\left(e^{x^{\alpha}} - 1\right)\right)} \mu_{\tilde{x}_i}(x)\, dx. \end{aligned} \tag{6}$$

The maximum likelihood estimates of the parameter α and β can be obtained by maximizing the log-likelihood $logL(\tilde{w}; \alpha, \beta)$ .Equating the partial derivative of the log-likelihood with respect to α and β to zero, the resulting equation is:

$$\begin{aligned} \frac{\partial logL(\tilde{w}; \alpha, \beta)}{\partial \alpha} &= \frac{m}{\alpha} - \beta \sum_{i=1}^{m} R_i c_{(i)}^{\alpha} e^{c_{(i)}^{\alpha}} logc_{(i)} \\ &+ \sum_{i=1}^{m} \frac{\int x^{\alpha-1} logx(1 + x^{\alpha} - \beta x^{\alpha} e^{x^{\alpha}}) exp(x^{\alpha} - \beta(e^{x^{\alpha}} - 1)) \mu_{\tilde{x}_i}(x) dx}{\int x^{\alpha-1} exp(x^{\alpha} - \beta(e^{x^{\alpha}} - 1)) \mu_{\tilde{x}_i}(x) dx}, \end{aligned} \tag{7}$$

$$\begin{aligned} \frac{\partial logL(\tilde{w}; \alpha, \beta)}{\partial \beta} &= \frac{m}{\beta} - \sum_{i=1}^{m} R_i (e^{c_{(i)}^{\alpha}} - 1) \\ &- \sum_{i=1}^{m} \frac{\int x^{\alpha-1} (e^{x^{\alpha}} - 1) exp(x^{\alpha} - \beta(e^{x^{\alpha}} - 1)) \mu_{\tilde{x}_i}(x) dx}{\int x^{\alpha-1} exp(x^{\alpha} - \beta(e^{x^{\alpha}} - 1)) \mu_{\tilde{x}_i}(x) dx}. \end{aligned} \tag{8}$$

Since there are no closed form of the solutions to the likelihood equations (7) and (8), an iterative numerical search procedure needs to be considered to obtain the MLEs. Next, we describe two widely practiced search procedures, namely, the finite difference method to determine the MLEs of the parameters α and β.

3 Bayesian estimation

In this section, we derive the Bayes estimates for the unknown parameters, reliability, and hazard rate functions under different loss functions based on the progressive Type-II censored fuzzy data. In the Bayesian estimation, an unknown parameter is assumed to behave as a random variable with a distribution commonly known as a prior probability distribution. Here, we consider the following independent *gamma* priors for all the parameters α and β given as follows:

- Prior for $\alpha : \pi_1(\alpha) = \frac{b^a}{\Gamma(a)} \alpha^{(a-1)} e^{-(\alpha b)}, \quad a,\ b > 0,$
- Prior for $\beta : \pi_2(\beta) = \frac{d^c}{\Gamma(c)} \beta^{(c-1)} e^{-(\beta d)}, \quad c,\ d > 0.$

with the parameter $\alpha \sim gamma(a,b)$ and $\beta \sim Gamma(c,d)$. Based on the above priors, the joint posterior density function of α and β given the data can be written as follows:

$$\begin{aligned} \pi(\alpha, \beta; \check{x}) &= K\beta^{m+c-1} \alpha^{m+a-1} \\ &.e^{-(\alpha b + \beta d)} e^{-\beta \sum_{i=1}^{m} R_i \left(e^{C_{(i)}^{\alpha}} - 1\right)} J_{11}, \end{aligned} \tag{9}$$

where

$$J_{11} = \prod_{i=1}^{m} \int x^{\alpha-1} exp[x^{\alpha} - \beta(e^{x^{\alpha}} - 1) \mu_{\tilde{x}_i}(x) dx.$$

K is the normalizing constant and can be evaluated as

$$\begin{aligned} K^{-1} &= \int_0^{\infty} \int_0^{\infty} \beta^{m+c-1} \alpha^{m+a-1} e^{-(\alpha b + \beta d)} \\ &e^{-(\alpha b + \beta d)} e^{-\beta \sum_{i=1}^{m} R_i \left(e^{C_{(i)}^{\alpha}} - 1\right)} J_{11} d\alpha\, d\beta. \end{aligned}$$

The Bayes estimates for unknown parameters, reliability function $R(t)$, and hazard rate $H(t)$ are considered with different loss functions given respectively as
Squared error loss function (SELF):

$$L(\hat{\theta} - \theta) = (\hat{\theta} - \theta)^2. \tag{10}$$

LINEX loss function (LLF):

$$Ł(\hat{\theta} - \theta) \propto exp(\delta\hat{\theta}) E_{\theta}[exp(-\delta\theta)] - \delta(\hat{\theta} - E_{\theta}(\theta)) - 1. \tag{11}$$

Firstly, if θ is the parameter to be estimated by an estimator $\hat{\theta}_{SL}$ then the square error loss function is defined as: $L(\hat{\theta} - \theta) = (\hat{\theta} - \theta)^2$. This loss function is symmetric in nature, that is, it gives equal weightage to both over and under estimation. The Bayes estimators of a function $u(\alpha, \beta)$ of the unknown parameters, reliability and hazard rate functions under the symmetric loss function are given as

$$\hat{u}_{SL} = E(u(\alpha, \beta) | \tilde{x}) = \frac{\int_0^{\infty} \int_0^{\infty} u(\alpha, \beta) \alpha^{m+a-1} \beta^{m+c-1} e^{-(\alpha b + \beta d)} \phi J_{11} d\alpha d\beta}{\int_0^{\infty} \int_0^{\infty} \alpha^{m+a-1} \beta^{m+c-1} e^{-(\alpha b + \beta d)} \phi J_{11} d\alpha d\beta},$$

where

$$\phi = e^{-\beta \sum_{i=1}^{m} R_i (e^{c_{(i)}^{\alpha}} - 1)},$$

$$\hat{R}(t)_{SL} = E(u = exp(-\beta(e^{t^{\alpha}} - 1)) | \tilde{x}) = \frac{\int_0^{\infty} \int_0^{\infty} e^{(-\beta(e^{t^{\alpha}} - 1))} \alpha^{m+a-1} \beta^{m+c-1} e^{-(\alpha b + \beta d)} \phi J_{11} d\alpha d\beta}{\int_0^{\infty} \int_0^{\infty} \alpha^{m+a-1} \beta^{m+c-1} e^{-(\alpha b + \beta d)} \phi J_{11} d\alpha d\beta},$$

$$\hat{H}(t)_{SL} = E(u = \alpha\beta t^{\alpha-1}e^{t^\alpha})|\tilde{x}) = \frac{\int_0^\infty\int_0^\infty \alpha\beta t^{\alpha-1}e^{t^\alpha}\alpha^{m+a-1}\beta^{m+c-1}e^{-(\alpha b+\beta d)}\phi J_{11}d\alpha d\beta}{\int_0^\infty\int_0^\infty \alpha^{m+a-1}\beta^{m+c-1}e^{-(\alpha b+\beta d)}\phi J_{11}d\alpha d\beta}.$$

