Detection Probability Maximization Scheme in Integrated Sensing and Communication Systems

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***Abstract*—Dual functional radar communication (DFRC) is a promising approach that provides a viable solution for the problem spectrum sharing between communication and radar applications. This paper studies a DFRC system with multiple communication users (CUs) and a radar target. The goal is devise beamforming vectors at the DFRC transmitter in such a way that the radar received signal to noise ratio (SNR) is maximized while the minimum data rate requirements of the individual CUs are satisfied. Even though the formulated optimization problem is non-convex, it is shown that it can be solved optimally through semi-definite relaxation (SDR). Also, it is observed that there is no need to transmit separate probing signal for the radar detection.**

***Index Terms*—Dual functional radar communication, detection probability, integrated sensing and communication, resource allocation.**

I. Introduction

Recently, the communication systems has been integrated with many types of other systems to broaden the capabilities of existing wireless communication infrastructure. More general examples of such integration include mobile computing, sensing, localization etc. In the same vein, research efforts are put to explore the possibilities of joining wireless communications with radar systems. These efforts lead to emergence of very interesting research field which is now known as integrated sensing and communication (ISAC) [1], [2].

It is expected that in 6G wireless networks sensing as a service will play a vital role [3]. Mainly, it is anticipated that the communication performance can be greatly improved if the information about destination detection, destination location and movement can be obtained [3]. With this information at hand, the access point can direct its beam towards the destination to enhance the communication performance. Therefore, dual functional radar communication (DFRC) systems deals with the possibility of using the same access point for transmission information to communication users (CUs) and detecting radar targets. Thus, resulting in integration gain and coordination gain [3].

With the sensing functionality integrated with the communication systems, the performance metrics also differ from those of the conventional communication systems. Most commonly used performance metrics for sensing, localization are based on Crámer-Rao bound (CRB) and detection probability. Specifically, for localization purposes, the goal is to devise the transmission schemes such that the performance is as close to the CRB as possible. Similarly, for detection purposes, the goal is to allocate transmission resources in such a way that the detection probability is maximized.

In terms of localization performance, recent works [4]–[8] provided CRB minimization schemes under different system setups. In [4], a hybrid approach with known/unknown placements of multiple sensors is presented to estimate the location of multiple targets. The design of various detectors based on the minimization of the CRB is presented in [5]. The works in [4], [5] do not consider the joint operation of sensing and communication. In terms of ISAC, [6] provides a CRB minimization scheme. Specifically, CRB is used as a performance metric of target estimation, and then a CRB minimization beamforming design is proposed which guarantees a pre-defined level of signal-to-interference-plus-noise ratio (SINR) for each CU. This work is further extended in [7] to design an energy efficient ISAC system with the help of antenna selection at the DFRC. In particular, l_0 norm is used to select the number of active antennas to reduce the energy consumption at the DFRC. More recently, an over-the-air computation aided DFRC is investigated in [8], and a beamforming design is proposed to encompass different performance aspects of the integrated sensing, communication and computation system. Interesting results related to the beamforming vectors are derived based on the shared and separated designs between computing, sensing and communication systems.

With regard to the detection probability maximization, it is well known that the probability of detection is an increasing function of the radar received signal to noise ratio (SNR) [9]. Much research effort is devoted to guarantee the constant modulus and similarity properties of the radar waveforms which include the sequential optimization algorithms (SOAs) in [10], the successive quadratically constrained quadratic programming (QCQP) in [11], the block coordinate descent (BCD) framework in [12], and the general majorizationminimization (MM) framework in [13]. In works [9]–[13], the authors make assumption about the apriori information about the target, which is generally impractical to obtain. Hence, an efficient beamforming design is proposed in [14] where no such assumption is made. It is concluded in [14] that in

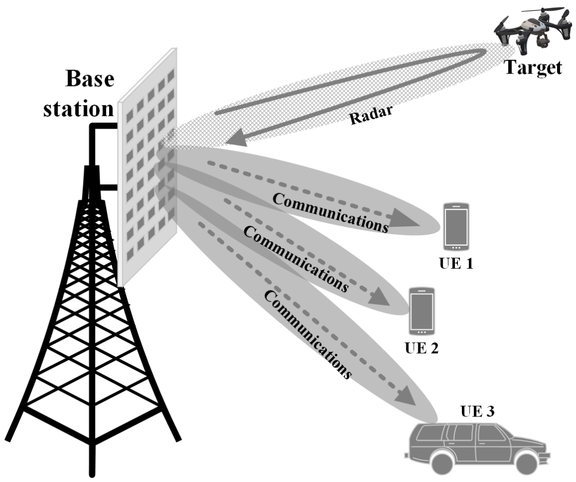

Fig. 1. The presumed multiuser donwlink integrated communications and sensing scenario with $K = 3$ CUs.

the presence of clutter, the use of dedicated probing signal can improve the detection performance of the radar. In order to reap the benefits of orthogonal frequency division multiple (OFDM) access in ISAC systems, a joint OFDM waveform design is proposed in [15] to increase the reflected signal power under interference and auto-correlation constraints. In order to reduce the complexity of this work, a low complexity design is proposed in [16] for single CU.

In this paper, we consider a DFRC with multiple CUs and a target. Then, the aim is to maximize the detection probability of the radar whilst satisfying the minimum rate requirements of the CUs. The maximization of the radar detection is achieved by properly designing the transmit beamforming vectors for the CUs. Specifically, it is observed that in considered system there is no need to transmit probing signal for target. This observation on the one hand reduces the complexity of the optimization algorithm while on the other hand reduces the feedback requirements of the overall system since otherwise the information related to probing signal is required at the CUs to perform interference cancellation.

II. System Model and problem formulation

The considered system model comprises of a multi-antenna DFRC, K single antenna CUs, and a radar target located at an angle θ from the DFRC. The scenario can be illustrated as Fig. 1.

