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On the systole growth in congruence quaternionic hyperbolic manifolds

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Abstract
We provide an explicit lower bound for the systole in principal congruence covers of compact quaternionic hyperbolic manifolds. We also prove the optimality of this lower bound.

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1 | INTRODUCTION

One the most important quantity associated with a Riemannian manifold M is the shortest geodesic length, which is called the *systole* of M. We will denote it by $\mathrm{sys}_1(M)$. Buser and Sarnak [2] constructed examples of hyperbolic surfaces S whose systole grows logarithmically with respect to the area:

$$\mathrm{sys}_1(S) \geqslant \frac{4}{3}\log(\mathrm{area}(S)) - c,$$

where c is a constant independent of S. Indeed these surfaces are congruence coverings of an arithmetic hyperbolic surface. In 1996, Gromov [5, Section 3.C.6] showed that for any regular congruence covering M_I of a compact arithmetic locally symmetric space M, there exists a constant

$C > 0$ so that

$$\mathrm{sys}_1(M_I) \geqslant C\log(\mathrm{vol}(M_I)) - d,$$

where d is independent of M_I. This method, however, does not provide an explicit value for the constant C. The knowledge of a precise value for this constant gives us not only geometrical information on the locally symmetric space, but it has also proven useful in applications to other contexts, see, for instance, the discussion in [1, Proposition 5.3], and [6, Section IV].

The optimal value of the constant C mainly depends on the Lie group type of the associated isometry group, and there are several cases where an explicit C are calculated. In 2007, Katz, Schaps and Vishne [7] generalized Buser and Sarnak's result to any compact arithmetic hyperbolic surface. They also proved that for compact arithmetic hyperbolic 3-manifolds the constant $C = \frac{2}{3}$ works. For real arithmetic hyperbolic n-manifold of the first type, these results were generalized by Murillo in [16], where he proved that the constant is equal to $\frac{8}{n(n+1)}$. In an appendix to this article, Dória and Murillo proved that this is the best possible constant in this case. A similar result for Hilbert modular varieties was obtained in [15].

Recently, Lapan, Linowitz and Meyer obtained a value for the constant C for a large class of arithmetic locally symmetric spaces, including real, complex and quaternionic hyperbolic orbifolds [10]. However, the values of the constants obtained in [10] are not optimal, as the comparison with the results mentioned above shows. See also [9]. This paper is dedicated to improving the constant for quaternionic hyperbolic spaces. The main result is the following.

Theorem 1.1. *Let $M = \Gamma\backslash\mathbf{H}^n_{\mathbb{H}}$ be a compact quaternionic hyperbolic orbifold, defined over the number field k. There exists a finite set S of prime ideals of $\mathcal{O}_k$ such that the principal congruence subgroup Γ_I associated with any ideal I prime to S satisfies*

$$\mathrm{sys}_1(M_I) \geqslant \frac{4}{(n+1)(2n+3)}\log\left(\mathrm{vol}(M_I)\right) - d,$$

where $M_I = \Gamma_I\backslash\mathbf{H}^n_{\mathbb{H}}$ and d is a constant independent of I.

The proof of Theorem 1.1 appears at the end of Section 5, after the needed preparation. Let us point out that the definition of Γ_I depends on the choice of an embedding; however, the result of Theorem 1.1 is not affected by this choice. For concrete Γ, the set of primes S can be made explicit; see Remark 4.6.

Note that when $n = 1$, $\mathbf{H}^1_{\mathbb{H}}$ is isometric to the four-dimensional real hyperbolic space, and the constant $\frac{2}{5}$ agrees with that of [16]. In Section 6, we generalize the argument of Dória and Murillo to prove that the constant $\frac{4}{(n+1)(2n+3)}$ is optimal; see Theorem 6.1.

2 | THE QUATERNIONIC HYPERBOLIC SPACE

2.1 | Hamiltonian quaternions

Let $\mathbb{H}$ be the $\mathbb{R}$-algebra of the Hamilton quaternions $q = q_0 + q_1 i + q_2 j + q_3 ij$ with q_i being real numbers, and $i^2 = -1$, $j^2 = -1$, $ij = -ji$. Any quaternion q has a *conjugate* $\overline{q} = q_0 - q_1 i - q_2 j - q_3 ij$, and the *norm* of q is given by $|q| = \sqrt{q\overline{q}}$. The *real part* of q is $\mathrm{Re}(q) = q_0$, and its *imaginary* *part* is $\mathrm{Im}(q) = q_1 i + q_2 j + q_3 ij$. We will consider the field $\mathbb{C}$ of complex numbers as the subring of $\mathbb{H}$ consisting of the quaternions of the form $q_0 + q_1 i$. We say that two quaternions p, q are *similar* if there exist a non-zero $r \in \mathbb{H}$ such that $q = rpr^{-1}$.

Lemma 2.1. *Any quaternion q is similar to a complex number with the same norm and the same real part.*

Proof. Let $p = \mathrm{Re}(q) + |\,\mathrm{Im}(q)|i$. If we take $r = \mathrm{Im}(q) + |\,\mathrm{Im}(q)|i$, a direct computation shows that $rpr^{-1} = q$. □

2.2 | Matrices over $\mathbb{H}$

Any matrix A with coefficients in $\mathbb{H}$ has a conjugate $\overline{A}$ whose entries are the conjugates of the corresponding entries of A, and has a transpose A^t which columns correspond to the rows of A in the classical way. It is clear that $(\overline{A})^t = \overline{(A^t)}$, and we denote that matrix by A^*. For matrices A and B of suitable sizes, we can verify that $(AB)^* = B^* A^*$, as in the complex case. A square matrix with coefficients in $\mathbb{H}$ is called *hermitian* if $A^* = A$. It is *unitary* if $AA^* = A^*A = I$, where I is the identity matrix.