Secondly, the loss function is linear exponential (LINEX) loss function, which was introduced by [53]. The sign and magnitude of the shape parameter ρ represent the direction and degree of symmetry respectively. The Bayes Estimator of θ which is denoted by $\hat{\theta}_{LL}$ under LINEX loss function that minimizes equation (11) is denoted as follows

$$\hat{\theta}_{LL} = \frac{-1}{\rho}ln[E_\theta(e^{-\rho\theta})],\ \rho\neq 0 \tag{12}$$

provided that $E_\theta(.)$exist and is finite. However, Bayesian estimation using the LINEX loss function is not frequently discussed, perhaps, because the estimator involves integral expressions, which are not analytically solvable, and one has to use numerical techniques. It can be observed that the above equations contain the ratio of integrals that cannot be obtained analytically and as a result, we make use of Tierney-Kadane's approximation procedures to evaluate the integrals involved. To facilitate these numerical computations, the author has written a program FORTRAN code, which is available on request

3.1 Tierney-Kadane's approximation

The Bayes estimate of $g(\alpha,\beta)$ can be written in the following expression.

$$\hat{g}(\alpha,\beta) = \frac{\int_0^\infty\int_0^\infty g(\alpha,\beta)\pi_1(\alpha)\pi_2(\beta)l(\alpha,\beta;\tilde{x})d\alpha d\beta}{\int_0^\infty\int_0^\infty \pi_1(\alpha)\pi_2(\beta)l(\alpha,\beta;\tilde{x})d\alpha d\beta}. \tag{13}$$

First, we rewrite the expression in (13) as

$$\hat{g}(\alpha,\beta) = \frac{\int_0^\infty\int_0^\infty g(\alpha,\beta)e^{Q(\alpha,\beta)}d\alpha d\beta}{\int_0^\infty\int_0^\infty e^{Q(\alpha,\beta)}d\alpha d\beta},$$

where $Q(\alpha,\beta) = \ln(\pi_1(\alpha)\pi_2(\beta)) + \ln(l(\alpha,\beta,\tilde{x}))$, $\delta(\alpha,\beta) = \frac{Q(\alpha,\beta)}{n}$, $\delta^*(\alpha,\beta) = \delta(\alpha,\beta) + \frac{\ln(g(\alpha,\beta))}{n}$, Then $\hat{g}(\alpha,\beta) = \frac{\int_0^\infty\int_0^\infty e^{n\delta^*(\alpha,\beta))}d\alpha d\beta}{\int_0^\infty\int_0^\infty e^{n\delta(\alpha,\beta)}d\alpha d\beta}$. Following [54] Eq. (13) can be approximated as the following form:

$$\hat{g}(\alpha,\beta) = \left[\frac{\det\Sigma^*}{\det\Sigma}\right]^{1/2}\exp\{n[\delta^*\left(\overline{\alpha}^*,\overline{\beta}^*\right) - \delta\left(\overline{\alpha},\overline{\beta}\right)]\}, \tag{14}$$

where $(\bar{\alpha}^*,\bar{\beta}^*)$ and $(\bar{\alpha},\bar{\beta})$ maximize $\delta^*(\alpha,\beta) and\ \delta(\alpha,\beta)$, respectively. Also, Σ^* and Σ are the negatives of the inverse Hessians of $\delta^*(\alpha,\beta) and \delta(\alpha,\beta)$ at $(\bar{\alpha}^*,\bar{\beta}^*)$and $(\bar{\alpha},\bar{\beta})$, respectively. In our case, we have

$$\delta = \frac{1}{n}[k+(a+m-1)ln\alpha+(c+m-1)ln\beta - (\alpha b+\beta d) - \beta\sum_{i=1}^m R_i\left(e^{c_{(i)}^\alpha}-1\right) + \sum_{i=1}^m ln\left[\int x^{\alpha-1}e^{\left(x^\alpha-\beta\left(e^{x^\alpha}-1\right)\right)}\mu_{\tilde{x}_i}(x)dx\right]\Bigg],$$

where, k is a constant; therefore, $(\bar{\alpha},\bar{\beta})$ can be obtained by solving the following two equation

$$\frac{\partial\delta}{\partial\alpha} = \frac{1}{n}\left[\frac{a+m-1}{\alpha} - b - \beta\sum_{i=1}^m R_i c_{(i)}^\alpha e^{c_{(i)}^\alpha}lnc_{(i)} + \sum_{i=1}^m \frac{\int\left(x^{\alpha-1}lnx\right)\left(1+x^\alpha-\beta x^\alpha e^{x^\alpha}\right)e^{\left(x^\alpha-\beta\left(e^{x^\alpha}-1\right)\right)}\mu_{\tilde{x}_i}(x)dx}{\int x^{\alpha-1}e^{\left(x^\alpha-\beta\left(e^{x^\alpha}-1\right)\right)}\mu_{\tilde{x}_i}(x)dx}\right],$$

$$\frac{\partial\delta}{\partial\beta} = \frac{1}{n}\left[\frac{c+m-1}{\beta} - d - \sum_{i1}^m R_i\left(e^{c_{(i)}^\alpha}-1\right) + \sum_{i=1}^m \frac{\int x^{\alpha-1}\left(1-e^{x^\alpha}\right)e^{\left(x^\alpha-\beta\left(e^{x^\alpha}-1\right)\right)}\mu_{\tilde{x}_i}(x)dx}{\int x^{\alpha-1}e^{\left(x^\alpha-\beta\left(e^{x^\alpha}-1\right)\right)}\mu_{\tilde{x}_i}(x)dx}\right],$$

and, from the second derivatives of $\delta(\alpha,\beta)$, the determinant of the negative of the inverse Hessian of $\delta(\alpha,\beta)$ at $(\bar{\alpha},\bar{\beta})$ is given by

$$\det\Sigma = \left(\delta_{11}\delta_{22} - {\delta_{12}}^2\right)^{-1},$$

where

$$\delta_{11} = -\frac{\partial^2\delta}{\partial\alpha^2} = \frac{1}{n}\Bigg[\frac{a+m-1}{\alpha^2} + \beta\sum_{i=1}^m R_i c_{(i)}^\alpha e^{c_{(i)}^\alpha}\left(lnc_{(i)}\right)^2\left(1+c_{(i)}^\alpha\right) - \sum_{i=1}^m\Bigg[\frac{\int\left[1+x^{2\alpha}+\left(\beta x^\alpha e^{x^\alpha}\right)^2+3x^\alpha\left(-\beta e^{x^\alpha}(1+x^\alpha)+1\right)\right]J_{22}\mu_{\tilde{x}_i}(x)dx}{\int x^{\alpha-1}e^{\left(x^\alpha-\beta\left(e^{x^\alpha}-1\right)\right)}\mu_{\tilde{x}_i}(x)dx} + \left[\frac{\int\left(1+x^\alpha-\beta x^\alpha e^{x^\alpha}\right)x^{\alpha-1}(lnx)e^{\left(x^\alpha-\beta\left(e^{x^\alpha}-1\right)\right)}\mu_{\tilde{x}_i}(x)dx}{\int x^{\alpha-1}e^{\left(x^\alpha-\beta\left(e^{x^\alpha}-1\right)\right)}\mu_{\tilde{x}_i}(x)dx}\right]^2\Bigg]\Bigg],$$