The DFRC uses N_t antennas for transmission of information to the CUs and N_r antennas for reception of reflection from the target to perform detection of the target. The beamforming vector, and information symbol for k-th CU are denoted by $\mathbf{u}_k \in \mathbb{C}^{N_t \times 1}$, s_k with $E[|s_k|^2] = 1$, respectively. Furthermore, we assume that DFRC employs a probing signal with beamforming vector $\mathbf{v} \in \mathbb{C}^{N_t \times 1}$. Hence, the overall transmitted symbol from the DFRC is $\mathbf{x} = \sum_{k=1}^{K} \mathbf{u}_k s_k + \mathbf{v} s_0$, where s_0 with $E[|s_0|^2] = 1$ is the symbol for probing signal. The channel between the DFRC and k-th CU is represented by $\mathbf{h}_k \in \mathbb{C}^{N_t \times 1}$. Next, we present the performance metrics for communication and radar systems, respectively.

A. Communication system performance metric

With the above assumptions, the received signal at the k-th user can be written as

$$y_k = \mathbf{h}_k^H \mathbf{u}_k s_k + \sum_{i=1, i\neq k}^{K} \mathbf{h}_k^H \mathbf{u}_i s_i + \mathbf{h}_k^H \mathbf{v} s_0 + \omega_k, \tag{1}$$

where $\omega_k \in \mathbb{C}$ is the additive white Gaussian noise (AWGN) at CU k with variance N_0. Therefore, the corresponding signal to interference plus noise ratio (SINR) is given as

$$\gamma_k^I = \frac{|\mathbf{h}_k^H \mathbf{u}_k|^2}{\sum_{i=1, i\neq k}^{K} |\mathbf{h}_k^H \mathbf{u}_i|^2 + |\mathbf{h}_k^H \mathbf{v}|^2 + N_o}. \tag{2}$$

In order to have a satisfactory communication performance, the k-th CU requires its SINR to be at least Γ_k. Mathematically, it can be represented as

$$\gamma_k \geq \Gamma_k. \tag{3}$$

B. Radar system performance metric

In this subsection, we present the radar system performance metric for two possible scenarios. In the first possibility, we assume that the reflected signals from the clutter components can be perfectly removed from the received signal. For this scenario the radar performance metric is radar SNR. In the second scenario, we assume that the clutter component can not be removed from the radar received signal. For this scenario, the radar performance metric is radar signal-to-clutter-plus-noise-ratio (SCNR).

1) Radar SNR with perfect clutter removal: In the considered system model, the received signal at the radar after the clutter removal can be written as

$$\mathbf{r}^{cr} = \alpha_0 \mathbf{a}_r(\theta_0) \mathbf{a}_t(\theta_0)^H \mathbf{x} + \mathbf{n} = \alpha \mathbf{A}(\theta_0) \mathbf{x} + \mathbf{n}, \tag{4}$$

where $\mathbf{A}(\theta) = \mathbf{a}_r(\theta)\mathbf{a}_t(\theta)^H \in \mathbb{C}^{N_r \times N_t}$, $\alpha \in \mathbb{C}$ is complex channel between target and radar which is independently distributed from $\mathbf{h}_k$'s, $\mathbf{n} \in \mathbb{C}^{N_r \times 1}$ is AWGN with $\mathbf{n} \sim \mathcal{CN}(\mathbf{0}, \mathbf{I})$, $\mathbf{a}_j(\theta) \in \mathbb{C}^{N_j \times 1}$ is the transmit are receive steering vectors for $j \in \{t, r\}$, respectively. The radar performs receive beamforming with vector $\mathbf{w}$ on the received signal, then the output of the radar receiver is given as

$$y_r = \mathbf{w}^H \mathbf{r} = \alpha \mathbf{w}^H \mathbf{A} \mathbf{x} + \mathbf{w}^H \mathbf{n}. \tag{5}$$

Subsequently, the radar signal to noise ratio can be written as

$$\gamma_r^{cr} = \frac{|\alpha \mathbf{w}^H \mathbf{A} \mathbf{x}|^2}{\mathbf{w}^H \mathbf{w}}. \tag{6}$$

It can be easily verified that the optimal receive beamforming vector, that maximizes the radar SNR, is $\mathbf{w}^* = \frac{\mathbf{A}(\theta)\mathbf{x}}{\mathbf{x}^H \mathbf{A}^H(\theta) \mathbf{A}(\theta) \mathbf{x}}$ the corresponding SNR is given as

$$\gamma_r^{cr}(\mathbf{w}^*) = |\alpha|^2 \mathbf{x}^H \mathbf{A}^H(\theta) \mathbf{A}(\theta) \mathbf{x}, \tag{7}$$

and the corresponding average radar SNR is given as

$$\bar{\gamma}_r^{cr} = |\alpha|^2 E\left[\mathbf{x}^H \mathbf{A}^H(\theta)\mathbf{A}(\theta)\mathbf{x}\right] = \sum_{k=1}^{K} \mathbf{u}_k^H \mathbf{\Phi}(\theta)\mathbf{u}_k + \mathbf{v}^H \mathbf{\Phi}(\theta)\mathbf{v}, \quad (8)$$

where $\mathbf{\Phi}(\theta) = |\alpha|^2 \mathbf{A}^H(\theta)\mathbf{A}(\theta)$ with largest eigenvalue ζ_{max} and corresponding eigenvector ϕ. It is widely known that the detection probability of the radar is directly proportional to the radar received SNR.