2.3 | A model for $\mathbf{H}^n_{\mathbb{H}}$

We denote by $\mathbb{H}^{n,1}$ the right $\mathbb{H}$-module $\mathbb{H}^{n+1}$ equipped with the standard hermitian product of signature $(n, 1)$, given by

$$\langle \mathbf{x}, \mathbf{y} \rangle = -\overline{x}_0 y_0 + \overline{x}_1 y_1 + \cdots + \overline{x}_n y_n.$$

We consider the subspaces

$$V_- = \{\mathbf{x} \in \mathbb{H}^{n,1} \mid \langle \mathbf{x}, \mathbf{x} \rangle < 0\},$$

$$V_0 = \{\mathbf{x} \in \mathbb{H}^{n,1} \setminus \{0\} \mid \langle \mathbf{x}, \mathbf{x} \rangle = 0\},$$

$$V_+ = \{\mathbf{x} \in \mathbb{H}^{n,1} \mid \langle \mathbf{x}, \mathbf{x} \rangle > 0\},$$

and the following map:

$$P : \mathbb{H}^{n,1} \setminus \{\mathbf{z} = (z_0, \dots, z_n) \mid z_0 = 0\} \to \mathbb{H}^n$$

$$\mathbf{z} = \begin{pmatrix} z_0 \\ z_1 \\ \vdots \\ z_n \end{pmatrix} \mapsto \begin{pmatrix} z_1 z_0^{-1} \\ z_2 z_0^{-1} \\ \vdots \\ z_n z_0^{-1} \end{pmatrix}.$$

It is clear that $P(\mathbf{z}) = P(\mathbf{w})$ if and only if $\mathbf{w} = \mathbf{z}\lambda$ for some $\lambda \in \mathbb{H}$.

The *quaternionic hyperbolic n-space* is defined as $\mathbf{H}^n_{\mathbb{H}} = PV_-$, and its ideal boundary is $\partial \mathbf{H}^n_{\mathbb{H}} = PV_0$. The hermitian product induces a Riemannian metric in $\mathbf{H}^n_{\mathbb{H}}$ given by (see [8]):

$$ds^2 = \frac{-4}{\langle \mathbf{z}, \mathbf{z} \rangle^2} \det \begin{pmatrix} \langle \mathbf{z}, \mathbf{z} \rangle & \langle \mathbf{z}, d\mathbf{z} \rangle \\ \langle d\mathbf{z}, \mathbf{z} \rangle & \langle d\mathbf{z}, d\mathbf{z} \rangle \end{pmatrix}.$$

This metric is normalized so that the sectional curvature is pinched between -1 and $-\frac{1}{4}$. The distance function $\rho(\cdot,\cdot)$ in $\mathbf{H}^n_{\mathbb{H}}$ induced by this Riemannian metric satisfies the formula

$$\cosh^2\left(\frac{\rho(z,w)}{2}\right) = \frac{\langle \mathbf{z},\mathbf{w}\rangle\langle \mathbf{w},\mathbf{z}\rangle}{\langle \mathbf{z},\mathbf{z}\rangle\langle \mathbf{w},\mathbf{w}\rangle}, \tag{2.1}$$

where $\mathbf{z}, \mathbf{w}$ are any preimages under P of z and w, respectively.

2.4 | The isometry group

The set of invertible right $\mathbb{H}$-linear transformations of $\mathbb{H}^{n+1}$ identifies with the set of invertible $(n+1)\times(n+1)$ matrices with entries in $\mathbb{H}$, denoted by $\mathrm{GL}_{n+1}(\mathbb{H})$. Let $\mathrm{Sp}(n,1)$ denote the subgroup of $\mathrm{GL}_{n+1}(\mathbb{H})$ that preserves the hermitian form $\langle\cdot,\cdot\rangle$. Equivalently,

$$\mathrm{Sp}(n,1) = \{A \in \mathrm{GL}_{n+1}(\mathbb{H}) | A^*JA = J\},$$

where $J = \mathrm{diag}(-1,1,\dots,1)$. The elements $A \in \mathrm{Sp}(n,1)$ act on $\mathbf{H}^n_{\mathbb{H}}$ as follows: $A(P(\mathbf{w})) = P(A(\mathbf{w}))$. This action preserves the Riemannian structure on $\mathbf{H}^n_{\mathbb{H}}$. In fact, the isometry group $\mathrm{Isom}(\mathbf{H}^n_{\mathbb{H}})$ is isomorphic to the quotient $\mathrm{PSp}(n,1) = \mathrm{Sp}(n,1)/\{\pm I\}$.

3 | EIGENVALUES AND TRANSLATION LENGTHS IN $\mathbf{Sp}(n,1)$

This section studies the algebraic properties of the eigenvalues in the quaternionic case (Section 3.1), and important implications for the translation lengths in $\mathbf{H}^n_{\mathbb{H}}$ (Proposition 3.8).

3.1 | Eigenvalues of quaternionic matrices

For matrices with coefficients in a general ring, there is no theory of eigenvalues. However, for division algebras there is a chance of developing this theory, and in the case of $M_n(\mathbb{H})$ we can trace back this to the work of Lee in the late 1940s [11].

Definition 3.1. Let A be an element in $M_n(\mathbb{H})$. An *(right) eigenvalue* of A is a complex number t such that

$$Av = vt \tag{3.1}$$

for some nonzero vector v in $\mathbb{H}^n$.

Remark 3.2. If $Av = vt$ for a quaternion number t, by Lemma 2.1 we can find a quaternion λ such that $\lambda t\lambda^{-1}$ is a complex number. Then $Av\lambda^{-1} = v\lambda^{-1}\lambda t\lambda^{-1}$, which shows that $\lambda t\lambda^{-1}$ is a complex eigenvalue.