and $J_{22} = x^{\alpha-1}(lnx)^2 e^{\left(x^\alpha-\beta\left(e^{x^\alpha}-1\right)\right)}$

$$\delta_{12} = -\frac{\partial^2\delta}{\partial\alpha\partial\beta} = \frac{1}{n}\Bigg[\sum_{i=1}^m R_i c_{(i)}^\alpha e^{c_{(i)}^\alpha}lnc_{(i)} - \sum_{i=1}^m\frac{\int x^{\alpha-1}(lnx)J_{33}e^{\left(x^\alpha-\beta\left(e^{x^\alpha}-1\right)\right)}\mu_{\tilde{x}_i}(x)dx}{\int x^{\alpha-1}e^{\left(x^\alpha-\beta\left(e^{x^\alpha}-1\right)\right)}\mu_{\tilde{x}_i}(x)dx} + \sum_{i=1}^m\frac{J_{44}\int x^{\alpha-1}lnx\left(1+x^\alpha-\beta x^\alpha e^{x^\alpha}\right)e^{\left(x^\alpha-\beta\left(e^{x^\alpha}-1\right)\right)}\mu_{\tilde{x}_i}(x)dx}{\left[\int x^{\alpha-1}e^{\left(x^\alpha-\beta\left(e^{x^\alpha}-1\right)\right)}\mu_{\tilde{x}_i}(x)dx\right]^2}\Bigg],$$

$$J_{33} = -x^\alpha e^{x^\alpha} + \left(1-e^{x^\alpha}\right)\left(1+x^\alpha-\beta x^\alpha e^{x^\alpha}\right),$$

$$J_{44} = \int x^{\alpha-1}\left(1-e^{x^\alpha}\right)e^{\left(x^\alpha-\beta\left(e^{x^\alpha}-1\right)\right)}\mu_{\tilde{x}_i}(x)dx$$

$$\delta_{22} = -\frac{\partial^2\delta}{\partial\beta^2} = \frac{1}{n}\Bigg[\frac{c+m-1}{\beta^2} - \sum_{i=1}^m\frac{\int x^{\alpha-1}(1-e^{x^\alpha})^2e^{\left(x^\alpha-\beta\left(e^{x^\alpha}-1\right)\right)}\mu_{\tilde{x}_i}(x)dx}{\int x^{\alpha-1}e^{\left(x^\alpha-\beta\left(e^{x^\alpha}-1\right)\right)}\mu_{\tilde{x}_i}(x)dx} + \sum_{i=1}^m\left[\frac{\int x^{\alpha-1}\left(1-e^{x^\alpha}\right)e^{\left(x^\alpha-\beta\left(e^{x^\alpha}-1\right)\right)}\mu_{\tilde{x}_i}(x)dx}{\int x^{\alpha-1}e^{\left(x^\alpha-\beta\left(e^{x^\alpha}-1\right)\right)}\mu_{\tilde{x}_i}(x)dx}\right]^2\Bigg].$$

Firstly, applying Tierney-Kadane's approximation procedure as in (14), to compute the Bayes estimator of α under SELF we take $g(\alpha,\beta)=\alpha$ and accordingly function $\delta^*(\alpha,\beta)$ becomes

$$\delta^*_{1\alpha}(\alpha,\beta)=\delta(\alpha,\beta)+\tfrac{\ln\alpha}{n}.$$

Then $\left(\bar{\alpha}^*,\bar{\beta}^*\right)$ are obtained by solving the following two non-linear equations

$$\frac{\partial\delta^*_{1\alpha}}{\partial\alpha}=\frac{\partial\delta}{\partial\alpha}+\frac{1}{n\alpha},\quad \frac{\partial\delta^*_{1\alpha}}{\partial\beta}=\frac{\partial\delta}{\partial\beta}$$

and, from the second derivatives of $\delta^*_{1\alpha}(\alpha,\beta)$, the determinant of the negative of the inverse Hessian of $\delta^*_{1\alpha}(\alpha,\beta)$ at $\left(\bar{\alpha}^*,\bar{\beta}^*\right)$ is given by

$$\det\Sigma^*_{1\alpha}=\left(\delta^*_{\alpha11}\delta^*_{\alpha22}-{\delta^*_{\alpha12}}^2\right)^{-1},$$

where

$$\delta^*_{\alpha11}=-\frac{\partial^2\delta^*_{1\alpha}}{\partial\alpha^2}=-\frac{\partial^2\delta_\alpha}{\partial\alpha^2}+\frac{1}{n\alpha^2}$$
$$\delta^*_{\alpha12}=\delta^*_{\alpha21}=-\frac{\partial^2\delta^*_{1\alpha}}{\partial\alpha\partial\beta}=-\frac{\partial^2\delta_\alpha}{\partial\alpha\partial\beta}$$
$$\delta^*_{\alpha22}=-\frac{\partial^2\delta^*_{1\alpha}}{\partial\beta^2}=-\frac{\partial^2\delta_\alpha}{\partial\beta^2},$$

thus, the approximate Bayes estimator of α under SELF is given by

$$\hat{\alpha}_{SL}=\left[\frac{\det\Sigma^*_{1\alpha}}{det\Sigma}\right]^{1/2}\exp\{n[\delta^*_{1\alpha}\left(\bar{\alpha}^*,\bar{\beta}^*\right)-\delta\left(\bar{\alpha},\bar{\beta}\right)]\}. \quad (15)$$

Similarly, we can derive the approximate Bayes estimator of β as

$$\hat{\beta}_{SL}=\left[\frac{\det\Sigma^*_{1\beta}}{det\Sigma}\right]^{1/2}\exp\{n[\delta^*_{1\beta}\left(\bar{\alpha}^*,\bar{\beta}^*\right)-\delta\left(\bar{\alpha},\bar{\beta}\right)]\}. \quad (16)$$

respectively. Next, we compute the Bayes estimator of survival function $R(t)$ under SELF. In this case, $g(\alpha,\beta)=exp(\beta(1-e^{t^\alpha}))$, then

$$\delta^*_{R(t)}(\alpha,\beta)=\delta(\alpha,\beta)+\frac{\beta(1-e^{t^\alpha})}{n}.$$