2) Radar SCNR without clutter removal: For this scenario, the radar received signal with a total of J clutter components can be written as

$$\mathbf{r} = \alpha_0 \mathbf{a}_r(\theta_0)\mathbf{a}_t(\theta_0)^H \mathbf{x} + \sum_{j=1}^{J} \alpha_i \mathbf{a}_r(\theta_j)\mathbf{a}_t(\theta_j)^H \mathbf{x} + \mathbf{n} \quad (9)$$

$$= \alpha_0 \mathbf{A}(\theta_0)\mathbf{x} + \sum_{j=1}^{J} \alpha_j \mathbf{A}(\theta_j)\mathbf{x} + \mathbf{n}. \quad (10)$$

Then, after performing the received combining, the radar SCNR can be written as

$$\gamma_r(\mathbf{w}) = \frac{|\alpha \mathbf{w}^H \mathbf{A}\mathbf{x}|^2}{\mathbf{w}^H \left(\sum_{j=1}^{J} |\alpha_j|^2 \mathbf{A}(\theta_j)\mathbf{x}\mathbf{x}^H \mathbf{A}^H(\theta_j) + \mathbf{I}\right) \mathbf{w}}. \quad (11)$$

The optimal value of $\mathbf{w}$ which maximizes (11) is given as

$$\mathbf{w}^* = \frac{(\mathbf{W}(\mathbf{x}))^{-1} \mathbf{A}(\theta_0)\mathbf{x}}{\mathbf{x}^H \mathbf{A}^H(\theta_0) (\mathbf{M}(\mathbf{x}) + \mathbf{I})^{-1} \mathbf{A}(\theta_0)\mathbf{x}}, \quad (12)$$

where $\mathbf{W}(\mathbf{x}) = \sum_{j=1}^{J} |\alpha_j|^2 \mathbf{A}(\theta_j)\mathbf{x}\mathbf{x}^H \mathbf{A}^H(\theta_j) + \mathbf{I}$. Putting (12) into (11) we get

$$\gamma_r(\mathbf{w}^*) = |\alpha_0|^2 \mathbf{x}^H \mathbf{A}^H(\theta_0)\mathbf{W}^{-1}(\mathbf{x})\mathbf{A}(\theta_0)\mathbf{x}, \quad (13)$$

with the corresponding average SCNR given as

$$\bar{\gamma}_r(\mathbf{w}^*) = |\alpha_0|^2 \sum_{k=1}^{K} \mathbf{u}_k^H \mathbf{A}^H(\theta_0)\mathbf{W}^{-1}(\mathbf{x})\mathbf{A}(\theta_0) \sum_{k=1}^{K} \mathbf{u}_k + |\alpha_0|^2 \mathbf{v}^H \mathbf{A}^H(\theta_0)\mathbf{W}^{-1}(\mathbf{x})\mathbf{A}(\theta_0)\mathbf{v}. \quad (14)$$

It is clear from (3), (8), (14) that the communication and radar performance depend on the beamforming vectors $\mathbf{u}_k, \mathbf{v}$. In the next subsection, we formulate an optimization problem to find the optimal values of $\mathbf{u}_k, \mathbf{v}$.

C. Problem formulation

In this paper, we are interested in maximizing the detection probability of the radar. As noted above, the radar detection probability is directly proportional to the radar SNR and SCNR depending on the clutter removal. Therefore, our aim is to maximize the radar SNR, SCNR whilst satisfying the communication requirements of the CUs. Overall, the mathematical formulation of the optimization problem for finding the appropriate beamforming vectors for perfect clutter removal scenario is given as follows

P1

$$\begin{aligned} \underset{\mathbf{u}_k, \mathbf{v}}{\text{maximize}} \quad & \sum_{k=1}^{K} \mathbf{u}_k^H \mathbf{\Phi}(\theta)\mathbf{u}_k + \mathbf{v}^H \mathbf{\Phi}(\theta)\mathbf{v} \\ \text{subject to} \quad & C1: \frac{|\mathbf{h}_k^H \mathbf{u}_k|^2}{\sum_{i=1, i\neq k}^{K} |\mathbf{h}_k^H \mathbf{u}_i|^2 + |\mathbf{h}_k^H \mathbf{v}|^2 + N_o} \geq \Gamma_k, \\ & C2: \sum_{k=1}^{K} |\mathbf{u}_k|^2 + |\mathbf{v}|^2 \leq P_{max}, \end{aligned} \quad (15)$$

and without clutter removal

P2

$$\begin{aligned} \underset{\mathbf{u}_k, \mathbf{v}}{\text{maximize}} \quad & |\alpha_0|^2 \sum_{k=1}^{K} \mathbf{u}_k^H \mathbf{A}^H \mathbf{W}^{-1}(\mathbf{x})\mathbf{A}\mathbf{u}_k + |\alpha_0|^2 \mathbf{v}^H \mathbf{A}^H \mathbf{W}^{-1}(\mathbf{x})\mathbf{A}\mathbf{v}. \\ \text{subject to} \quad & C1, C2. \end{aligned} \quad (16)$$

In **P1**, **P2** the objective is to maximize the average SNR, SCNR of the radar system. The constraints $C1$ guarantee that the data rate requirements of the CUs are met and $C2$ makes sure that the total transmitted power is no more than the maximum allowed transmit power. For **P1**, the objective is a convex function and the constraints $C1$ are non-convex, **P1** is a non-convex optimization problem and hence difficult to solve. For **P2**, the constraints are non-convex for a multiple CU scenario.

In the following sections, we discuss the proposed solution strategies for problems **P1**, **P2**.

III. Proposed solution

In this section, we present SDR based solution for the optimization problem **P1**, **P2**. Although both problems **P1**, **P2** are non-convex, in the following we show that problem **P1** can be solved optimally and the global optimal solution is presented in closed form. On the other hand, an iterative algorithm is used to find a suboptimal solution for problem **P2**.