Let A be an element in $M_n(\mathbb{H})$. We can write $A = A_1 + jA_2$, where $A_1, A_2 \in M_n(\mathbb{C})$, and consider the map

$$f : M_n(\mathbb{H}) \to M_{2n}(\mathbb{C})$$
$$A \mapsto \begin{pmatrix} A_1 & -\overline{A_2} \\ A_2 & \overline{A_1} \end{pmatrix}. \tag{3.2}$$

The next theorem summarizes Lee's results that are relevant to us. Eigenvalues are counted *with* multiplicities.

Theorem 3.3 (Lee). *The map f is an isomorphism of rings from $M_n(\mathbb{H})$ into its image in $M_{2n}(\mathbb{C})$. Moreover for any $A \in M_n(\mathbb{H})$ we have:*

(1) *the eigenvalues of A corresponds exactly to those of $f(A)$, and they fall into n pairs of complex conjugate numbers;*
(2) $f(A^*) = f(A)^*$.

Proof. See [11]: Section 4, and Theorems 2 and 5. □

Corollary 3.4. *Let $t_1, \overline{t_1}, \dots, t_n, \overline{t_n}$ denote the eigenvalues of $A \in M_n(\mathbb{H})$. Then for the trace the following holds:*

$$\mathrm{Re}(\mathrm{tr}(A)) = \frac{t_1 + \overline{t_1} + \cdots + t_n + \overline{t_n}}{2}. \tag{3.3}$$

In particular, $\mathrm{Re}(\mathrm{tr}(A))$ *is a conjugation invariant.*

Proof. A direct computation shows that $\mathrm{Re}(\mathrm{tr}(A)) = \frac{\mathrm{tr}(f(A))}{2}$, so that the result follows from Theorem 3.3. □

Corollary 3.5. *The eigenvalues of A and A^* coincide.*

Proof. Since $f(A^*) = f(A)^*$, the eigenvalues of A^* are the eigenvalues of $f(A)^*$, which are the complex conjugates of the eigenvalues of $f(A)$. But the set of these eigenvalues is invariant by complex conjugation. □

Corollary 3.6. *If $A \in M_n(\mathbb{H})$ is unitary ($A^*A = I$), then the eigenvalues of A have all norm equal to 1.*

Proof. If A is unitary, then $f(A)$ is a unitary matrix in $M_{2n}(\mathbb{C})$ by Theorem 3.3 (2). It is known that eigenvalues of unitary complex matrices have norm equal to 1. □

3.2 | Translation lengths in Sp$(n, 1)$

We start with the following easy observation.

Lemma 3.7. *Let $A \in \mathrm{Sp}(n, 1)$. If $t \in \mathbb{C}$ is an eigenvalue of A, then so is t^{-1}.*

Proof. If t is an eigenvalue of A, then clearly t^{-1} is an eigenvalue of A^{-1}. Now, the equation $A^*JA = J$ implies $A^{-1} = J^{-1}A^*J$, and thus A^{-1} has the same eigenvalues as A^*. But these are the same as the eigenvalues of A by Corollary 3.5. It follows that t^{-1} is an eigenvalue of A. □

We can now prove the main result of this section. For the results concerning the algebra of Hermitian spaces, we refer the reader to [19].

Proposition 3.8. *Let $A \in \mathrm{Sp}(n,1)$, and assume that A leaves invariant a geodesic of $\mathbf{H}^n_{\mathbb{H}}$.*

(1) *There are exactly 4 eigenvalues of A with norm different from 1. If t is one of such eigenvalues, the other such eigenvalues are given by $\bar{t}, t^{-1}$ and $\bar{t}^{-1}$.*
(2) *Assume that $|t| > 1$ in (1). Then, the translation length ℓ_A of A along the geodesic satisfies the equation*

$$\ell_A = 2\ln(|t|).$$

Proof. Being of real rank 1, the Lie group $\mathrm{Sp}(n,1)$ acts transitively on the set of geodesics of its associated symmetric space $\mathbf{H}^n_{\mathbb{H}}$. Therefore, after conjugation, we may assume that A fixes the geodesic curve $\alpha(s) = (\tanh s, 0\dots,0)$. Let $\mathbf{v}_0 = (1,-1,\dots,0)$ and $\mathbf{v}_1 = (1,1,0,\dots,0)$, so that $P(\mathbf{v}_0)$ and $P(\mathbf{v}_1)$ are the limit points of α in $\partial\mathbf{H}^n_{\mathbb{H}}$. In particular those limit points are fixed by A, that is, there exist $\lambda, \beta \in \mathbb{H}$ such that

$$A\mathbf{v}_0 = \mathbf{v}_0\lambda,$$

$$A\mathbf{v}_1 = \mathbf{v}_1\beta.$$

In terms of the standard basis $\mathbf{e}_0,\dots,\mathbf{e}_n$ of $\mathbb{H}^{n,1}$ we obtain

$$A\mathbf{e}_0 = \mathbf{e}_0\frac{\beta+\lambda}{2} + \mathbf{e}_1\frac{\beta-\lambda}{2}; \tag{3.4}$$

$$A\mathbf{e}_1 = \mathbf{e}_0\frac{\beta-\lambda}{2} + \mathbf{e}_1\frac{\beta+\lambda}{2}. \tag{3.5}$$

