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A splitting theorem for homotopy equivalent smooth 4-manifolds

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RIASSUNTO: *Si prova un teorema di decomposizione per* 4*-varietà chiuse, connesse, lisce ed omotopicamente equivalenti che rappresenta una estensione parziale di un recente risultato ottenuto in* [2] *al caso non semplicemente connesso. Si studia poi il problema di approssimare* (*modulo omotopie*) *una equivalenza di omotopia tra* 4*-varietà lisce e chiuse mediante un omeomorfismo topologico* (*Problema di Borel in dimensione* 4). *In particolare, si ottiene una nuova dimostrazione del teorema di unicità* (*modulo omeomorfismi topologici*) *delle* 4*-varietà lisce, chiuse ed asferiche con gruppo fondamentale buono.*

ABSTRACT: *We prove a decomposition theorem for closed connected homotopy equivalent smooth four-manifolds, which partially extends a recent result of* [2] *to the non-simply connected case. Then we study the question of when a homotopy equivalence between closed smooth* 4*-manifolds is homotopic to a topological homeomorphism. In particular, we obtain a new proof of the well-known uniqueness of closed aspherical smooth* 4*-manifolds with good fundamental groups.*

1 – Introduction

Recently, CURTIS, FREEDMAN, HSIANG, and STONG [2] have proved the following decomposition theorem for h-cobordant smooth simply-connected 4-manifolds.

KEY WORDS AND PHRASES: *Four-manifolds – Homotopy equivalences – Immersions – Homotopy type – Homeomorphism type – Covering maps – Aspherical manifolds*
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Theorem 1.1. *Let M^4 and N^4 be h-cobordant simply-connected closed smooth 4-manifolds. There exist decompositions $M = M_0 \cup_\Sigma M_1$ and $N = N_0 \cup_\Sigma N_1$, where M_0 and N_0 are compact contractible smooth 4-manifolds with boundary Σ so that (M_1, Σ) and (N_1, Σ) are diffeomorphic simply-connected bordered 4-manifolds.*

The proof of this theorem may be extended to show that if $M_1, \dots,$ M_r are h-cobordant simply-connected closed smooth 4-manifolds, then there are splittings $M_i = M_0 \cup_\Sigma Y_i$, where Y_i is compact and contractible and M_0 is simply-connected. Other related results about decompositions of 4-manifolds with special fundamental groups can be found in two further papers of Stong (see [23] and[24]).

The goal of the present note is to prove a partial extension of theorem 1.1 for homotopy equivalent smooth 4-manifolds without any restriction on the fundamental group.

Our main result is the following

Theorem 1.2. *Let M^4 and N^4 be closed connected smooth oriented 4-manifolds and let $h\colon M \to N$ be an orientation preserving homotopy equivalence. Then there are a map $f\colon M \to N$, and bordered manifold decompositions $M = W \cup_V W'$ (where W' is connected) and $N = U' \cup_{\partial U} U$ such that*

(1) *f is homotopic to h;*
(2) *U is a regular (connected) neighborhood of $f(D^4)$, where D^4 is a small 4-disc in M;*
(3) *$f|\colon (W, V = \partial W) \to (U', \partial U)$ is a diffeomorphism;*
(4) *$f|\colon (W', V = \partial W') \to (U, \partial U)$ is a degree one map.*

The proof is based on the Smale-Hirsch immersion theory [16], [20] or equivalently on the Phillips submersion theorems [19]. The classification of vector bundles over 4-complexes by Dold and Whitney [3] implies that the homotopy equivalence h is tangential, i.e. there is a bundle map $b\colon TM \to TN$, covering h, such that $b_x\colon T_xM \underset{\cong}{\to} T_{h(x)}N$ is an isomorphism on each fiber. Therefore we obtain an immersion (submersion) of the open manifold $M \backslash \overset{\circ}{B}{}^4$ into N, where B^4 is a (closed) small 4-disc in M. Then we modify $M \backslash \overset{\circ}{B}{}^4$ to a compact submanifold with boundary such that we can apply Ehreshmann's theorem [4] to get a covering map. Finally we study the structure of the induced decomposition to change, up to homotopy, the named covering to a map $f\colon M \to N$ satisfying the properties of the theorem. We also study the problem of f being homotopic to a topological homeomorphism. This is related to the Borel conjecture in dimension 4. In particular, we obtain an alternative proof (without the use of WALL's surgery sequence [25]) of the validity of the conjecture for smooth aspherical closed 4-manifolds with good fundamental groups (compare also [11] and [12]). More precisely, any orientation preserving homotopy equivalence between such manifolds is homotopic to a topological homeomorphism.

As general references for 3- and 4-manifold topology see [15] and [11], respectively. Concepts and notations from piecewise-linear and algebraic topology are standard, and can be found for example in [21] and [22].

2 – Homotopy equivalences in dimension 4

Let M^4 and N^4 be closed connected smooth 4-manifolds, and denote by TM and TN their tangent bundles. For simplicity we will assume that M and N are oriented. The first lemma is an observation which follows from the DOLD-WHITNEY classification of $\mathrm{SO}(n)$-bundles over 4-dimensional complexes (see [3]). It states that any homotopy equivalence in dimension 4 is tangential.