A. Probposed solution for problem *P1*

In order to solve **P1**, we use the semi-definite relaxation (SDR) technique. In this regard, we introduce following variables $\mathbf{U}_k = \mathbf{u}_k \mathbf{u}_k^H, \mathbf{V} = \mathbf{v}\mathbf{v}^H, \mathbf{H}_k = \mathbf{h}_k \mathbf{h}_k^H$. Then, **P1** can be equivalently written as

P2

$$\begin{aligned}\underset{\mathbf{U}_k,\mathbf{V}}{\text{maximize}}\quad & \sum_{k=1}^{K}\text{Tr}(\mathbf{\Phi}\mathbf{U}_k)+\text{Tr}(\mathbf{\Phi}\mathbf{V})\\ \text{subject to}\quad & \tilde{C}1:\ \frac{\text{Tr}(\mathbf{H}_k\mathbf{U}_k)}{\text{Tr}\left(\mathbf{H}_k\left(\sum_{i=1,i\neq k}^{K}\mathbf{U}_i+\mathbf{V}\right)\right)+N_o}\geq\Gamma_k,\\ & \tilde{C}2:\ \sum_{k=1}^{K}\text{Tr}(\mathbf{U}_k)+\text{Tr}(\mathbf{V})\leq P_{max},\\ & \tilde{C}3:\ \mathbf{U}_k\succeq\mathbf{0},\\ & \tilde{C}4:\ \mathbf{V}\succeq\mathbf{0},\\ & \tilde{C}5:\ \text{rank}(\mathbf{U}_k)\leq 1,\\ & \tilde{C}6:\ \text{rank}(\mathbf{V})\leq 1,\end{aligned}\tag{17}$$

where for simplicity of notation we have used $\mathbf{\Phi}(\theta)=\mathbf{\Phi}$. Since constraints $\tilde{C}5$, $\tilde{C}6$ are non-convex, **P2** is non-convex. However, if we remove the constraints $\tilde{C}5$, $\tilde{C}6$, then the relaxed problem is a semi-definite program (SDP), whose solution can be easily found through some off the shelf tools such as CVX.

After relaxing the constraints $\tilde{C}5$, $\tilde{C}6$ in **P2**, we denote the new problem as **P2-rel** given as

P2-rel

$$\begin{aligned}\underset{\mathbf{U}_k,\mathbf{V}}{\text{maximize}}\quad & \sum_{k=1}^{K}\text{Tr}(\mathbf{\Phi}\mathbf{U}_k)+\text{Tr}(\mathbf{\Phi}\mathbf{V})\\ \text{subject to}\quad & \tilde{C}1:\ \frac{\text{Tr}(\mathbf{H}_k\mathbf{U}_k)}{\text{Tr}\left(\mathbf{H}_k\left(\sum_{i=1,i\neq k}^{K}\mathbf{U}_i+\mathbf{V}\right)\right)+N_o}\geq\Gamma_k,\\ & \tilde{C}2:\ \sum_{k=1}^{K}\text{Tr}(\mathbf{U}_k)+\text{Tr}(\mathbf{V})\leq P_{max},\\ & \tilde{C}3:\ \mathbf{U}_k\succeq\mathbf{0},\\ & \tilde{C}4:\ \mathbf{V}\succeq\mathbf{0}\end{aligned}\tag{18}$$

Then, Lemma 1 below provides a useful result about **P2-rel**.

Lemma 1. *The optimal solution of **P2-rel** has following properties*

- $\text{rank}(\mathbf{U}_k^*)=1$,
- $\mathbf{V}^*=\mathbf{0}$.

Proof. Before we proceed with the proof, let us consider the following optimization problem

P3

$$\begin{aligned}\underset{\mathbf{X}}{\text{maximize}}\quad & \text{Tr}(\mathbf{A}\mathbf{X})\\ \text{subject to}\quad & \tilde{C}7:\ \text{Tr}(\mathbf{X})\leq S\end{aligned}\tag{19}$$

where $\mathbf{A}$ is a positive semi-definite matrix. It can be easily deduced that the optimal value of **P3** is $\lambda_{max}S$, where λ_{max} is the largest eigenvalue of the matrix $\mathbf{A}$. Also, it is clear that the optimal value of problem **P2-rel** cannot be greater than $\lambda_{max}S$ since **P3** is a relaxation of **P2-rel**, which can be easily verified by replacing $\mathbf{X}=\sum_{k=1}^{K}\mathbf{U}_k+\mathbf{V}$, $S=P_{max}$ and removing the SINR constraints from **P2-rel**. We will use this result related to **P3** later in the proof.

Next, we note that **P2-rel** is a convex SDP, and hence the duality gap is zero. Therefore, we consider the Lagrangian of **P2-rel**, which is given as

$$\mathcal{L}(\mathbf{U}_k,\mathbf{V},\eta_k,\omega)=\omega P_{max}+\text{Tr}\left(\sum_{k=1}^{K}\mathbf{U}_k\mathbf{M}_k+\mathbf{T}\mathbf{V}\right)-N_0\sum_{k=1}^{K}\eta_k,\tag{20}$$

where η_k is the dual variable associated with the k-th SINR constraint, and ω is the dual variable corresponding to the sum power constraint. Furthermore, $\mathbf{M}_k,\mathbf{T}$ are defined as

$$\mathbf{T}=\mathbf{\Phi}-\omega\mathbf{I}-\sum_{k=1}^{K}\eta_k\mathbf{H}_k\tag{21}$$

$$\mathbf{M}_k=\mathbf{\Phi}-\omega\mathbf{I}+\frac{\eta_k}{\Gamma_k}\mathbf{H}_k-\sum_{i=1,i\neq k}^{K}\eta_i\mathbf{H}_i.\tag{22}$$

Hence, the dual problem of **P2-rel** can be written as

$$\begin{aligned}\underset{\omega,\eta_k}{\text{minimize}}\quad & \max_{\mathbf{U}_k,\mathbf{V}}\mathcal{L}\left(\mathbf{U}_k,\mathbf{V},\eta_k,\omega\right)\\ \text{subject to}\quad & \tilde{C}8:\ \omega\geq 0,\eta_k\geq 0,\end{aligned}\tag{23}$$

with the corresponding complementary conditions given as

$$\text{Tr}(\mathbf{M}_k^*\mathbf{U}_k^*)=0,\qquad \text{Tr}(\mathbf{T}^*\mathbf{V}^*)=0,\tag{24}$$

where $\mathbf{M}_k^*,\mathbf{T}^*$ are the corresponding values for η_k^*,ω^*. The problem (16) can be written as