It also shows that the right $\mathbb{H}$-submodule V of $\mathbb{H}^{n,1}$ generated by $\{\mathbf{v}_0, \mathbf{v}_1\}$ coincides with the right $\mathbb{H}$-submodule of $\mathbb{H}^{n,1}$ generated by $\{\mathbf{e}_0, \mathbf{e}_1\}$, and A leaves V invariant. Since the Hermitian form restricted to V has signature (1,1), its restriction to $V^\perp$ has signature $(n-1, 0)$. Now, the fact that A preserves the Hermitian form implies that its restriction to the submodule $V^\perp$ is a unitary transformation. Since $\mathbf{e}_2,\dots,\mathbf{e}_n$ are orthogonal to $\mathbf{e}_0$ and $\mathbf{e}_1$ (with respect to the hermitian form J), they generate $V^\perp$, and in a suitable basis the matrix A has the form

$$A = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & B \end{pmatrix}$$

where $B \in M_{n-1}(\mathbb{H})$ is unitary. By Corollary 3.6, the eigenvalues of B have norm 1; it thus remains to study the eigenvalues of the matrix

$$\begin{pmatrix} \lambda & 0 \\ 0 & \beta \end{pmatrix}. \tag{3.6}$$

Since A preserves the geodesic $\alpha(s) = (\tanh s, 0, \dots, 0)$, and $\alpha(0) = P(\mathbf{e}_0)$, it follows that $A(P(\mathbf{e}_0)) = (\tanh s_0, \dots, 0)$ for some real number $s_0 > 0$. Moreover, the equality

$$(\tanh s_0, 0, \dots, 0) = P((\cosh s_0, \sinh s_0, 0, \dots, 0))$$

implies the existence of $w \in \mathbb{H}$ such that

$$A(\mathbf{e}_0) = \begin{pmatrix} \cosh s_0 \\ \sinh s_0 \\ \vdots \\ 0 \end{pmatrix} w.$$

Since A preserves the Hermitian form, w has quaternion norm equal to 1. Comparing with (3.5), we obtain the equations

$$\frac{\beta - \lambda}{2} = \sinh s_0 w \quad \text{and} \quad \frac{\beta + \lambda}{2} = \cosh s_0 w,$$

which imply that $\beta = e^{s_0} w$, and $\lambda = e^{-s_0} w$. In particular, λ and β have norm different from 1, and they satisfy the relations $\overline{\lambda} = \beta^{-1}, \overline{\beta} = \lambda^{-1}$. Since quaternions are similar to complex numbers with the same real part and norm (Lemma 2.1), we may assume that λ and β are complex numbers, and then eigenvalues of A. See Remark 3.2. Moreover, similarity in $\mathbb{H}$ commutes with conjugation and preserves the norm, and then we obtain that the eigenvalues of the matrix (3.6) have the form t, t^{-1}, $\overline{t}$, $\overline{t}^{-1}$ with $t \in \mathbb{C}$ such that $|t| > 1$.

The translation length ℓ_A of A is equal to the distance between $P(\mathbf{e}_0)$ and $P(A\mathbf{e}_0)$. Using the distance formula (2.1), a direct computation shows that

$$\ell_A = \rho(P(\mathbf{e}_0), A(P(\mathbf{e}_0))) = 2s_0 = 2\log(|\beta|) = 2\log(|t|),$$

proving the second statement. □

Corollary 3.9. *Let $A \in \mathrm{Sp}(n,1)$ leaving invariant a geodesic in $\mathbf{H}^n_{\mathbb{H}}$. Then*

$$e^{\frac{\ell_A}{2}} \geqslant \frac{|\operatorname{Re}(\operatorname{tr}(A))|}{n+1}.$$

Proof. Let $t_1, \overline{t}_1, \dots, t_{n+1}, \overline{t}_{n+1}$ be the eigenvalues of A. Assume that t_1 (or $\overline{t}_1$) is the eigenvalue with largest norm. By Proposition 3.8, we have $\ell_A = 2\log(|t_1|)$, and applying Corollary 3.4, we obtain

$$\begin{aligned} \frac{|\operatorname{Re}(\operatorname{tr}(A))|}{n+1} &= \frac{|t_1 + \overline{t}_1 + \cdots + t_{n+1} + \overline{t}_{n+1}|}{2(n+1)} \\ &\leqslant \frac{|t_1| + |\overline{t}_1| + \cdots + |t_{n+1}| + |\overline{t}_{n+1}|}{2(n+1)} \end{aligned}$$

$$\leqslant |t_1|$$
$$= e^{\frac{\ell_A}{2}}. \qquad \square$$

4 | ARITHMETIC SUBGROUPS OF Sp$(n, 1)$

4.1 | Arithmeticity of lattices

Combined work of Margulis, Corlette, and Gromov-Schoen shows that any lattice Γ in $\mathrm{Sp}(n,1)$ is arithmetic. That is, there exists a number field k with degree $[k : \mathbb{Q}] = d$ and an absolutely simple algebraic k-group $\mathbf{G}$ such that

$$\mathbf{G}(k \otimes_{\mathbb{Q}} \mathbb{R}) \cong \mathrm{Sp}(n,1) \times K, \tag{4.1}$$

where K is a compact group, and Γ is commensurable with $\mathbf{G}(\mathcal{O}_k) = \mathbf{G} \cap \mathrm{GL}_m(\mathcal{O}_k)$ for a fixed embedding $\mathbf{G} \to \mathrm{GL}_m$. A group $\mathbf{G}$ satisfying (4.1) is called *admissible*.

The condition (4.1) implies that k is a totally real number field, $\mathbf{G}$ is a simply connected algebraic k-group of type C_{n+1}, and by fixing an embedding $k \subset \mathbb{R}$ we may assume that $\mathbf{G}(\mathbb{R}) = \mathrm{Sp}(n,1)$. Moreover, Γ is cocompact if and only if $k \neq \mathbb{Q}$ (see [4, Proposition 2.8]).

4.2 | Admissible groups

By the classification of simple algebraic groups any admissible k-group $\mathbf{G}$ is isomorphic to a unitary group $\mathbf{U}(V, h)$, where D is a quaternion algebra over k, and V is the right D-module D^{n+1} equipped with a nondegenerate Hermitian form h which is sesquilinear with respect to the standard involution of D. More precisely, we have the following.