LEMMA 2.1. *Let $h\colon M \to N$ be an orientation preserving homotopy equivalence. Then there is a fiberwise isomorphism $b\colon TM \to TN$ such that the following diagram*

$$\begin{array}{ccc} TM & \xrightarrow{\;b\;} & TN \\ \downarrow & & \downarrow \\ M & \xrightarrow[\;h\;]{} & N \end{array}$$

commutes, i.e. *h is tangential. In other words, TM and $h^*(TN)$ are isomorphic as* $\mathrm{SO}(4)$*-bundles.*

PROOF. According to [3] one has to show that the second Stiefel-Whitney classes, the Euler classes, and the first Pontrjagin classes of TM and $h^*(TN)$ coincide, i.e. $TM \cong h^*(TN)$ as SO(4)-bundles if and only if

$$\begin{aligned} w_2(TM) &= w_2(h^*(TN)) \\ e(TM) &= e(h^*(TN)) \\ p_1(TM) &= p_1(h^*(TN)). \end{aligned}$$

Now w_2 is an invariant of the homotopy type. Identifying

$$H_0(M;\mathbb{Z}) = \mathbb{Z} = H_0(N;\mathbb{Z})$$

we have the formulae for the Euler characteristics

$$\begin{aligned} \chi(M) &=< e(TM), [M] > \\ \chi(N) &=< e(TN), [N] >, \end{aligned}$$

where $[M]$ and $[N]$ denote the fundamental classes of M and N, respectively.

Since $\chi(M)$ and $\chi(N)$ are equal, we obtain

$$\begin{aligned} < e(TM), [M] > &=< e(TN), [N] > \\ &=< e(TN), h_*[M] > \\ &=< h^*(e(TN)), [M] > \\ &=< e(h^*(TN)), [M] >, \end{aligned}$$

and hence $e(TM) = e(h^*(TN)) \in H^4(M;\mathbb{Z}) \cong \mathbb{Z}$. Regarding the first Pontrjagin class p_1 we proceed as above using the Hirzebruch signature formula

$$\mathrm{Sig}(M) = \frac{1}{3} < p_1(M), [M] >$$

and the fact that $\mathrm{Sig}(M) = \mathrm{Sig}(N)$. □

The second observation we make follows from the SMALE-HIRSCH immersion theory (see for example [16] and [20]). Roughly it states that if X^n and Y^m are smooth connected manifolds of dimension n and m respectively satisfying the conditions:

(1) $n \leq m$;

(2) $X = \{\text{ closed } n-\text{ disc}\} \cup \{\text{ handles of index } < m\}$,
then the differential map

$$d\colon \operatorname{Imm}(X, Y) \to \operatorname{Max}(TX, TY)$$

is a weak homotopy equivalence. Here $\operatorname{Imm}(X, Y)$ denotes the space of immersions of X in Y with the C^1-topology, and $\operatorname{Max}(TX, TY)$ the space of tangent bundle monomorphisms between them with the compact-open topology, i.e. $\operatorname{Max}(TX, TY)$ is the set of bundle maps

$$\begin{array}{ccc} TX & \xrightarrow{b} & TY \\ \downarrow & & \downarrow \\ X & \xrightarrow[\bar{b}]{} & Y \end{array}$$

such that $b_x\colon T_xX \to T_{\bar{b}(x)}Y$ is injective for any $x \in X$. This result implies that any bundle map $(b, \bar{b})$ is homotopic to an immersion. We apply this fact to our smooth closed 4-manifolds M and N. Since the dimensions are both 4, hence any immersion is also a submersion, we can equally well apply the Phillips theorem (see [19], theorem A) to obtain the following result.

Lemma 2.2. *Let M and N be closed connected smooth oriented 4-manifolds and $h\colon M \to N$ an orientation preserving homotopy equivalence. Then there exists a map $f\colon M \to N$ such that*
(1) *f is homotopic to h;*
(2) *$f|_{M\setminus \mathring{D}^4}\colon M\setminus \mathring{D}^4 \to N$ is an immersion, where D^4 is a small 4-disc in M.*

Proof. From lemma 2.1 we obtain a bundle map $b\colon TM \to TN$ such that the diagram

$$\begin{array}{ccc} h^*(TN) & \xrightarrow{c} & TN \\ \cong\downarrow & & \| \\ TM & \xrightarrow{b} & TN \\ \downarrow & & \downarrow \\ M & \xrightarrow{h} & N \end{array}$$

commutes, where c denotes the canonical map. By [16] (or [19]) we obtain an immersion $f\colon M\backslash B^4 \to N$ such that f is homotopic to $h|_{M\backslash B^4}$. Let $B^4_\epsilon \subset M$ denote a closed 4-disc containing B^4 in its interior, but only an "ϵ-little" bigger. Then the restriction $f|_{M\backslash \overset{\circ}{B^4_\epsilon}}$ is homotopic to $h|_{M\backslash \overset{\circ}{B^4_\epsilon}}$. This implies that f can be extended to M. Obviously, the resulting map, also denoted by $f\colon M \to N$, is homotopic to h as required. □