P4

$$\begin{aligned}\underset{\omega,\eta_k}{\text{minimize}}\quad & \omega P_{max}-N_0\sum_{k=1}^{K}\eta_k\\ \text{subject to}\quad & \tilde{C}9:\ \mathbf{M}_k\preceq 0,\\ & \tilde{C}10:\ \mathbf{T}\preceq 0,\\ & \tilde{C}11:\ \omega\geq 0,\\ & \tilde{C}11:\ \eta_k\geq 0.\end{aligned}\tag{25}$$

We have following proposition for **P4** when the optimal values of all the η_k's in **P4** are zero.

Proposition 1. *If none of the $\eta_k>0$ in the optimal solution of **P4** then we can set $\mathbf{V}^*=0,\mathbf{U}_k^*=\frac{P_{max}}{\sum_{i=1}^{K}p_i}p_k\phi\phi^H$ to maximize the objective value of **P2-rel**, where p_k's are obtained from solving the following linear constraints*

$$p_k\,\text{Tr}(\mathbf{H}_k\mathbf{U}_k)\geq\Gamma_k\left(\sum_{i=1,i\neq k}^{K}p_i\,\text{Tr}(\mathbf{H}_k\mathbf{U}_i)+N_0\right),\tag{26}$$

$$\sum_{k=1}^{K}p_k\leq P_{max}.\tag{27}$$

Proof. First note that when all $\eta_k = 0$, then the optimal value of **P4** is $\zeta_{max}P_{max}$. Now if we use $\mathbf{V}^* = 0, \mathbf{U}_k^* = \frac{P_{max}}{\sum_{i=1}^{K} p_i} p_k \phi\phi^H$, we will have same objective value for **P2-rel** and the guarantee to satisfy the rate constraints and sum power are satisfied due to the choice of p_k's through (17), (18). Thus, the proposition is proved. □

Since for the special case of $\eta_k = 0, \forall k$, we have $\text{rank}(\mathbf{U}_k^*) = 1, \text{rank}(\mathbf{V}^*) = 0$, and thus the Lemma 1 is proved for this special case when all the η_k's are zero. In the rest of this paper, we assume that there is at least one $\eta_k > 0$.

Next, we consider the possibility that there exist a subset $\mathcal{K}$ such that $\eta_{\hat{k}}^* > 0, \forall \ \hat{k} \in \mathcal{K}$. From () and using the fact that $\text{rank}(\mathbf{U}_k^*) \geq 1$ due to SINR constraints, it can be easily deduced that $\text{rank}(\mathbf{M}_k^*) \leq N_t - 1$. Also, based on $\text{Tr}(\mathbf{T}^*\mathbf{V}^*) = 0$ we have that

$$\eta_{\hat{k}}^* \left(1 + \frac{1}{\Gamma_{\hat{k}}}\right) \text{Tr}(\mathbf{H}_{\hat{k}}\mathbf{V}^*) + \text{Tr}(\mathbf{T}^*\mathbf{V}^*) = \text{Tr}(\mathbf{M}_{\hat{k}}^*\mathbf{V}^*) \leq 0, \tag{28}$$

where the last inequality is a result of the fact that $\mathbf{M}_{\hat{k}}^* \preceq \mathbf{0}$ and $\mathbf{V}^* \succeq \mathbf{0}$. This means $\text{Tr}(\mathbf{H}_{\hat{k}}\mathbf{V}^*) = 0$ which implies

$$\mathbf{H}_{\hat{k}}\mathbf{V}^* = \mathbf{0}, \tag{29}$$

since both $\mathbf{H}_{\hat{k}}, \mathbf{V}^*$ are positive-semidefinite matrices. It follows that

$$(\mathbf{\Phi} - \omega^*\mathbf{I} - \sum_{k \in \mathcal{K}} \eta_k^* \mathbf{H}_k)\mathbf{V}^* = \text{Tr}(\mathbf{T}^*\mathbf{V}^*) = 0, \tag{30}$$

which means $(\mathbf{\Phi} - \omega^*\mathbf{I})\mathbf{V}^*$. Thus, $\mathbf{V}^*$ must be orthogonal to $\mathbf{H}_k$ as well as $(\mathbf{\Phi} - \omega^*\mathbf{I})$. In what follows, we make user of the following identities about rank of matrices

$$\text{rank}(\mathbf{AB}) \leq \min(\text{rank}(\mathbf{A}), \text{rank}(\mathbf{B})), \tag{31}$$
$$\text{rank}(\mathbf{A} + \mathbf{B}) \leq \text{rank}(\mathbf{A}) + \text{rank}(\mathbf{B}), \tag{32}$$
$$\text{rank}(\mathbf{A} - \mathbf{B}) \geq \text{rank}(\mathbf{A}) - \text{rank}(\mathbf{B}), \tag{33}$$
$$\text{rank}(-\mathbf{A}) = \text{rank}(\mathbf{A}), \tag{34}$$

it can be established that $\text{rank}(\mathbf{\Phi}) = 1, \text{rank}(\mathbf{H}_k) = 1, \text{rank}(\mathbf{\Phi} - \omega^*\mathbf{I}) \geq N_t - 1$. Since we have assumed that $\mathbf{H}_k$ and $\mathbf{\Phi}$ are independent, their combined dimensions are N_t. Hence, we conclude that $\mathbf{V}^* = \mathbf{0}$ since it is orthogonal to the total number of dimensions N_t. Hence, the second property in the Lemma 1 is proved for all possible cases.