Proposition 4.1. *Let $k \subset \mathbb{R}$ be a totally real number field. Any admissible k-group $\mathbf{G}$ is of the form $\mathbf{U}(V, h_a)$, where $a \in \mathcal{O}_k$ and*

$$h_a(x, y) = -a\overline{x}_0 y_0 + \sum_{i=1}^{n} \overline{x}_i y_i. \tag{4.2}$$

Proof. It follows from [4, Proposition 2.6] that $\mathbf{G}$ depends only on D, and not on the choice of h provided the latter has the correct signature over the different embeddings $\sigma : k \to \mathbb{R}$. Then it suffices to choose a positive $a \in \mathcal{O}_k$ with $\sigma(a) < 0$ for all nontrivial embeddings $\sigma : k \to \mathbb{R}$; such an element exists by weak approximation (see [18, Section 1.2.2]). $\square$

4.3 | A group scheme structure

We fix an order $\mathcal{O}_D$ of the quaternion algebra D, and consider the lattice $L = \mathcal{O}_D^{n+1}$ in V. Choosing h as in (4.2) we obtain a Hermitian module (L, h) (more generally it suffices to take h with integral coefficients). For any ring extension $\mathcal{O}_k \to A$, we consider the unitary group

$$\mathbf{G}^L(A) = \mathbf{U}(L \otimes_{\mathcal{O}_k} A, h). \tag{4.3}$$

Then $\mathbf{G}^L$ defines an affine group scheme over $\mathcal{O}_k$ with generic fiber $\mathbf{G}$; it is a closed subgroup of the group scheme $\mathfrak{End}_{\mathcal{O}_D}(L)^\times$ defined by taking invertible endomorphisms (see [3, Section II.2.6]). In particular, the arithmetic subgroup $\Gamma = \mathbf{G}^L(\mathcal{O}_k)$ can be seen as a subgroup of the matrix group $\mathrm{GL}_{n+1}(\mathcal{O}_D)$.

For any ideal $I \subset \mathcal{O}_k$, we define the principal congruence subgroup Γ_I as the kernel of the natural map $\mathbf{G}^L(\mathcal{O}_k) \to \mathbf{G}^L(\mathcal{O}_k/I)$.

Proposition 4.2. *The subgroup Γ_I corresponds to the kernel of the map $\Gamma \to \mathrm{GL}_{n+1}(\mathcal{O}_D/I\mathcal{O}_D)$.*

Proof. In view of the definition (4.3), this follows directly from the isomorphism $\mathcal{O}_D \otimes_{\mathcal{O}_k} \mathcal{O}_k/I \cong \mathcal{O}_D/I\mathcal{O}_D$. □

4.4 | Localizations

We will denote by $\mathcal{V} = \mathcal{V}_\mathrm{f} \cup \mathcal{V}_\infty$ the set of (finite and infinite) places of k. For any $v \in \mathcal{V}$, the symbol k_v denotes the completion of k with respect to v, and $D_v = D \otimes_k k_v$. It follows from the admissibility of $\mathbf{G} = \mathbf{U}(V, h)$ that $D_v \cong \mathbb{H}$ for each $v \in \mathcal{V}_\infty$; see [4, Corollary 2.5]. For $v \in \mathcal{V}_\mathrm{f}$, we denote by $\mathfrak{o}_v$ the valuation ring of k_v, and by π_v a uniformizer in $\mathfrak{o}_v$.

For $L = \mathcal{O}_D^{n+1}$ and $e \in \mathbb{N}$, we set $P_v^{(e)} = \ker(\mathbf{G}^L(\mathfrak{o}_v) \to \mathbf{G}^L(\mathfrak{o}_v/\pi_v^e))$. Note that $P_v^{(0)} = \mathbf{G}^L(\mathfrak{o}_v)$, and the latter is a hyperspecial parahoric subgroup of $\mathbf{G}(k_v)$ for all but finitely many $v \in \mathcal{V}_\mathrm{f}$; see [20, Section 3.9.1]. We shall use the notation $P_v = P_v^{(0)}$. The group $\Gamma = \mathbf{G}^L(\mathcal{O}_k)$ can thus be written as $\mathbf{G}(k) \cap \prod_{v \in \mathcal{V}_\mathrm{f}} P_v$. Let $I \subset \mathcal{O}_k$ be an ideal with prime factorization $I = \prod_{v \in \mathcal{V}_\mathrm{f}} \mathfrak{p}_v^{e_v}$. Then it follows from the Chinese reminder theorem that

$$\Gamma_I = \mathbf{G}(k) \cap \prod_{v \in \mathcal{V}_\mathrm{f}} P_v^{(e_v)}. \tag{4.4}$$

The following lemma is a well-known consequence of the strong approximation property for $\mathbf{G}$.

Lemma 4.3. *For the index, the following equality holds*

$$[\Gamma : \Gamma_I] = \prod_{v \in \mathcal{V}_\mathrm{f}} [P_v : P_v^{(e_v)}].$$

Proof. For two subgroups A, B of a common group, there is a bijection between $A/A \cap B$ and AB/B. The result follows with $A = \Gamma = \mathbf{G}(k) \cap \prod P_v$ and $B = \prod P_v^{(e_v)}$, noting that $\mathbf{G}(k)B$ is the whole adelic group $\mathbf{G}(\mathbb{A}_\mathrm{f})$ by strong approximation. □

Lemma 4.4. *Assume that $P_v = \mathbf{G}^L(\mathfrak{o}_v)$ is parahoric hyperspecial. Then*

$$[P_v : P_v^{(e)}] = q_v^{e(n+1)(2n+3)} \prod_{j=1}^{n+1} \left(1 - \frac{1}{q_v^{2j}}\right),$$

where q_v denotes the order of the residue field $\mathfrak{f}_v = \mathfrak{o}_v/\pi_v$.