3 – Decomposition properties

Let us assume that we have a homotopy equivalence $f\colon M^n \to N^n$ of closed connected smooth oriented n-manifolds such that

$$f|_{M^n\backslash \overset{\circ}{D^n}}\colon M^n\backslash \overset{\circ}{D^n} \to N^n$$

is an immersion. Let $U \subset N$ be a (connected compact) regular neighborhood of $f(D^n) \subset N$. Since $f|_{M\backslash \overset{\circ}{D^n}}$ is an immersion, the map f is transverse regular on ∂U. For simplicity we shall assume that ∂U is connected. The proof of our result can be verified with minor changes also for the case in which ∂U has more than one component. In this case of course, the complement $U' = N\backslash U$ has also several components (precisely, one more than the number of components of ∂U).

Let us denote

$$f^{-1}(\partial U) = V = \bigcup_{k=1}^{d} V_k.$$

Obviously, V is a submanifold of M of dimension $n-1$ and $V_1, \ldots, V_d$ are its connected components.

LEMMA 3.1. *The complement $M\backslash V$ decomposes into $d+1$ connected components.*

PROOF. Since $V \subset M$ is a nice submanifold, we have, by the Alexander duality (see [22], p. 296), $H_0(M\backslash V) \cong H^n(M, V)$. For convenience, we shall supress the integral homology coefficients. The exact sequence of the pair (M, V) gives

$$H^{n-1}(M) \to H^{n-1}(V) \to H^n(M, V) \to H^n(M) \to 0.$$

Since $H^{n-1}(V) \cong \oplus_d \mathbb{Z}$ and $H^n(M) \cong \mathbb{Z}$, it suffices to prove that

$$H^{n-1}(M) \to H^{n-1}(V)$$

is the zero map. Because $H^{n-1}(V)$ is $\mathbb{Z}$-free, it follows from the Universal coefficient theorem that we must show $H_{n-1}(V) \to H_{n-1}(M)$ is zero. But this follows from the diagram

$$\begin{array}{ccc} H_{n-1}(V) & \longrightarrow & H_{n-1}(M) \\ {\scriptstyle (f|_V)_*}\downarrow & & \cong \downarrow {\scriptstyle f_*} \\ H_{n-1}(\partial U) & \longrightarrow & H_{n-1}(N) \end{array}$$

as $H_{n-1}(\partial U) \to H_{n-1}(N)$ is null. In fact, $[\partial U]$ goes to zero because ∂U bounds in N. □

Let us denote $W_1, \dots, W_r, W_1', \dots, W_s'$ the closures of the connected components of $M \backslash V$, i.e. $r + s = d + 1$, such that

$$\begin{array}{ll} f(W_i) \subset N \backslash \overset{\circ}{U}, & i = 1, \dots, r \\ f(W_j') \subset U, & j = 1, \dots, s. \end{array}$$

We observe that $N \backslash U$ is connected. In fact, $H_0(N \backslash \partial U) \cong H^n(N, \partial U)$ and $H_{n-1}(\partial U) \to H_{n-1}(N)$ is zero, hence $H^{n-1}(N) \to H^{n-1}(\partial U)$ is zero too.

Now the exact sequence

$$0 \to H^{n-1}(\partial U) \to H^n(N, \partial U) \to H^n(N) \to 0$$

yields $H_0(N \backslash \partial U) \cong \mathbb{Z} \oplus \mathbb{Z}$ as requested. We observe that $f|_{W_i}$, $f|_{V_k}$, and $f|_{\partial W_i}$ are immersions, hence they are covering maps by the Ehresmann theorem (see [4] and its relative version as stated in [13], pp. 16-17). The same is true for $f|_{W_j'}$, except for the component which contains D^n. This component is unique because otherwise $D^n \cap V \neq \emptyset$. Let $D^n \subset W_1' \subset M$. Now we are going to show that $f(W_i) = N \backslash \overset{\circ}{U}$ and $f(W_j') = U$. Since $f|_{M \backslash \overset{\circ}{D^n}}$ is a submersion (immersion), the restrictions $f|_{W_i}$ and $f|_{W_j'}$ are open maps, hence $f(W_i) \subset N \backslash \overset{\circ}{U}$ and $f(W_j') \subset U$ are open-closed subsets as requested. Finally, we observe that W_i, W'_j, and V_k are all compact sets, hence the maps (induced by f)

$$\begin{aligned} W_i &\to N\backslash\overset{\circ}{U} && i = 1, \dots, r \\ V_k &\to \partial U && k = 1, \dots, d \\ W'_j &\to U && j = 2, \dots, s \end{aligned}$$

are all finite coverings. In particular, we have homology isomorphisms

$$\begin{aligned} H_*(W_i;\mathbb{Q}) &\underset{\cong}{\to} H_*(N\backslash\overset{\circ}{U};\mathbb{Q}) \\ H_*(V_k;\mathbb{Q}) &\underset{\cong}{\to} H_*(\partial U;\mathbb{Q}) \\ H_*(W'_j;\mathbb{Q}) &\underset{\cong}{\to} H_*(U;\mathbb{Q}). \end{aligned}$$

The following lemma says that a finite covering $W'_j \to U$, $j = 2, \dots, s$ does not exist.