To prove the first property in Lemma 1, we note that in order to satisfy (17), $\mathbf{V}^*$ must be a zero matrix. Therefore, $\mathbf{T}^*$ must be full rank, i.e. $\text{rank}(\mathbf{T}^*) = N_t$. Also, we have

$$\mathbf{T}^* = \mathbf{M}_k^* - \eta_k^* \left(1 + \frac{1}{\Gamma_k}\right) \mathbf{H}_k. \tag{35}$$

Multiplying $\mathbf{U}_k^*$ on both sides of (28) and using (17) we get

$$\mathbf{T}^*\mathbf{U}_k^* = -\eta_k^* \left(1 + \frac{1}{\Gamma_k}\right) \mathbf{H}_k\mathbf{U}_k^*, \tag{36}$$

and since $\mathbf{T}^*$ is full rank, we must have $\text{rank}(\mathbf{T}^*\mathbf{U}_k^*) = \text{rank}(\mathbf{U}_k^*)$ and hence from (29), we have

$$\text{rank}(\mathbf{U}_k^*) = \text{rank}\left(-\eta_k^* \left(1 + \frac{1}{\Gamma_k}\right) \mathbf{H}_k\mathbf{U}_k^*\right) \leq 1, \tag{37}$$

where we have used (24) and the fact that $\text{rank}(\mathbf{H}_k) = 1$. Combining (30) with the earlier observation that $\text{rank}(\mathbf{U}_k^*) \geq 1$, we conclude $\text{rank}(\mathbf{U}_k^*) = 1$. This completes the proof of the first property in Lemma 1. □

Lemma 1 not only shows that the obtained solution for **P2-rel** is the optimal solution for **P2** but also helps in reducing the complexity of finding solution by reducing the number of optimization variables through noting that $\mathbf{V}^* = \mathbf{0}$. According to Proposition 1, the notion is to avoid that part of the interference at the CUs which is caused by the probing signal since it cannot be cancelled at the CUs as they do not have the apriori information about the probing signal. Thus, the optimal transmission strategy is to adjust the beamforming for the CUs only in such a way that the radar SNR is maximized.

Based on the result of Lemma 1, in the rest of this subsection, we assume that $\mathbf{v}^* = \mathbf{0}$. Next, we use the results presented above to find a close-form solution for the optimization problem **P1**. The Lagrangian of **P1** with $\mathbf{v}^* = \mathbf{0}$ can be written as

$$\mathcal{L}(\mathbf{u}_k, \eta_k, \omega) = -\sum_{k=1}^{K} \mathbf{u}_k^H \mathbf{\Phi} \mathbf{u}_k + \omega(\sum_{k=1}^{K} |\mathbf{u}_k|^2 - P_{max}) + \sum_{k=1}^{K} \eta_k \left(\Gamma_k \sum_{i=1, i \neq k}^{K} |\mathbf{h}_k^H \mathbf{u}_i|^2 + \Gamma_k N_o - |\mathbf{h_k}^H \mathbf{u}_k|^2\right) \tag{38}$$

and the corresponding dual problem can be written as

P1-d

$$\underset{\eta_k, \omega}{\text{maximize}} \quad \min_{\mathbf{u}_k} \mathcal{L}(\mathbf{u}_k, \eta_k, \omega) \tag{39}$$

The following lemma presents a useful result for the problem **P1**, **P1-d**.

Lemma 2. *The optimal values of **P1**, **P1-d** are equal i.e. the duality gap for problem **P1** is zero.*

Proof. Let us denote the optimal value of **P1** as x^* and that of **P1-d** as y^*. It is clear that $y^* \geq x^*$ due to the fact that the optimal value of the dual of a maximization problem provides an upper bound to the optimal value of primal problem.

In order to show that $y^* \leq x^*$, note that the duality gap between SDR of **P1** and its dual is zero since its a convex optimization problem and the Slater conditions are satisfied. Hence, we must have $x_{SDRP1}^* = y_{SDRP1D}^* \geq y^*$. Also from Lemma 1 we have that SDR relaxation of **P1** is tight i.e. $x^* = x_{SDRP1}^*$. Therefore, we conclude that $x^* \geq y^*$. This completes the proof. □

(wrote on 19-4-2022)Based on Lemma 2, we can solve **P1-d** to obtain the solution of **P1** in closed-form. Before proceeding further, we present an important property of the solution for problem **P1** in following lemma.

Lemma 3. *Problem **P1** has following property.*

- *If $\mathbf{u}_k^*, \forall k \in \{1, \cdots K\}$ is the optimal solution of problem **P1** then $\mathbf{u}_k^* e^{j\theta_k}$ is also an optimal solution for problem **P1**.*

Proof. To prove this property, we note that if $\mathbf{u}_k$ is an optimal solution, then $\mathbf{u}_k^* e^{j\theta_k}$ is also an optimal solution since it does not violate any constraints of **P1** and achieves the same objective value. This means we can apply any phase rotation to $\mathbf{u}_k$ to make sure that $\mathbf{h}_k^H \mathbf{u}_k$ is a real positive value. □

With the help of Lemma 2 and Lemma 3, we can obtain the optimal solution of the problem $\min_{\mathbf{u}_k} \mathcal{L}(\mathbf{u}_k, \eta_k, \omega)$ in the following lemma.