Now compute $\left(\bar{\alpha}^*,\bar{\beta}^*\right)$ by solving the following two nonlinear equations

$$\frac{\partial\delta^*_{R(t)}}{\partial\alpha}=\frac{\partial\delta}{\partial\alpha}-\frac{\beta t^\alpha e^{t^\alpha}lnt}{n},\quad \frac{\partial\delta^*_{R(t)}}{\partial\beta}=\frac{\partial\delta}{\partial\beta}+\frac{(1-e^{t^\alpha})}{n}$$

and, from the second derivatives of $\delta^*_{R(t)}(\alpha,\beta)$, the determinant of the negative of the inverse Hessian of $\delta^*_{R(t)}(\alpha,\beta)$ at $\left(\bar{\alpha}^*,\bar{\beta}^*\right)$ is given by

$$\det\Sigma^*_{R(t)}=\left(\delta^*_{R(t)11}\delta^*_{R(t)22}-{\delta^*_{R(t)12}}^2\right)^{-1},$$

where

$$\delta^*_{R(t)11}=-\frac{\partial^2\delta^*_{R(t)}}{\partial\alpha^2}=-\frac{\partial^2\delta_{R(t)}}{\partial\alpha^2}+\frac{\beta lnt}{n}[t^\alpha e^{t^\alpha}lnt+t^{2\alpha}e^{t^\alpha}lnt]$$
$$\delta^*_{R(t)12}=\delta^*_{R(t)21}=-\frac{\partial^2\delta^*_{R(t)}}{\partial\alpha\partial\beta}=-\frac{\partial^2\delta_{R(t)}}{\partial\alpha\partial\beta}+\frac{1}{n}t^\alpha e^{t^\alpha}lnt$$
$$\delta^*_{R(t)22}=-\frac{\partial^2\delta^*_{R(t)}}{\partial\beta^2}=-\frac{\partial^2\delta_{R(t)}}{\partial\beta^2},$$

thus, the Bayes estimator of reliability $R(t)$ under SELF is given by

$$\hat{R}(t)_{SL}=\left[\frac{\det\Sigma^*_{R(t)}}{det\Sigma}\right]^{1/2}\exp\{n[\delta^*_{R(t)}\left(\bar{\alpha}^*,\bar{\beta}^*\right)-\delta\left(\bar{\alpha},\bar{\beta}\right)]\}. \quad (17)$$

Similarly, the Bayes estimator of hazard rate $H(t)$ under SELF is given by

$$\hat{H}(t)_{SL}=\left[\frac{\det\Sigma^*_{H(t)}}{det\Sigma}\right]^{1/2}\exp\{n[\delta^*_{H(t)}\left(\bar{\alpha}^*,\bar{\beta}^*\right)-\delta\left(\bar{\alpha},\bar{\beta}\right)]\}. \quad (18)$$

Secondly, in order to obtain the Bayes estimators of α and β under LINEX loss function we replacing $g(\alpha,\beta)$ by $e^{-\rho\alpha}$ and $e^{-\rho\beta}$ respectively, and accordingly function $\delta^*(\alpha,\beta)$ take the form:

$$\left.\begin{array}{l}\delta^*_{2\alpha}(\alpha,\beta)=\delta(\alpha,\beta)-\frac{\rho\alpha}{n},\ g(\alpha,\beta)=e^{-\rho\alpha}\\ \delta^*_{2\beta}(\alpha,\beta)=\delta(\alpha,\beta)-\frac{\rho\beta}{n},\ g(\alpha,\beta)=e^{-\rho\beta}\end{array}\right\}$$

using the same manner as in Eqs. (15) and (16). Therefore, the approximate Bayes estimate of α and β based on LINEX loss function are:

$$\hat{\alpha}_{LL}=\frac{-1}{\rho}\, ln\left(\left[\frac{\det\Sigma^*_{2\alpha}}{det\Sigma}\right]^{1/2}\exp\{n[\delta^*_{2\beta}\left(\bar{\alpha}^*,\bar{\beta}^*\right)-\delta\left(\bar{\alpha},\bar{\beta}\right)]\}\right), \quad (19)$$

$$\hat{\beta}_{LL}=\frac{-1}{\rho}\, ln\left(\left[\frac{\det\Sigma^*_{2\beta}}{det\Sigma}\right]^{1/2}\exp\{n[\delta^*_{2\beta}\left(\bar{\alpha}^*,\bar{\beta}^*\right)-\delta\left(\bar{\alpha},\bar{\beta}\right)]\}\right). \quad (20)$$

Finally, the approximate Bayes estimator of $R(t)$ and $H(t)$, under LINEX loss function can then be obtained in a straightforward manner.

4 Simulation Study

In this section, we study the performance of the ML and the Bayes estimators based on the mean squared errors (MSEs) of the unknown parameters as well as the reliability and hazard functions of the WED based on progressive Type-II censoring scheme from fuzzy lifetime data. To evaluate the Bayes estimates using Tierney-Kadane's approximation method under square error and LINEX loss functions. In our simulation study, we have generated 1000 replications for each sample size, n = 20, 40, and 80 from WED with α= (0.502, 1.002) and β = (1.002, 2.007) based on progressive Type-II censoring from fuzzy lifetime data with uncensored levels m equal to [n/2] and [3n/4]. The hyper-parameter (a, b, c, d) are taken for informative prior means so that are exactly equal to the true values of the parameters: (1.02, 6.05, 1.5, 3.9), (1.02, 6.05, 1.8, 2.2), (1.5, 3.9, 1.5, 3.9) and (1.5, 3.9, 1.8, 2.2). The loss parameter is represented by ($\rho = \pm 2$). Then, using the method as proposed by [51], each realization of the generated samples was fuzzifed by employing fuzzy information system $(\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_K)$ from the WED, which corresponds to the following membership functions

$$\mu_{\tilde{x}_1}(x) = \begin{cases} 1 & x \le h, \text{ h=0.05} \\ \frac{x-h}{h} & h < x \le 2h \\ 0 & otherwise \end{cases}$$

$$\mu_{\tilde{x}_1}(x) = \begin{cases} \frac{x-(i-1)h}{h} & (i-1)h \le x < ih \\ \frac{(i+1)h-x}{h} & ih \le x < (i+1)h \qquad i = 2, \ldots, k-1 \\ 0 & otherwise \end{cases}$$

$$\mu_{\tilde{x}_1}(x) = \begin{cases} \frac{x-kh}{h} & ih < x \le (k+1)h \\ 1 & x > (k+1)h \\ 0 & otherwise \end{cases}$$

and censoring scheme, progressively censored samples from the WED were generated, using the method proposed by [27], as follows:

1. Generate Z_i from $U(0,1) for i = 1, \ldots, m$
2. For given values of the progressive censoring scheme $(R_1, \ldots, R_m)$
3. Set $V_i = Z_i^{\frac{1}{a_i}}$, $a_i = i + \sum_{j=m-i+1}^{m} R_j$, $i = 1, \ldots, m$.
4. Set $U_i = 1 - V_{m-i+1} V_{m-i+2} \ldots V_m, i = 1, \ldots, m$.
5. Thus, $X_i = f^{-1}(U_i), i = 1, \ldots, m$, is the desired progressive type-II censored sample from the WED. For each of the censoring cases, we calculate the ML and Bayesian estimates based on the mean squared errors (MSEs) of the unknown parameters, reliability $R(t)$, and hazard $H(t)$ rate functions under progressive Type-II censoring scheme from fuzzy lifetime data. For reliability and hazard characteristics, We take The time t = 0.5 and 0.8.