LEMMA 3.2. *With the above notation, $s = 1$ (hence $r = d$),* i.e. *there is only one component, W'_1 say, such that $f(W'_1) = U$.*

PROOF. Assume $s > 1$, i.e. for example $W'_2 \to U$ is a covering map. There is a subset W_i such that $\partial W_i \cap \partial W'_2 \neq \emptyset$. Suppose $V_k \subset \partial W_i \cap \partial W'_2$ and consider $W := W_i \cup V_k \cup W'_2$. Then f induces a map of triples (also denoted by f)

$$f\colon (W, W_i, W'_2) \to (N, N\backslash\overset{\circ}{U}, U).$$

We set $\overset{\wedge}{W_i} = W_i \cup V_k$ and $\overset{\wedge}{W'_2} = W'_2 \cup V_k$, hence $W = \overset{\wedge}{W_i} \cup \overset{\wedge}{W'_2}$. Let us consider the Mayer-Vietoris sequences with homology $\mathbb{Q}$-coefficients which are associated to the above triples

$$\begin{array}{ccccccccc} \dots \to & H_q(V_k) & \to & H_q(\overset{\wedge}{W_i}) \oplus H_q(\overset{\wedge}{W'_2}) & \to & H_q(W) & \to & H_{q-1}(V_k) & \to \dots \\ & \cong\downarrow & & \cong\downarrow & & \downarrow & & \cong\downarrow & \\ \dots \to & H_q(\partial U) & \to & H_q(N\backslash\overset{\circ}{U}) \oplus H_q(U) & \to & H_q(N) & \to & H_{q-1}(\partial U) & \to \dots \end{array}$$

where the vertical homomorphisms are induced by f. Now the five lemma implies that $H_q(W;\mathbb{Q}) \underset{\cong}{\to} H_q(N;\mathbb{Q})$. Since $H_n(W;\mathbb{Q}) \cong 0$ (because W

is a bordered manifold) we have a contradiction. Therefore it must be $s = 1$ (hence $r = d$), i.e. there can be only a map $W_1' \to U$. ∎

LEMMA 3.3. *The induced homomorphism*

$$f_*\colon \Pi_1(W_1') \to \Pi_1(U)$$

is surjective.

PROOF. We have to show that the map

$$f|\colon (W_1', \partial W_1') \to (U, \partial U)$$

has degree one. Then the result follows from [1], proposition 1.2. Now we have an orientation preserving homotopy equivalence $f\colon M \to N$. Let $u_0 \in U$ be a regular value of f, and suppose $f^{-1}(u_0) = \{x_i\colon i = 1, \dots, m\}$. Then we have

$$1 = \deg(f) = \sum_{i=1}^{m} \deg(f; x_i) = \deg(f|_{W_i'}),$$

where $\deg(f; x_i)$ is the local degree of f at x_i. ∎

Notation: In the sequel, we will denote W_1' by W'.

THEOREM 3.4. *Let $f\colon M^n \to N^n$ be as above and let $P \subset N$ be a closed connected oriented submanifold with $P \cap f(D^n) = \emptyset$. Then $Q = f^{-1}(P)$ is a submanifold of M such that any connected component of Q is diffeomorphic to P.*

PROOF. Since $P \cap f(D^n) = \emptyset$, the map f is transverse regular to P, and hence the preimage $Q = f^{-1}(P)$ is a submanifold of M. Let $\nu(P)$ and $\nu(Q)$ denote the normal fibrations, so we have a fiberwise isomorphism

$$df|_{\nu(Q)}\colon \nu(Q) \to \nu(P).$$

Since

$$df|_Q\colon TM|_Q = TQ \oplus \nu(Q) \to TN|_P = TP \oplus \nu(P)$$

is a fiberwise isomorphism, it follows that $df\colon TQ \to TP$ is a fiberwise isomorphism too, i.e. $f\colon Q \to P$ is a submersion. By EHRESMANN's theorem [4], it must be a covering map. Let $C \subset Q$ be a connected component, so the restriction $f|_C\colon C \to P$ is also a covering map. Let us denote by $N(C)$, $N(Q)$, and $N(P)$ the regular neighborhoods of C, Q, and P, respectively. Then the Thom spaces $T\nu(C)$, $T\nu(Q)$, and $T\nu(P)$ can be identified with $N(C)/\partial N(C)$, $N(Q)/\partial N(Q)$, and $N(P)/\partial N(P)$, respectively. Let us consider the following diagram of inclusions and maps

$$\begin{array}{ccccccc}
M & \longrightarrow & (M, M\backslash N(C)) & \longleftarrow & (N(C), \partial N(C)) & \longleftarrow & C \\
\| & & \cup & & \cap & & \cap \\
M & \longrightarrow & (M, M\backslash N(Q)) & \longleftarrow & (N(Q), \partial N(Q)) & \longleftarrow & Q \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} & & {\scriptstyle f|}\downarrow & & \downarrow{\scriptstyle f|} \\
N & \longrightarrow & (N, N\backslash N(P)) & \longleftarrow & (N(P), \partial N(P)) & \longleftarrow & P.
\end{array}$$