Lemma 4. *The optimal solution of* $\min_{\mathbf{u}_k} \mathcal{L}(\mathbf{u}_k, \eta_k, \omega)$ *for any given* ω, η_k *is*

$$\mathbf{u}_k^*(\omega, \{\eta_k\}) = \sqrt{p_k} \frac{\left[\omega \mathbf{I} - \mathbf{\Phi} + \sum_{i=1, i\neq k}^K \eta_i \mathbf{h}_i \mathbf{h}_i^H\right]^{-1} \mathbf{h}_k}{\left\| \left[\omega \mathbf{I} - \mathbf{\Phi} + \sum_{i=1, i\neq k}^K \eta_i \mathbf{h}_i \mathbf{h}_i^H\right]^{-1} \mathbf{h}_k \right\|}, \tag{40}$$

where

$$p_k = \frac{\Gamma_k \mathbf{h}_k^H \left(\sum_{i=1, i\neq k}^K p_i \hat{\mathbf{u}}_i \hat{\mathbf{u}}_i^H\right) \mathbf{h}_k + \Gamma_k N_0}{|\mathbf{h}_k^H \hat{\mathbf{u}}_k|^2}, \tag{41}$$

and

$$\hat{\mathbf{u}}_k = \frac{\left[\omega \mathbf{I} - \mathbf{\Phi} + \sum_{i=1, i\neq k}^K \eta_i \mathbf{h}_i \mathbf{h}_i^H\right]^{-1} \mathbf{h}_k}{\left\| \left[\omega \mathbf{I} - \mathbf{\Phi} + \sum_{i=1, i\neq k}^K \eta_i \mathbf{h}_i \mathbf{h}_i^H\right]^{-1} \mathbf{h}_k \right\|} \tag{42}$$

Proof. Taking the derivative of (38) and putting it equal to zero we get

$$\mathbf{u}_k = \left[\omega \mathbf{I} - \mathbf{\Phi} + \sum_{i=1, i\neq k}^K \eta_i \mathbf{h}_i \mathbf{h}_i^H\right]^{-1} \eta_k \mathbf{h}_k \mathbf{h}_k^H \mathbf{u}_k. \tag{43}$$

Now according to Lemma 3, $\mathbf{h}_k^H \mathbf{u}_k$ is a scalar value. Therefore, the optimal value of $\mathbf{u}_k$ is parallel to $\left[\omega \mathbf{I} - \mathbf{\Phi} + \sum_{i=1, i\neq k}^K \eta_i \mathbf{h}_i \mathbf{h}_i^H\right]^{-1} \mathbf{h}_k$ since $\eta_k \geq 0$. Hence, we have

$$\mathbf{u}_k = \sqrt{p_k} \frac{\left[\omega \mathbf{I} - \mathbf{\Phi} + \sum_{i=1, i\neq k}^K \eta_i \mathbf{h}_i \mathbf{h}_i^H\right]^{-1} \mathbf{h}_k}{\left\| \left[\omega \mathbf{I} - \mathbf{\Phi} + \sum_{i=1, i\neq k}^K \eta_i \mathbf{h}_i \mathbf{h}_i^H\right]^{-1} \mathbf{h}_k \right\|}. \tag{44}$$

Hence, (40) is proved. In order to find the values of p_k's, first we note that none of the η_k is zero and all of the η_k's are bounded from above. This statement can be proved with the help of (41) as follows. From (41) we can conclude

$$\eta_k = \frac{N_0 \Gamma_k}{\mathbf{h}_k^H \left(\omega I - \mathbf{\Phi} + \sum_{i=1, i\neq k}^K \eta_i \mathbf{h}_i \mathbf{h}_k^H\right)^{-1} \mathbf{h}_k}. \tag{45}$$

Now recall from the discussion in the proof of Lemma 1 that $\text{rank}(\omega \mathbf{I} - \mathbf{\Phi}) \geq N_t - 1$ and since $\mathbf{h}_k$'s and steering vectors are independent, $\omega \mathbf{I} - \mathbf{\Phi} + \sum_{i=1, i\neq k}^K \eta_i \mathbf{h}_i \mathbf{h}_i^H$ is a full rank positive definite matrix. Therefore, from (43) we have $\eta_k > 0$ and bounded from above. Using this observation, and the existence of zero duality gap it is easy to see that the SINR constraints must be met with equality

$$\Gamma_k \sum_{i=1, i\neq k}^K |\mathbf{h}_k^H \mathbf{u}_i|^2 + \Gamma_k N_o = |\mathbf{h_k}^H \mathbf{u}_k|^2. \tag{46}$$

From (44) we can obtain the value of p_k's and the proof is completed. □

With the help of Lemma 4, the algorithmic complexity of finding the solution for problem **P1** is greatly reduced. This is due to the fact that for the primal problem **P1** the number of optimization variables are $2 \times N_t \times K + 2 \times N_t$. However, for solving the dual problem the search should be performed on K real variables. Also, we can use the gradient descent to find the optimal values of $\omega, \{\eta_k\}$.

*B. Proposed solution for problem **P2***

When clutter removal is not possible, the objective function becomes more complex than that of problem **P1** due to the involvement of $\mathbf{W}(\mathbf{x}) = \sum_{j=1}^J |\alpha_j|^2 \mathbf{A}(\theta_j) \mathbf{x} \mathbf{x}^H \mathbf{A}^H(\theta_j) + \mathbf{I}$. In order to address this issue, we use an iterative approach where in each iteration $\mathbf{x}$ in $\mathbf{W}(\mathbf{x})$ is replaced by optimal value of $\mathbf{x}$ in the previous iteration. Hence, in the $l+1$-th iteration we set

$$\mathbf{\Phi}' = \mathbf{W}(\mathbf{x}) = \mathbf{W}(\mathbf{x}^l) \tag{47}$$

where $\mathbf{x}^l$ is the optimal solution achieved in the l-th iteration. The iterative procedure is repeated until a sufficient convergence criteria is met or the maximum number of iteration is reached. Therefore, during the $l+1$-th iteration the optimization problem **P2** is modified as **P2'**