From the simulation results in Tables 1 − 6, some of the points are quite clear based on these estimates and the others have been summarized in the following main points:

1. The estimated MSEs values of all the parameters decrease as sample size n and the censored level m increase.
2. The estimated values for the MSEs based on Tierney-Kadane's approximation are smaller than the maximum likelihood estimators.
3. The estimated values for the MSEs for the reliability function will increase with increasing lifetime t at $\alpha =$ 0.5 while decreasing with increasing lifetime t at $\alpha = 1$ as shown according to the characteristics of the WED.
4. The estimated values for the MSEs for the hazard rate function will increase with increasing values of α and β for all lifetime t values and these values increase with increasing lifetime t.

Finally, we can conclude that the Bayes estimates by using Tierney-Kadane's approximation are better than the maximum likelihood estimates for the parameters α and β, reliability and hazard functions of WED.

Table 1. The mean square errors (MSEs) of the MLE and Bayes methods for α under squared error (SL) and LINEX loss functions ($LL(\rho)$) with ($\rho = \pm 2$) based on progressive Type-II censoring from fuzzy lifetime data for different choices n, m, α and β.

Approaches				MLE	MSE (Tierney-Kadane's)		
n	m	α	β	MSE	SL	LL(-2)	LL(2)
20	10	0.502	1.002	0.0478	0.0236	0.0224	0.0275
			2.007	0.0757	0.0384	0.0368	0.0858
		1.002	1.002	0.1638	0.0680	0.0584	0.1147
			2.007	0.1125	0.0884	0.0716	0.0275
	15	0.502	1.002	0.0294	0.0198	0.0192	0.0217
			2.007	0.0395	0.0369	0.0354	0.0406
		1.002	1.002	0.0957	0.0493	0.0468	0.0576
			2.007	0.1006	0.0667	0.0552	0.0831
40	20	0.502	1.002	0.0169	0.0096	0.0099	0.0099
			2.007	0.0612	0.0170	0.0165	0.0195
		1.002	1.002	0.0478	0.0298	0.0295	0.0328
			2.007	0.0489	0.0315	0.0267	0.0383
	30	0.502	1.002	0.0115	0.0096	0.0095	0.0101
			2.007	0.0246	0.0222	0.0213	0.0241
		1.002	1.002	0.0313	0.0228	0.0234	0.0238
			2.007	0.0376	0.0275	0.0244	0.0320
80	40	0.502	1.002	0.0079	0.0074	0.0082	0.0066
			2.007	0.0167	0.0109	0.0122	0.0102
		1.002	1.002	0.0179	0.0133	0.0147	0.0126
			2.007	0.0189	0.0120	0.0133	0.0114
	60	0.502	1.002	0.0059	0.0048	0.0049	0.0049
			2.007	0.0166	0.0121	0.0118	0.0128
		1.002	1.002	0.0142	0.0118	0.0123	0.0115
			2.007	0.0188	0.0120	0.0128	0.0129

Table 2. The mean square errors (MSEs) of the MLE and Bayes methods for β under squared error (SL) and LINEX loss functions ($LL(\rho)$) with ($\rho = \pm 2$) based on progressive Type-II censoring from fuzzy lifetime data for different choices n, m, α and β.

Approaches				MLE	MSE (Tierney-Kadane's)		
n	m	α	β	MSE	SL	LL(-2)	LL(2)
20	10	0.502	1.002	0.0941	0.1025	0.0753	0.1392
			2.007	1.3110	1.0164	0.6953	1.3224
		1.002	1.002	1.1624	0.0722	0.0535	0.1023
			2.007	1.3384	0.5439	0.3052	0.8870
	15	0.502	1.002	0.1588	0.0460	0.0374	0.0609
			2.007	0.7518	0.5153	0.3129	0.7315
		1.002	1.002	0.2511	0.0425	0.0402	0.0534
			2.007	0.5292	0.3173	0.2582	0.5234
40	20	0.502	1.002	0.0913	0.0552	0.0462	0.0694
			2.007	1.2578	0.7007	0.5087	0.9405
		1.002	1.002	0.1195	0.0361	0.0364	0.0233
			2.007	0.8756	0.2398	0.2498	0.1484
	30	0.502	1.002	0.0293	0.0268	0.0243	0.0313
			2.007	0.2780	0.3657	0.2611	0.4914
		1.002	1.002	0.0569	0.0259	0.0278	0.0270
			2.007	0.3194	0.1542	0.1711	0.2266
80	40	0.502	1.002	0.0345	0.0381	0.0351	0.0425
			2.007	0.5740	0.3916	0.3478	0.5328
		1.002	1.002	0.0403	0.0230	0.0244	0.0233
			2.007	0.2901	0.1223	0.1819	0.1484
	60	0.502	1.002	0.0142	0.0177	0.0164	0.0195
			2.007	0.2169	0.2482	0.2001	0.3115
		1.002	1.002	0.0219	0.0152	0.0157	0.0154
			2.007	0.1257	0.0878	0.1051	0.0999

Table 3. The mean square errors (MSEs) of the MLE and Bayes methods for R(t) under squared error (SL) and LINEX loss functions ($LL(\rho)$) with ($\rho = \pm 2$) at $t = 0.5$ based progressive Type-II censoring from fuzzy lifetime data for different choices n, m, α and β.

Approaches				MLE	MSE (Tierney-Kadane's)		
n	m	α	β	MSE	SL	LL(-2)	LL(2)
20	10	0.502	1.002	0.0060	0.0133	0.0151	0.0108
			2.007	0.0364	0.0407	0.0434	0.0334
		1.002	1.002	0.0158	0.0052	0.0056	0.0047
			2.007	0.0133	0.0130	0.0141	0.0042
	15	0.502	1.002	0.0085	0.0052	0.0057	0.0097
			2.007	0.0058	0.0105	0.0114	0.0045
		1.002	1.002	0.0117	0.0050	0.0050	0.0087
			2.007	0.0088	0.0053	0.0056	0.0052
40	20	0.502	1.002	0.0057	0.0079	0.0087	0.0069
			2.007	0.0355	0.0292	0.0306	0.0256
		1.002	1.002	0.0072	0.0035	0.0036	0.0035
			2.007	0.0061	0.0057	0.0060	0.0046
	30	0.502	1.002	0.0039	0.0029	0.0031	0.0027
			2.007	0.0028	0.0062	0.0065	0.0055
		1.002	1.002	0.0057	0.0036	0.0035	0.0038
			2.007	0.0040	0.0025	0.0025	0.0023
80	40	0.502	1.002	0.0044	0.0063	0.0066	0.0058
			2.007	0.0337	0.0198	0.0204	0.0182
		1.002	1.002	0.0034	0.0023	0.0024	0.0024
			2.007	0.0031	0.0027	0.0028	0.0025
	60	0.502	1.002	0.0021	0.0019	0.0020	0.0019
			2.007	0.0022	0.0039	0.0040	0.0036
		1.002	1.002	0.0032	0.0024	0.0035	0.0025
			2.007	0.0021	0.0015	0.0015	0.0014

Table 4. The mean square errors (MSEs) of the MLE and Bayes methods for R(t) under squared error (SL) and LINEX loss functions ($LL(\rho)$) with ($\rho = \pm 2$) at $t = 0.8$ based progressive Type-II censoring from fuzzy lifetime data for different choices n, m, α and β.