Here we assume that $f^{-1}(N(P)) = N(Q)$, so f maps $M\backslash N(Q)$ into $N\backslash N(P)$ as $M\backslash f^{-1}(N(P)) = f^{-1}(N\backslash N(P))$. This can be arranged by using the regular neighborhood theorem (see for example [21]). The above diagram induces the following diagram involving integral cohomology groups

$$\begin{array}{ccccccc}
0 \leftarrow H^n(M) \cong \mathbb{Z} & \underset{\cong}{\leftarrow} & H^n(M, M\backslash N(C)) & \underset{\cong}{\rightarrow} & H^n(N(C), \partial N(C)) & \underset{\cong}{\leftarrow} & H^q(C) \cong \mathbb{Z} \\
\| & & \downarrow & & \uparrow & & \uparrow \\
0 \leftarrow H^n(M) \cong \mathbb{Z} & \underset{\text{epi}}{\leftarrow} & H^n(M, M\backslash N(Q)) & \underset{\cong}{\rightarrow} & H^n(N(Q), \partial N(Q)) & \underset{\cong}{\leftarrow} & H^q(Q) \cong \oplus_\alpha \mathbb{Z} \\
{\scriptstyle f^*}\uparrow{\scriptstyle \cong} & & \uparrow{\scriptstyle f^*} & & {\scriptstyle f^*}\uparrow & & \uparrow{\scriptstyle f^*} \\
0 \leftarrow H^n(N) \cong \mathbb{Z} & \underset{\cong}{\leftarrow} & H^n(N, N\backslash N(P)) & \underset{\cong}{\rightarrow} & H^n(N(P), \partial N(P)) & \underset{\cong}{\leftarrow} & H^q(P) \cong \mathbb{Z}.
\end{array}$$

Here $q := \dim Q = \dim P$, $H^n(M, M\backslash N(C)) \underset{\cong}{\rightarrow} H^n(N(C), \partial N(C))$ is the excision isomorphism, $H^q(C) \underset{\cong}{\rightarrow} H^n(N(C), \partial N(C))$ is the Thom isomorphism, and α is the number of connected components of Q. The lower right square commutes because $\nu(Q)$ is induced by $\nu(P)$ via df. Note also that

$$f^*\colon H^n(N, N\backslash N(P)) \cong \mathbb{Z} \to H^n(M, M\backslash N(Q)) \cong \oplus_\alpha \mathbb{Z}$$

maps a generator into $(\epsilon_1, \ldots, \epsilon_\alpha)$, where $\epsilon_i = \pm 1$. The diagram shows that the composition $C \to Q \to P$ is of degree one. This implies that $\Pi_1(C) \to \Pi_1(P)$ is surjective (see [1]), so it must be an isomorphism because $C \to P$ is a covering map. Therefore, C is diffeomorphic to P as claimed. $\square$

Corollary 3.5. *Each V_k is diffeomorphic to ∂U, for $k = 1, \ldots, d$.*

Let us consider the finite covering map $f|_{W_i} : W_i \to N \setminus \mathring{U}$, $i = 1, \ldots, r$. Note that ∂W_i is a union of components V_k, each diffeomorphic to ∂U via $f|_{V_k} : V_k \to \partial U$ as explained by corollary 3.5. So the number of components of ∂W_i corresponds to the order of the covering map $f|_{W_i} : W_i \to N \setminus \mathring{U}$.

Proposition 3.6. *For each $i = 1, \ldots, r$, the covering map $f|_{W_i}$ is a diffeomorphism.*

Proof. We consider the following diagram of maps and inclusions

$$\begin{array}{ccccc}
M & \longrightarrow & (M, M\setminus W_i) & \longleftarrow & (W_i, \partial W_i) \\
\| & & \uparrow & & \downarrow \\
M & \longrightarrow & (M, M\setminus f^{-1}(N\setminus U)) = (M, f^{-1}(U)) & \longleftarrow & (f^{-1}(N\setminus U), V) \\
{\scriptstyle f}\downarrow & & {\scriptstyle f}\downarrow & & {\scriptstyle f}\downarrow \\
N & \underset{j}{\longrightarrow} & (N, U) & \longleftarrow & (N\setminus U, \partial U)
\end{array}$$