Proof. If $\mathbf{G}^L(\mathfrak{o}_v)$ is hyperspecial, then $\mathbf{G}_{k_v}$ must be split, and $\mathbf{G}^L_{\mathfrak{o}_v}$ is the Chevalley group scheme of type C_{n+1} (see [20, Section 3.4.2]). In particular, $\mathbf{G}^L_{\mathfrak{o}_v}$ is smooth, and the reduction map $P_v \to \mathbf{G}^L(\mathfrak{o}_v/\pi_v^e)$ is surjective. In case $e = 1$, the index is thus given by the order of $\mathbf{G}^L(\mathfrak{f}_v)$, which can be found, for instance, in [17, Table 1]. For $e > 1$, this order must be multiplied by the order of $\ker(\mathbf{G}^L(\mathfrak{o}_v/\pi_v^e) \to \mathbf{G}^L(\mathfrak{f}_v))$, which by induction over e equals $q_v^{(e-1)\dim(\mathbf{G})}$. For the type C_{n+1} we have $\dim(\mathbf{G}) = (n+1)(2n+3)$, and the result follows. □

4.5 | An upper bound for the index

We keep the notation introduced above. We denote by $\mathrm{N}(I)$ the norm of an ideal $I \subset \mathcal{O}_k$, that is, the order of $\mathcal{O}_k/I$.

Proposition 4.5. *Let $\Gamma = \mathbf{G}^L(\mathcal{O}_k)$. There exists a finite set $S \subset \mathcal{V}_\mathrm{f}$ such that for any ideal $I \subset \mathcal{O}_k$ prime to S the following holds:*

$$[\Gamma : \Gamma_I] \leqslant \mathrm{N}(I)^{(n+1)(2n+3)}.$$

Proof. Let S be the set of places v such that $P_v = \mathbf{G}^L(\mathfrak{o}_v)$ is not hyperspecial. Let $I = \prod_{v\in\mathcal{V}_\mathrm{f}} \mathfrak{p}_v^{e_v}$ be any ideal with $e_v = 0$ for each $v \in S$. From (4.4) and Lemma 4.4, we obtain

$$\begin{aligned}[\Gamma : \Gamma_I] &= \prod_{v\in\mathcal{V}_\mathrm{f}} [P_v : P_v^{(e_v)}] \\ &\leqslant \prod_{v\in\mathcal{V}_\mathrm{f}} q_v^{e_v(n+1)(2n+3)}.\end{aligned}$$

But the latter equals $\mathrm{N}(I)^{(n+1)(2n+3)}$ since $q_v = \mathrm{N}(\mathfrak{p}_v)$. □

Remark 4.6. Proposition 4.5 holds for any arithmetic subgroup of $\mathbf{G}(k)$. In the case $\Gamma = \mathbf{G}^L(\mathcal{O}_k)$, we can have some control on the set S. Let us assume that $L = \mathcal{O}_D^{n+1}$ with $\mathcal{O}_D$ a *maximal* order, and consider the integral form $h = h_a$ given in (4.2). Then it follows from [4, Lemmas 5.1 and 5.5] that S can be taken to be the set of places v where either

- D_v ramifies;
- or $\mathfrak{p}_v$ divides the coefficient a.

5 | BOUNDING THE SYSTOLE FROM BELOW

This section deals with the computations that provide a lower bound for the trace in a congruence subgroup. This material is then used in Section 5.3 for bounding the systole of the corresponding manifolds, in particular for proving Theorem 1.1.

We essentially keep the notation of the preceding section: Γ will denote the arithmetic subgroup $\mathbf{G}^L(\mathcal{O}_k)$, where $\mathbf{G} = \mathbf{U}(V, h)$ is an admissible k-group with $h = h_a$ as in (4.2). It will be important to work with the matrix representation with coefficients in D (the quaternion algebra over k).

That is, we embed Γ as a subgroup of

$$\mathbf{G}(k) = \{C \in \mathrm{GL}_{n+1}(D) \mid C^* J C = J\},$$

where $J = \mathrm{diag}(-a, 1, \dots, 1)$. In particular the trace $\mathrm{tr}(C)$ has the same meaning as in Section 3. In accordance with the notation of Sections 2 and 3, we use the convention that the rows (respectively, columns) of the matrices are indexed from 0 to n.

5.1 | Two lemmas

Recall that we have fixed an embedding $k \subset \mathbb{R}$, which we refer to as the *trivial* embedding (or trivial Archimedean place). The symbol I_{n+1} denotes the identity matrix in $\mathrm{GL}_{n+1}(D)$.

Lemma 5.1. *Assume $k \neq \mathbb{Q}$. For any $C \in \Gamma$ different from $\pm I_{n+1}$, we have $|\operatorname{Re}(\mathrm{tr}(C))| \neq n + 1$.*

Proof. Suppose that $|\operatorname{Re}(\mathrm{tr}(C))| = n + 1$. Since $k \neq \mathbb{Q}$ there exists a nontrivial embedding $\sigma : k \to \mathbb{R}$, for which $|\operatorname{Re}(\mathrm{tr}(C^\sigma))| = n + 1$. By the admissibility condition, we have $C^\sigma \in \mathrm{Sp}(n + 1)$, so that C^σ is unitary (in the quaternionic sense). Corollary 3.6 implies that the eigenvalues of C^σ are all complex numbers of norm equal to one. With $|\operatorname{Re}(\mathrm{tr}(C^\sigma))| = n + 1$, it follows that these eigenvalues are either all equal to 1, or all equal to -1. Since unitary matrices are diagonalizable, we obtain $C^\sigma = \pm I_{n+1}$, so that $C = \pm I_{n+1}$. □