Approaches				MLE	MSE (Tierney-Kadane's)		
n	m	α	β	MSE	SL	LL(-2)	LL(2)
20	10	0.502	1.002	0.0116	0.0231	0.0258	0.0187
			2.007	0.0416	0.0439	0.0457	0.0351
		1.002	1.002	0.0142	0.0133	0.0151	0.0107
			2.007	0.0145	0.0236	0.0248	0.0182
	15	0.502	1.002	0.0064	0.0077	0.0088	0.0066
			2.007	0.0079	0.0124	0.0133	0.0106
		1.002	1.002	0.0092	0.0061	0.0069	0.0053
			2.007	0.0096	0.0093	0.0099	0.0077
40	20	0.502	1.002	0.0061	0.0113	0.0124	0.0098
			2.007	0.0384	0.0275	0.0285	0.0236
		1.002	1.002	0.0065	0.0051	0.0057	0.0045
			2.007	0.0030	0.0083	0.0090	0.0071
	30	0.502	1.002	0.0029	0.0036	0.0039	0.0032
			2.007	0.0041	0.0069	0.0073	0.0063
		1.002	1.002	0.0044	0.0029	0.0031	0.0027
			2.007	0.0017	0.0033	0.0035	0.0029
80	40	0.502	1.002	0.0042	0.0068	0.0072	0.0063
			2.007	0.0342	0.0154	0.0158	0.0139
		1.002	1.002	0.0033	0.0133	0.0027	0.0024
			2.007	0.0014	0.0025	0.0027	0.0023
	60	0.502	1.002	0.0016	0.0020	0.0021	0.0024
			2.007	0.0021	0.0040	0.0041	0.0038
		1.002	1.002	0.0023	0.0017	0.0017	0.0017
			2.007	0.0008	0.0012	0.0012	0.0011

Table 5. The mean square errors (MSEs) of the MLE and Bayes methods for $H(t)$ under squared error (SL) and LINEX loss functions ($LL(\rho)$) with ($\rho = \pm 2$) at $t = 0.5$ based progressive Type-II censoring from fuzzy lifetime data for different choices n, m, α and β.

Approaches				MLE	MSE (Tierney-Kadane's)		
n	m	α	β	MSE	SL	LL(-2)	LL(2)
20	10	0.502	1.002	0.4584	0.3636	0.3265	0.6452
			2.007	4.3844	3.0097	4.0456	4.5712
		1.002	1.002	1.5659	0.2536	0.1869	0.5021
			2.007	3.4898	1.7801	3.2633	4.2112
	15	0.502	1.002	0.4881	0.1998	0.1858	0.3712
			2.007	2.6861	2.0000	2.6482	3.3178
		1.002	1.002	0.4772	0.1437	0.1226	0.2477
			2.007	1.6590	1.0739	2.5442	2.8597
40	20	0.502	1.002	0.2984	0.1629	0.1813	0.3400
			2.007	4.0245	1.8857	4.0332	3.6310
		1.002	1.002	0.2701	0.1113	0.1037	0.1911
			2.007	2.0550	0.7449	0.9726	2.3905
	30	0.502	1.002	0.0931	0.0975	0.1061	0.1860
			2.007	1.2049	1.3583	2.7367	2.5313
		1.002	1.002	0.1392	0.0748	0.0686	0.0888
			2.007	0.8166	0.4734	0.5523	1.4968
80	40	0.502	1.002	0.1007	0.0889	0.1065	0.1640
			2.007	1.6463	0.9283	3.8515	2.6405
		1.002	1.002	0.0981	0.0625	0.0567	0.0665
			2.007	0.7020	0.3321	0.4429	0.9776
	60	0.502	1.002	0.0466	0.0536	0.0639	0.0952
			2.007	0.8933	0.8554	2.6449	1.8555
		1.002	1.002	0.0560	0.0418	0.0374	0.0370
			2.007	0.3324	0.2414	0.3040	0.6769

Table 6. The mean square errors (MSEs) of the MLE and Bayes methods for $H(t)$ under squared error (SL) and LINEX loss functions $(LL(\rho))$ with ($\rho = \pm 2$) at $t = 0.8$ based progressive Type-II censoring from fuzzy lifetime data for different choices n, m, α and β.

Approaches				MLE	MSE (Tierney-Kadane's)		
n	m	α	β	MSE	SL	LL(-2)	LL(2)
20	10	0.502	1.002	0.6376	0.3902	0.7203	0.7106
			2.007	4.5816	3.0825	4.5881	4.6359
		1.002	1.002	10.3341	0.6749	0.8033	1.5500
			2.007	19.0500	4.4074	14.3104	10.6875
	15	0.502	1.002	0.3269	0.2401	0.4773	0.4530
			2.007	3.1610	2.2405	3.4228	3.6003
		1.002	1.002	2.2292	0.4079	0.4218	0.9209
			2.007	10.8102	2.9026	8.7467	8.3031
40	20	0.502	1.002	0.4980	0.1774	0.5567	0.3915
			2.007	4.2418	1.8960	4.3163	3.7654
		1.002	1.002	0.9752	0.3006	0.6197	0.7183
			2.007	7.5314	1.8716	4.6019	7.0746
	30	0.502	1.002	0.1374	0.1238	0.1552	0.2495
			2.007	3.0700	1.5627	3.2681	2.8737
		1.002	1.002	0.4180	0.2071	0.3111	0.3921
			2.007	2.7883	1.3552	3.3865	5.2201
80	40	0.502	1.002	0.1519	0.1042	0.1312	0.1870
			2.007	3.8684	0.9772	4.1032	2.7846
		1.002	1.002	0.1449	0.1685	0.7196	0.2594
			2.007	2.4565	0.9100	3.3829	3.4574
	60	0.502	1.002	0.0668	0.0675	0.0944	0.0136
			2.007	2.8823	1.0049	3.1382	2.1960
		1.002	1.002	0.3025	0.1102	0.1226	0.0159
			2.007	1.1071	0.7362	1.6458	2.8137