which induces the following diagram in homology with $\mathbb{Z}$-coefficients

$$\begin{array}{ccccc}
H_n(M) & \to & H_n(M, M\setminus W_i) & \underset{\cong}{\leftarrow} & H_n(W_i, \partial W_i) \\
\| & & \uparrow & & \downarrow \\
H_n(M) & \to & H_n(M, f^{-1}(U)) & \underset{\cong}{\leftarrow} & H_n(f^{-1}(N\setminus U), V) \cong \oplus_{i=1}^{r} H_n(W_i, \partial W_i) \cong \oplus_r \mathbb{Z} \\
{\scriptstyle f_*}\downarrow & & {\scriptstyle f_*}\downarrow & & {\scriptstyle f_*}\downarrow \\
H_n(N) & \underset{j_*}{\longrightarrow} & H_n(N, U) & \underset{\cong}{\leftarrow} & H_n(N\setminus U, \partial U).
\end{array}$$

Obviously, the above isomorphisms are given by excision. The homomorphism j_* is bijective. In fact, the diagram

$$\begin{array}{ccc} H_n(N,U) & \xrightarrow{\partial_*} & H_{n-1}(U) \\ \cong\big\uparrow & & \big\uparrow \\ H_n(N\backslash U,\partial U) & \xrightarrow[\cong]{\partial_*} & H_{n-1}(\partial U) \end{array}$$

implies that $\partial_*\colon H_n(N,U) \cong \mathbb{Z} \to H_{n-1}(U)$ is the zero map. Now the exact homology sequence of the pair (N,U)

$$0 \cong H_n(U) \to H_n(N) \to H_n(N,U) \cong \mathbb{Z} \to 0$$

yields the isomorphism $j_*\colon H_n(N) \underset{\cong}{\to} H_n(N,U)$. Of course, the same holds for $H_n(M) \underset{\cong}{\to} H_n(M, M\backslash W_i)$, but we can do without it. Furthermore, note that

$$H_n(M) \cong \mathbb{Z} \to H_n(M, f^{-1}(U)) \cong \oplus_r \mathbb{Z}$$

sends the fundamental class $[M]$ into $(\epsilon_1, \ldots, \epsilon_r)$, where $\epsilon_i = \pm 1$, and $H_n(M, f^{-1}(U)) \to H_n(M, M\backslash W_i)$ is the projection to the i-factor of the direct sum. By going down and then to the right of the diagram, the class $[M]$ goes to a generator of $H_n(N\backslash U, \partial U)$. Hence a generator of $H_n(W_i, \partial W_i)$ must map to a generator of $H_n(N\backslash U, \partial U)$, i.e. the restriction $f|\colon (W_i, \partial W_i) \to (N\backslash \mathring{U}, \partial U)$ is of degree 1. Since $f|_{W_i}\colon W_i \to N\backslash \mathring{U}$ is a finite covering map, it must be a diffeomorphism as claimed. ∎

Remark. The proof of proposition 3.6 shows again that

$$f|\colon (W_1', \partial W_1') \to (U, \partial U)$$

is of degree 1.

Corollary 3.7. $d = 1$, i.e. *there are a diffeomorphism* $W_1 \to N\backslash \mathring{U}$ *and a degree one map* $W_1' \to U$. *In particular,* $\Pi_1(W_1') \to \Pi_1(U)$ *is onto.*

Summarizing we have proved our decomposition theorem.

4 – Towards a classification

Let us assume M, N, and $f\colon M \to N$ as in theorem 1.2. In this section we study the problem of f being homotopic to a topological homeomorphism from M to N. Since the set $f(D^4) \subset U \subset N$ could be very bad, we have to add some hypothesis to control its singularities. Under the hypothesis we show that W' (and also U) is a topological 4-disc. For this of course we have to use the FREEDMAN theorem (see [9] and [11]).

PROPOSITION 4.1. *Suppose ∂U ($\cong V$) is a 3-sphere. Then f is homotopic to a topological homeomorphism $M \to N$.*

PROOF. Let us consider the closed 4-manifold $\overline{M} = W \cup B^4$, i.e. we cup off the boundary $\partial W = V \cong \mathbb{S}^3$ with a 4-disc B^4 instead of W'. Then $f|_W$ extends to a degree one map $\overline{f}\colon \overline{M} \to N$. It follows that $\overline{f}_*\colon \Pi_1(\overline{M}) \to \Pi_1(N)$ is surjective by [1]. To the triple (W, W', V) we can apply the Van Kampen theorem, and obtain the following commutative diagram

$$\begin{array}{ccc}
\Pi_1(W) * \Pi_1(W') & \xrightarrow[\cong]{} & \Pi_1(M) \\
{\scriptstyle \text{mono}}\uparrow & & \cong\downarrow{\scriptstyle f_*} \\
\Pi_1(W) * 1 & & \Pi_1(N) \\
\cong\downarrow & & \| \\
\Pi_1(\overline{M}) & \xrightarrow[\overline{f}_*]{} & \Pi_1(N).
\end{array}$$