Lemma 5.2. *Let $C = (c_{ij})$ be an element in $\mathbf{G}(k)$ and write $c_{ii} = 1 + y_i$. For every nontrivial embedding $\sigma : k \to \mathbb{R}$ and each $i = 0, \dots, n$, we have $|\sigma(|c_{ii}|^2)| \leqslant 1$ and $|\sigma(\operatorname{Re}(y_i))| \leqslant 2$.*

Proof. The equation $C^* J C = J$ implies that the columns of C satisfy the equations

$$-a|c_{00}|^2 + \sum_{i=1}^{n} |c_{i0}|^2 = -a, \tag{5.1}$$

$$-a|c_{0j}|^2 + \sum_{i=1}^{n} |c_{ij}|^2 = 1, \text{ for } j = 1, \dots, n. \tag{5.2}$$

where $|x|^2 = x\bar{x}$ denotes the quaternion norm of x in D. Since all the coefficients c_{ij} lie in D, the norm $|c_{ij}|^2$ is an element of k. Let $\sigma : k \to \mathbb{R}$ be a nontrivial embedding. Applying σ to (5.1), we obtain

$$\begin{aligned} -\sigma(a)\sigma(|c_{00}|^2) &\leqslant -\sigma(a)\sigma(|c_{00}|^2) + \sum_{i=1}^{n} \sigma(|c_{i0}|^2) \\ &= -\sigma(a). \end{aligned}$$

Hence $|\sigma(|c_{00}|^2)| \leqslant 1$. Similarly, applying σ to (5.2), we obtain that $|\sigma(|c_{ii}|^2)| \leqslant 1$ for $i = 1, \dots, n$. Now, if $D = (\frac{\delta,\gamma}{k})$ and $c = x_0 + x_1 i + x_2 j + x_3 ij$, then $|c|^2 = x_0^2 - \delta x_1^2 - \gamma x_2^2 + \delta\gamma x_3^2$. Since D^σ is a division algebra, we have $\sigma(\delta) < 0$ and $\sigma(\gamma) < 0$, and thus

$$\begin{aligned}\sigma(|c|^2) &= \sigma(x_0)^2 - \sigma(\delta)\sigma(x_1)^2 - \sigma(\gamma)\sigma(x_2)^2 + \sigma(\delta)\sigma(\gamma)\sigma(x_3)^2 \\ &\geqslant \sigma(x_0)^2 \\ &= \sigma(\mathrm{Re}(c))^2.\end{aligned}$$

In particular, $\sigma(|c_{ii}|^2) \leqslant 1$ implies $|\sigma(\mathrm{Re}(c_{ii}))| \leqslant 1$, from which one deduces $|\sigma(\mathrm{Re}(y_i))| \leqslant 2$. □

5.2 | Bounding the trace

We want to bound the trace of a congruence subgroup Γ_I, for $I \subset \mathcal{O}_k$ some ideal. In the matrix representation, we have the following description:

$$\Gamma_I = \{(c_{ij}) \in \Gamma \mid c_{ii} - 1 \in I\mathcal{O}_D, c_{ij} \in I\mathcal{O}_D \text{ for } i \neq j\}.$$

We recall that the element $a \in \mathcal{O}_k$ appears (with negative sign) as the unique nontrivial coefficient of the Hermitian form $h = h_a$ that determines $\mathbf{G}$.

Lemma 5.3. *Let $C \in \Gamma_I$, and write $c_{ii} = 1 + y_i$. Then*

$$2a \sum_{i=0}^{n} \mathrm{Re}(y_i) \in I^2. \tag{5.3}$$

Proof. We first replace $c_{00} = 1 + y_0$ in (5.1) to obtain

$$-a(2\,\mathrm{Re}(y_0) + |y_0|^2) + \sum_{i=1}^{n} |c_{i0}|^2 = 0. \tag{5.4}$$

For $C \in \Gamma_I$ we have $y_0 \in I\mathcal{O}_D$ and $c_{i0} \in I\mathcal{O}_D$ for $i > 0$. From (5.4), it follows that $2a\,\mathrm{Re}(y_0) \in I^2$. By replacing $c_{ii} = 1 + y_i$ in (5.2), the same argument shows that $2\,\mathrm{Re}(y_i) \in I^2$ for $i > 0$. Since $a \in \mathcal{O}_k$ we have that $2a\,\mathrm{Re}(y_i) \in I^2$ for all $i = 0, \dots, n$, and thus the same holds for their sum. □

In the following, the symbol $\mathrm{N}(\cdot)$ denotes either the norm $\mathrm{N}_{k/\mathbb{Q}}(\cdot)$ for elements of k, or the norm of ideals in $\mathcal{O}_k$. Recall that for a principal ideal $I = (\alpha)$, one has $\mathrm{N}(I) = |\,\mathrm{N}(\alpha)|$ unless $\alpha = 0$.

Corollary 5.4. *Let Γ_I be defined over the number field k of degree $d > 1$, and let $C \in \Gamma_I$ different from $\pm I_{n+1}$. Then*

$$\left|\mathrm{N}\left(\sum_{i=0}^{n} \mathrm{Re}(y_i)\right)\right| \geqslant \frac{\mathrm{N}(I)^2}{2^d\,\mathrm{N}(a)},$$

where $c_{ii} = 1 + y_i$.