5 Application Examples

5.1 Vinyl chloride data application

The first data set represents the Vinyl chloride, a known human carcinogen, exposure to this compound should be avoided as much as possible, and its level should be kept as low as technically possible It is known that the concentration of vinyl chloride in drinking water of 0.5 mg/liter is being associated with an increased risk of liver and Brain tumors for exposure beginning at adulthood and will double cancer risk for continuous exposure from birth. Therefore, we consider the data set used by [55] which represents 34 data points in mg/L from the vinyl chloride obtained from clean upgrade monitoring wells as:
$5.1, 1.2, 1.3, 0.6, 0.5, 2.4, 0.5, 1.1, 8.0, 0.8, 0.4, 0.6, 0.9, 0.4, 2.0, 0.5, 5.3, 3.2, 2.7, 2.9, 2.5, 2.3, 1.0, 0.2, 0.1, 0.1, 1.8, 0.9, 2.0, 4.0, 6.8, 1.2, 0.4, 0.2$. We found the WED was a good fit for this data set as shown in [56] studied the concentration of the vinyl chloride in the water of these wells based on this data set by finding estimates of the parameters, which represent the scale and shape of the concentration using WED. We assume that imprecision of the failure times of the concentration of the vinyl chloride in the water of these wells based on this data set in the form of fuzzy numbers, $\tilde{x}_i = (a_i, x_i, b_i)$, where $a_i = 0.05x_i$ and $b_i = 0.03x_i$ with the membership functions

$$\mu_{\tilde{x}}(x) = \begin{cases} \frac{x-(x_i-a_i)}{a_i} & x_i - a_i \le x < x_i \\ \frac{x_i+b_i-x}{b_i} & x_i \le x < x_i + b_i, \end{cases} \qquad i = 1, \ldots, n \tag{21}$$

we determine the average and mean square errors of the concentration of the vinyl chloride in the water of these wells based on progressive Type-II censoring scheme from fuzzy lifetime data. Table (7) shows the ML and Bayes estimators for $\alpha, \beta, R(t)$, and $H(t)$ of WED under progressive Type-II censoring scheme in the form of fuzzy numbers. For reliability and hazard characteristics, We take the time t = 0.5 and 1. For Bayesian estimation, informative prior with hyper-parameter are taken to be $a = 2, b = 3$ and $c = 4, d = 2$. The shape parameter ρ for LINEX loss function was set to be $\rho = \pm 2$. From Figures (1, 2), the pdf of the marginal posterior distributions of α and β are plotted under progressive Type-II censoring scheme with uncensored levels m equal $[3n/4]$ based on the fuzzy lifetime data. We conclude the Bayesian densities for α and β are positively skewed with right tail and this means that the vinyl chloride concentration will decrease with increasing time which ensures the simulations results.

Table 7. The estimates $(\hat{\theta}_o)$ and mean square errors (MSEs) of the MLE and Bayes methods for $\alpha, \beta, R(t)$, and $H(t)$ at $t = 0.5, 1$ under squared error Loss (SL) and LINEX loss functions $(LL(\rho))$ with $(\rho = \pm 2)$ at the hyper parameters $(A = 2, B = 3, C = 4, D = 2)$ based on progressive Type-II censoring from fuzzy Vinyl Chloride Data.

Approaches		MLE	Tierney-Kadane's		
m	*Par.*	$\hat{\theta}_{ML}$	$\hat{\theta}_{SL}$	$\hat{\theta}_{LL(-2)}$	$\hat{\theta}_{LL(2)}$
n/2	α	6.898E-01 (2.36-02)	6.491E-01 (1.95E-02)	6.608E-01 (2.21E-02)	6.385E-01 (1.74E-02)
	β	3.453E-01 (1.85E-04)	4.179E-01 (1.04E-03)	4.277E-01 (1.29E-03)	4.175E-01 (1.42E-03)
	R(t = 0.5)	7.434E-01 (5.74E-04)	6.891E-01 (1.70E-04)	6.935E-01 (1.99E-04)	6.835E-01 (1.34E-04)
	R(t = 1)	5.525E-01 (1.70E-04)	4.943E-01 (6.57E-04)	5.006E-01 (5.60E-04)	4.879E-01 (7.48E-04)
	H(t = 0.5)	5.489E-01 (3.61E-03)	6.724E-01 (1.11E-02)	6.672E-01 (1.04E-01)	6.254E-01 (8.23E-02)
	H(t = 1)	6.474E-01 (2.69E-02)	7.588E-01 (4.71E-02)	7.597E-01 (2.24E-01)	7.003E-01 (1.80E-01)
3n/4	α	6.317E-01 (9.10E-03)	5.917E-01 (6.74E-03)	5.984E-01 (7.42E-03)	5.853E-01 (6.17E-03)
	β	3.185E-01 (1.74E-04)	3.797E-01 (3.56E-05)	3.867E-01 (2.54E-05)	3.726E-01 (5.26E-05)
	R(t = 0.5)	7.492E-01 (8.84E-04)	6.994E-01 (5.47E-04)	7.032E-01 (5.68E-04)	6.947E-01 (5.21E-04)
	R(t = 1)	5.786E-01 (1.68E-04)	5.258E-01 (3.47E-05)	5.310E-01 (4.44E-05)	5.205E-01 (2.77E-05)
	H(t = 0.5)	4.952E-01 (4.03E-05)	5.915E-01 (5.98E-04)	5.853E-01 (2.27E-02)	5.595E-01 (1.65E-02)
	H(t = 1)	5.469E-01 (4.02E-03)	6.223E-01 (6.48E-03)	6.149E-01 (7.94E-02)	5.889E-01 (6.84E-02)

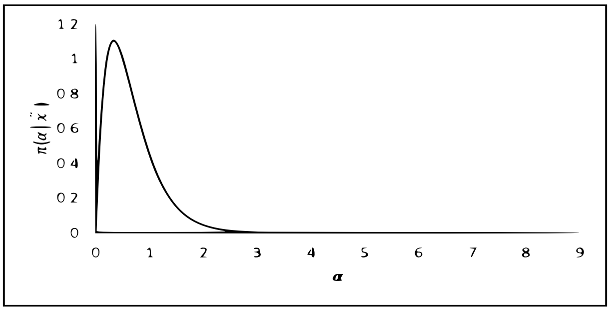

Fig. 1: Plot the posterior density function $\pi(\alpha|\tilde{x})$for fuzzy Vinyl Chloride Data.

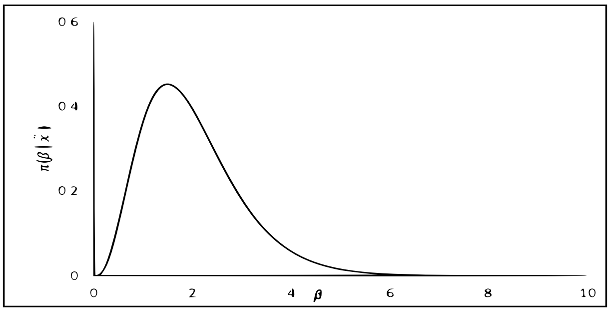

Fig. 2: Plot the posterior density function $\pi(\beta|\tilde{x})$ for fuzzy Vinyl Chloride Data.

5.2 COVID-19 data application

The second data set, we contribute modestly to the subject by applying this WED to analyze the daily data set of confirmed deaths for COVID-19 in Egypt from December 22, 2020, to February 16, 2021. This data set obtained from the following email address: http://covid.gov.Eg/Coronatracker.com/Country/Egypt. It is given as follows: 37, 42, 51, 49, 43, 53, 61, 54, 56, 55, 56, 54, 64, 58, 55, 57, 54, 56, 57, 55, 52, 55, 58, 59, 52, 54, 56, 55, 58, 51, 54, 52, 49, 57, 53, 55, 48, 54, 48, 46, 53, 44, 47, 53, 52, 48, 44, 47, 48, 52, 53, 53, 42, 36, 59, 56, 51.