Since $\overline{f}_*$ is surjective, we conclude from the diagram that $\overline{f}_*$ is an isomorphism on Π_1, and hence $\Pi_1(W) * \Pi_1(W') \cong \Pi_1(W) * 1$, i.e. $\Pi_1(W')$ vanishes. Furthermore, the degree one property of $\overline{f}$ implies that the induced homology homomorphism $\overline{f}_*\colon H_2(\overline{M}) \to H_2(N)$ is surjective (see [1]). Now $H_2(W) \cong H_2(\overline{M})$, so from the commutativity of the diagram

$$\begin{array}{ccc}
H_2(W) & \xrightarrow{\text{epi}} & H_2(N) \\
\downarrow & & \| \\
H_2(M) & \xrightarrow[\cong]{f_*} & H_2(N),
\end{array}$$

it follows that $H_2(W) \to H_2(M)$ is surjective. The Mayer-Vietoris sequence of the triple $(W, W', V \cong \mathbb{S}^3)$ gives then

$$0 \cong H_2(V) \to H_2(W) \oplus H_2(W') \to H_2(M) \to 0,$$

hence $H_2(W') \cong 0$ and $H_2(W) \underset{\cong}{\to} H_2(M)$. Thus FREEDMAN's theorem ([9] and [11]) applies to give a homeomorphism $W' \cup B^4 \cong \mathbb{S}^4$, i.e. $W' \underset{\mathrm{Top}}{\cong} B^4$. The isomorphisms

$$H_2(U') \cong H_2(W) \cong H_2(M) \cong H_2(N) \cong H_2(U') \oplus H_2(U)$$

also yield $H_2(U) \cong 0$, and hence $U \underset{\mathrm{Top}}{\cong} B^4$. This makes it now possible to extend the diffeomorphism $f|_W : W \to U' = N \backslash \mathring{U}$ to a topological homeomorphism $M \to N$ which is homotopic to f (and h). ∎

REMARK. Suppose V is a homology 3-sphere which is Π_1-*null* in W (resp. W'), i.e. loops in the image are contractible in W (resp. W'). Then we can repeat the proof above substituting the 4-disc B^4 with a contractible compact 4-manifold having V as its boundary.

For aspherical closed 4-manifolds with *good* fundamental groups we can apply the controlled embedding theorem (see [9] and [10]) for engulfing $f(D^4)$ into a topological 4-disc. So theorem 1.2 and proposition 4.1 give an alternative proof (without the use of WALL's surgery exact sequence [25]) of the following well-known result (see for example [11], p. 205).

THEOREM 4.2. *Any closed connected oriented aspherical smooth 4-manifold with good fundamental group is determined, up to topological homeomorphism, by its fundamental group.*

Recall that the term *good* is used to refer to fundamental groups for which the embedding theorem ([9] and [10]) is known. Examples are given by the *amenable groups*, i.e. the smallest class of groups containing finite and cyclic groups, which is closed under direct limits, subgroups, quotients, and group extensions. Recently, FREEDMAN and TEICHNER [12] have expanded the class of known good groups to contain all groups of *subexponential growth* and still closed under extensions and direct limits. This is really an effective expansion because it is known that amenable groups grow either polynomially or exponentially. However, there are (uncountably many) groups of intermediate growth, i.e. groups that grow faster than any polynomial but slower than any exponential function (see [14]).

The analogue of theorem 4.2 was proved for higher dimensions by Farrell and Jones ([5] and [7]). They have also proved the uniqueness result for *hyperbolic* manifolds, another aspherical class, in dimensions greater than 4 ([6] and [8]). However, hyperbolic groups are not known to be good, so it remains an open problem the uniqueness of hyperbolic manifolds in dimension 4.

Proposition 4.3. *Let M^4 and N^4 be aspherical, and suppose that ∂U is Π_1-null in N. Then M is stably homeomorphic to N,* i.e. *$M\#k(\mathbb{S}^2\times\mathbb{S}^2)$ is topologically homeomorphic to $N\#\ell(\mathbb{S}^2\times\mathbb{S}^2)$ for some integers k, $\ell\geq 0$.*

Proof. Since f is transverse regular to ∂U, the preimage $V=f^{-1}(\partial U)$ is a (connected) submanifold of M with a product neighborhood $V\times[-\epsilon,\epsilon]$. Let $j\colon V\to M$ and $j'\colon U\to N$ be the inclusions. From the diagram

$$\begin{array}{ccc} \Pi_1(M) & \xrightarrow[\cong]{f_*} & \Pi_1(N) \\ {\scriptstyle j_*}\uparrow & & \uparrow{\scriptstyle j'_*=0} \\ \Pi_1(V) & \xrightarrow[\cong]{(f|_V)_*} & \Pi_1(\partial U), \end{array}$$

it follows that $j_*=0$, so $j^*w_1(M)=0$. Because V is orientable, there is a framed link $L\subset V$ such that surgery on L in V gives $\mathbb{S}^3$ ([15] and [18]). The framings of the components of L in V extend to framings in M. Let us consider