$$\begin{aligned} \underset{\mathbf{u}_k, \mathbf{v}}{\text{maximize}} \quad & |\alpha_0|^2 \sum_{k=1}^K \mathbf{u}_k^H \mathbf{A}^H \mathbf{\Phi}' \mathbf{A} \mathbf{u}_k + |\alpha_0|^2 \mathbf{v}^H \mathbf{A}^H \mathbf{\Phi}' \mathbf{A} \mathbf{v}. \\ \text{subject to} \quad & C1: \frac{|\mathbf{h}_k^H \mathbf{u}_k|^2}{\sum_{i=1, i\neq k}^K |\mathbf{h}_k^H \mathbf{u}_i|^2 + |\mathbf{h}_k^H \mathbf{v}|^2 + N_o} \geq \Gamma_k, \\ & C2: \sum_{k=1}^K |\mathbf{u}_k|^2 + |\mathbf{v}|^2 \leq P_{max}, \end{aligned} \tag{48}$$

with the corresponding SDR relaxation given as **P2'-SDR**

$$\begin{aligned} \underset{\mathbf{U}_k, \mathbf{V}}{\text{maximize}} \quad & \sum_{k=1}^K \text{Tr}(\mathbf{\Phi}' \mathbf{U}_k) + \text{Tr}(\mathbf{\Phi}' \mathbf{V}) \\ \text{subject to} \quad & \tilde{C}1 - \tilde{C}4. \end{aligned} \tag{49}$$

Since in each iteration the optimal solutions depend on the channel gains h_k's of the CUs, which can cause dependence of $\mathbf{\Phi}'$ on the channel gains. Thus the independence argument used in the proof of Lemma 1 cannot be used here for showing that $\text{rank}(\mathbf{U}_k) = 1$. The following lemma presents the solution structure for problem **P2'-SDR**.

Lemma 5. *There exists an optimal solution for **P2'-SDR** such that*

- $\mathbf{V}^* = \mathbf{0}$,

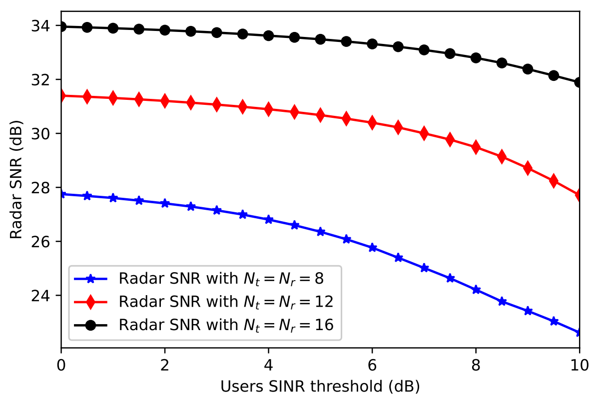

Fig. 2. Harvested power with respect to SINR threshold when $R = 20$ m.

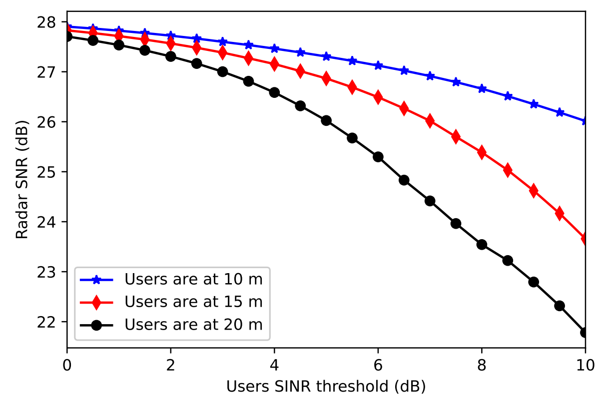

Fig. 3. Harvested power with respect to SINR threshold when $R = 20$ m.

- $\text{rank}(\mathbf{U}_k) = 1$.

Proof. In order to prove that $\mathbf{V}^* = \mathbf{0}$ assume that the optimal solution to **P2'-SDR** is $\{\mathbf{U}_k^*\}$, and $\mathbf{V}^* \neq \mathbf{0}$. Then, we can obtain another solution $\{\mathbf{U'}_k^*\}, \mathbf{V'}^* = \mathbf{0}$ of **P2'-SDR** such that $\mathbf{U'}_{\hat{k}}^* = \mathbf{U}_k^* + \mathbf{V}^*$ for some $k \in \{1, \cdots, K\}$, $\mathbf{U'}_k^* = \mathbf{U}_k^*, \forall k \in \{1, \cdots, K\} \setminus \hat{k}$ and $\mathbf{V'}^* = \mathbf{0}$. By doing so, we can easily observe that the optimal value for the solution $\{\mathbf{U'}_k^*\}, \mathbf{V'}^* = \mathbf{0}$ is exactly the same as that achieved by $\{\mathbf{U}_k^*\}, \mathbf{V}^* \neq \mathbf{0}$. Furthermore, it is easy to verify that all the constraints are satisfied for modified solution. This proves the first statement of this lemma. In order to proceed further, we set $\mathbf{V} = \mathbf{0}$ in **P2'-SDR**. Then **P2'SDR** can be written as **P2'-SDR**

$$\begin{aligned} \underset{\mathbf{U}_k}{\text{maximize}} \quad & \sum_{k=1}^{K} \text{Tr}(\mathbf{\Phi}' \mathbf{U}_k) \\ \text{subject to} \quad & \tilde{C}1 - \tilde{C}3. \end{aligned} \tag{50}$$

Using a well known result in [], it can be shown that there exist rank one solutions for all $\mathbf{U}_k$'s for **P2'SDR**. □

IV. Numerical Results

In this section, we provide the simulation results. The important simulation parameters are provided in Table I below. We demonstrate two types of results. First, we show the radar SNR with respect to the SINR thresholds of the individual CUs for different number of N_t, N_r. Second, we show the radar SNR with respect to the SINR thresholds of the individual CUs for different distances between the DFRC transmitter and CUs.

V. Conclusions

This work focuses on beamforming optimization at the DFRC transmitter where the objective is to maximize the radar detection probability while satisfying the communication data rate requirements of CUs. The formulated problem is shown to be non-convex. Then, an SDR based scheme is proposed to solve the formulated problem optimally. It is also concluded that there is no need to transmit separate probing signal for target detection.

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