We found the WED was a good fit for this data set shown in [56]. Assume that imprecision of the failure times of the COVID-19 deaths in Egypt from December 22, 2020, to February 16, 2021, based on this data set is formulated by fuzzy numbers, $\tilde{x}_i = (a_i, x_i, b_i)$, where $a_i = 0.05x_i$ and $b_i = 0.03x_i$, and use the membership functions in (21). We obtain the average and mean square errors of the α, β, $R(t)$, and $H(t)$ based on progressive Type-II censoring scheme from fuzzy COVID-19 Death Data. For reliability and hazard characteristics, we take the time t = 36 and 42. From Table (8), the MLE and Bayes estimates for β to this WED are nearly zero, ensuring that the mean square error is small and the estimated model efficiency decreases (reliability decrease and hazard increase).In Figures (3,4), the pdf of the marginal posterior distributions of α and β are plotted to study their behavior under progressive Type-II censoring scheme with uncensored levels m equal $[3n/4]$ based on the fuzzy lifetime data. We conclude the Bayesian densities for α and β are positively skewed with a right tail as shown in Figures (3, 4), which indicates the number of COVID-19 deaths in Egypt will decrease rapidly with increasing the time of the epidemic. Thus, these results indicate that WED is very efficient for modeling the COVID-19 data sets.

Table 8. The estimates $(\hat{\theta}_o)$ and mean square errors (MSEs) of the MLE and Bayes methods for $\alpha, \beta, R(t)$, and $H(t)$ at $t = 36, 42$ under squared error Loss (SL) and LINEX loss functions $(LL(\rho))$ with $(\rho = \pm 2)$ at the hyper parameters $(A = 2, B = 3, C = 4, D = 2)$ based on progressive Type-II censoring from fuzzy COVID-19 Death Data.

Approaches		MLE	Tierney-Kadane's		
m	*Par.*	$\hat{\theta}_{ML}$	$\hat{\theta}_{SL}$	$\hat{\theta}_{LL(-2)}$	$\hat{\theta}_{LL(2)}$
n/2	α	**6.839E-01 (2.45E-04)**	**6.560E-01 (1.81E-04)**	**6.756E-01 (3.26E-05)**	**6.724E-01 (1.08E-05)**
	β	**2.253E-07 (1.99E-14)**	**3.294E-07 (5.30E-014)**	**5.715E-07 (4.89E-012)**	**1.078E-06 (2.91-012)**
	R(t = 36)	**9.758E-01 (4.29E-05)**	**9.585E-01 (4.05E-04)**	**9.506E-01 (9.09E-04)**	**9.833E-01 (2.71E-06)**
	R(t = 42)	**9.102E-01 (1.44E-03)**	**7.329E-01 (2.16E-02)**	**9.146E-01 (4.11E-03)**	**9.220E-01 (7.79E-05)**
	H(t = 36)	**5.406E-03 (1.33E-06)**	**1.188E-02 (4.90E-05)**	**7.882E-03 (2.87E-05)**	**7.864E-03 (6.24E-06)**
	H(t = 42)	**6.744E-03 (8.90E-06)**	**3.091E-02 (1.00E-04)**	**2.705E-02 (9.89E-05)**	**2.696E-02 (4.96E-05)**
3n/4	α	**6.991E-01 (2.03E-07)**	**6.649E-01 (2.05E-05)**	**6.748E-01 (2.37E-05)**	**6.699E-01 (6.96E-07)**
	β	**9.020E-08 (3.87E-017)**	**9.947E-07 (1.89E-013)**	**1.753E-06 (1.06E-012)**	**8.305E-07 (3.81E-012)**
	R(t = 36)	**9.814E-01 (8.63E-07)**	**9.834E-01 (2.24E-05)**	**9.753E-01 (2.88E-05)**	**9.809E-01 (6.11E-07)**
	R(t = 42)	**9.375E-01 (1.14E-04)**	**9.686E-01 (7.88E-03)**	**9.518E-01 (7.21E-04)**	**9.372E-01 (4.12E-05)**
	H(t = 36)	**4.470E-03 (4.64E-08)**	**3.078E-03 (3.27E-06)**	**5.990E-03 (1.20E-05)**	**6.037E-03 (4.51E-07)**
	H(t = 42)	**1.719E-02 (6.01E-07)**	**1.876E-02 (4.58E-06)**	**2.204E-02 (3.04E-06)**	**2.204E-02 (4.50E-06)**

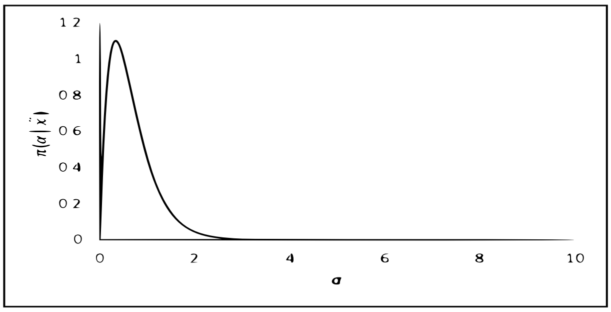

Fig. 3: Plot the posterior density function $\pi(\alpha|\tilde{x})$ for fuzzy COVID-19 Death Data.

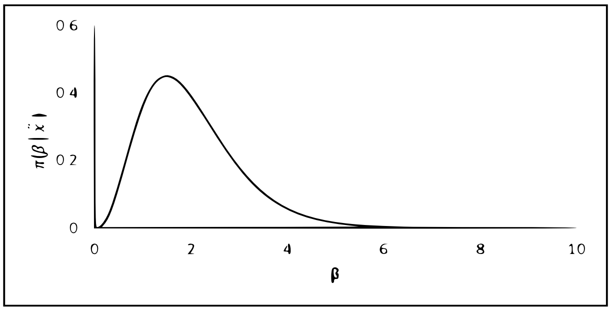

Fig. 4: Plot the posterior density function $\pi(\beta|\tilde{x})$ for fuzzy COVID-19 Death Data.

6 Conclusion

In this paper, we have considered the ML, and Bayes estimates for the parameters $\alpha, \beta, R(t)$, and $H(t)$ of the WED based on progressive Type-II censoring scheme from fuzzy lifetime data. The MLEs of the unknown parameters is computed by using the finite difference method. Also, approximate Bayes estimators under squared error and LINEX loss functions were obtained by using Tierney-Kadane's method. the approximate Bayes estimators are compared with the ML in terms of MSE by using the Monte Carlo simulation method. Based on the simulation study, we see that; Tierney-Kadane's approximation procedure gives the most precise parameter estimates as shown by MSEs in Tables 1-6. Moreover, we applied the proposed methods to analyze real data applications including the COVID-19 pandemic, which concluded that the number of COVID-19 deaths in Egypt is declining in the next few months.

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