$$X=M\times[0,1]\cup(\mu D^2\times D^2\times[-\epsilon,\epsilon]),$$

where μ is the number of components of L, and the union is taken along the boundary subset $L\times D^2\times[-\epsilon,\epsilon]\times\{1\}$. Note that if $w_2(M)=0$, then we may choose the framed link L so that $w_2(X)=0$ (see [17]). Then $\partial X=M\cup M'$, where M' is the result of surgery on L in M. The map

f extends to a map $F\colon X \to N$ such that $(F|_{M'})_*\colon \Pi_1(M') \to \Pi_1(N)$ is an isomorphism, and $(F|_{M'})^{-1}(\partial U) = \mathbb{S}^3$. Since the components of L are null-homotopic in M, they may be isotoped into disjoint discs, and so M is homeomorphic to the connected sum $M'\#\mu(\mathbb{S}^2 \times \mathbb{S}^2)$. The asphericity of M' and N implies that $F|_{M'}$ is a homotopy equivalence. Moreover, we can adjust the construction of the extension of f in order to maintain the properties of theorem 1.2 for $F|_{M'}$. Thus we obtain $M' \underset{\text{Top}}{\cong} N$, and hence M is stably homeomorphic to N. $\square$

REFERENCES

[1] W. Browder: *Poincaré spaces, their normal fibrations and surgery*, Invent. Math., **17** (1977), 191-202.

[2] C.L. Curtis – M.H. Freedman – W. C. Hsiang – R. Stong: *A decomposition theorem for h-cobordant smooth simply-connected compact* 4*-manifolds*, Invent. Math., **123** (1996), 343-348.

[3] A. Dold – H. Whitney: *Classification of oriented sphere bundles over a* 4*-complex*, Ann. of Math., **69** (1959), 667-677.

[4] C. Ehresmann: *Sur le espaces fibrés différentiables*, C. R. Hebd. Séanc. Acad. Sci. Paris, **224** (1947), 1611-1612.

[5] F.T. Farrell – L.E. Jones: *The surgery L-groups of poly-*(*finite or cyclic*) *groups*, Invent. Math., **91** (1988), 559-586.

[6] F.T. Farrell – L.E. Jones: *A topological analogue of Mostow's rigidity theorem*, J. Amer. Math. Soc., **2** (1989), 257-370.

[7] F.T. Farrell – L.E. Jones: *Classical aspherical manifolds*, Conf. Board of the Math. Sci., Amer. Math. Soc., **75** (1990), Providence, R. I.

[8] F.T. Farrell – L.E. Jones: *Rigidity and other topological aspects of compact nonpositively curved manifolds*, Bull. Amer. Math. Soc., **22** (1990), 59-64.

[9] M.H. Freedman: *The topology of four-dimensional manifolds*, J. Diff. Geom., **17** (1982), 357-453.

[10] M.H. Freedman: *The disk theorem for four-dimensional manifolds*, in "Proc. Int. Congress Math. Warsaw" (1983), 647-663.

[11] M.H. Freedman – F. Quinn: *Topology of* 4*-Manifolds*, Princeton Univ. Press, Princeton, N. J. 1990.

[12] M.H. Freedman – P. Teichner: 4*-Manifold topology I: Subexponential groups*, Invent. Math., **122** (1995), 509-529.

[13] C. Godbillon: *Feuilletages Études géometriques*, Progress in Math., **98** (1991), Birkhäuser Verlag, Basel.

[14] R.I. Grigorchuk: *Degrees of growth of finitely generated groups, and the theory of invariant means*, Math. USSR Izvestiya, **25** (1985), 259-300.

[15] J. Hempel: 3-*Manifolds*, Ann. of Math. Studies 86, Princeton Univ. Press, Princeton, N. J. 1976.

[16] M.W. Hirsch: *Immersions of manifolds*, Trans. Amer. Math. Soc., **93** (1959), 242-276.

[17] S.J. Kaplan: *Constructing framed* 4*-manifolds with given almost framed boundaries*, Trans. Amer. Math. Soc., **254** (1979), 237-263.

[18] W.B.R. Lickorish: *A representation of orientable combinatorial* 3*-manifolds*, Ann. of Math., **76** (1962), 531-538.

[19] A. Phillips: *Submersions of open manifolds*, Topology, **6** (1966), 171-206.

[20] V. Poénaru: *Homotopy theory and differentiable singularities*, Lect. Notes in Math. **197** Springer-Verlag, (1970), 106-132.

[21] C.P. Rourke – B.J. Sanderson: *Introduction to piecewise linear topology*, Springer Verlag, Berlin- New York, 1972.

[22] E.H. Spanier: *Algebraic Topology*, Mc Graw-Hill, New York, 1966.

[23] R. Stong: *Uniqueness of connected sum decomposition in dimension 4*, Topology Appl., **56** (1994), 277-291.

[24] R. Stong: *A structure theorem and a splitting theorem for simply-connected smooth* 4*-manifolds*, Math. Research Lett., **2** (1995), 497-503.

[25] C.T.C. Wall: *Surgery on Compact Manifolds*, Academic Press, London- New York, 1970.

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