Proof. We have $\sum_{i=0}^{n} \mathrm{Re}(y_i) \neq 0$ by Lemma 5.1. The result follows then immediately by applying $\mathrm{N}(\cdot)$ on (5.3). ☐

Proposition 5.5. *Let k be of degree $d > 1$, and $I \subset \mathcal{O}_k$ be a proper nontrivial ideal. For any $C \in \Gamma_I$ different from $\pm I_{n+1}$ we have*

$$|\,\mathrm{Re}(\mathrm{tr}(C))| \geqslant \frac{\mathrm{N}\,(I)^2}{2^{2d-1}(n+1)^{d-1}\,\mathrm{N}(a)} - n - 1.$$

Proof. By Lemma 5.2, we have

$$\begin{aligned} \left|\mathrm{N}\left(\sum_{i=0}^{n}\mathrm{Re}(y_i)\right)\right| &= \left|\sum_{i=0}^{n}\mathrm{Re}(y_i)\right|\left|\prod_{\sigma\neq id}\sigma\left(\sum_{i=0}^{n}\mathrm{Re}(y_i)\right)\right| \\ &\leqslant \left|\sum_{i=0}^{n}\mathrm{Re}(y_i)\right|\cdot 2^{d-1}(n+1)^{d-1}. \end{aligned}$$

With Corollary 5.4, we obtain

$$\left|\sum_{i=0}^{n}\mathrm{Re}(y_i)\right| \geqslant \frac{\mathrm{N}\,(I)^2}{2^{2d-1}(n+1)^{d-1}\,\mathrm{N}(a)}.$$

Now, since $\mathrm{Re}(\mathrm{tr}(C)) = n + 1 + \sum_{i=0}^{n} \mathrm{Re}(y_i)$, we have

$$\begin{aligned} |\,\mathrm{Re}(\mathrm{tr}(C))| &\geqslant \left|\sum_{i=0}^{n}\mathrm{Re}(y_i)\right| - n - 1 \\ &\geqslant \frac{\mathrm{N}\,(I)^2}{2^{2d-1}(n+1)^{d-1}\,\mathrm{N}(a)} - n - 1. \end{aligned}$$

☐

5.3 | Bounding the systole

We can now use the preceding results to obtain a lower bound for the systole of $\Gamma_I\backslash\mathbf{H}^n_{\mathbb{H}}$ in terms of the norm of the ideal I.

Proposition 5.6. *Let $M = \Gamma\backslash\mathbf{H}^n_{\mathbb{H}}$ be a compact arithmetic orbifold with $\Gamma = \mathbf{G}^L(\mathcal{O}_k)$. If Γ_I is a principal congruence subgroup associated to an ideal $I \subset \mathcal{O}_k$, then*

$$\mathrm{sys}_1(M_I) \geqslant 4\log(\mathrm{N}(I)) - c,$$

where $M_I = \Gamma_I\backslash\mathbf{H}^n_{\mathbb{H}}$ is the associated congruence cover of M, and c is a constant independent of I.

Proof. Let $A \in \Gamma_I$ be a matrix corresponding to a shortest closed geodesic in M_I, so that its translation length ℓ_A equals $\mathrm{sys}_1(M_I)$. By Corollary 3.9 and Proposition 5.5, we obtain

$$\begin{aligned}\ell_A &\geqslant 2\log\left(\frac{|\operatorname{Re}(\operatorname{tr}(A))|}{n+1}\right)\\ &\geqslant 2\log\left(\frac{\mathrm{N}(I)^2}{2^{2d-1}(n+1)^d\,\mathrm{N}(a)}-1\right)\\ &\geqslant 2\log\left(\frac{\mathrm{N}(I)^2}{2\cdot 2^{2d-1}(n+1)^d\,\mathrm{N}(a)}\right)\\ &= 4\log(\mathrm{N}(I)) - 2\log\left(2^{2d}(n+1)^d\,\mathrm{N}(a)\right)\end{aligned}$$

if $\mathrm{N}(I)^2 \geqslant 2^{2d}(n+1)^d\,\mathrm{N}(a)$. Since there exist only finitely many ideals $I \subset \mathcal{O}_k$ with bounded norm, the result follows by enlarging the constant c if necessary. □

We can now prove the main result.

Proof of Theorem 1.1. Let $M = \Gamma\backslash\mathbf{H}^n_{\mathbb{H}}$ be a compact quaternionic orbifold. Then $\Gamma \subset \mathbf{G}(k)$ for some admissible k-group $\mathbf{G}$ with $k \neq \mathbb{Q}$. On the other hand, by [10, Proposition 2.2] we can replace the study of Γ with any subgroup commensurable with it, in particular, we may assume that $\Gamma = \mathbf{G}^L(\mathcal{O}_k)$ as above. By Proposition 4.5, there exist a finite set S of prime ideals of $\mathcal{O}_k$ such that any ideal $I \subset \mathcal{O}_k$ with no prime factors in S satisfies

$$[\Gamma : \Gamma_I] \leqslant \mathrm{N}(I)^{(n+1)(2n+3)}.$$

From Proposition 5.6, we obtain

$$\operatorname{sys}_1(M_I) \geqslant \frac{4}{(n+1)(2n+3)}\log([\Gamma : \Gamma_I]) - c,$$

for some constant c independent of I. The result then follows with the equality $\operatorname{vol}(M_I) = \operatorname{vol}(M)[\Gamma : \Gamma_I]$. □

6 | OPTIMALITY OF THE CONSTANT

In this section, we show that the constant $\frac{4}{(n+1)(2n+3)}$ is sharp, using similar arguments as in the Appendix of [16]. The precise result is the following.

Theorem 6.1. *Let $k \subset \mathbb{R}$ be totally real with $k \neq \mathbb{Q}$, and let $\mathbf{G}$ be an admissible k-group, so that $\mathbf{G}(\mathbb{R}) = \mathrm{Sp}(n,1)$. Then there exists an arithmetic subgroup $\Gamma \subset \mathbf{G}(k)$ such that for any sequence of prime ideals $\mathfrak{p} \subset \mathcal{O}_k$ the principal congruence subgroups $\Gamma_{\mathfrak{p}}$ satisfy*

$$\operatorname{sys}_1(M_{\mathfrak{p}}) \leqslant \frac{4}{(n+1)(2n+3)}\log\left(\operatorname{vol}(M_{\mathfrak{p}})\right) + d',$$

where $M_{\mathfrak{p}} = \Gamma_{\mathfrak{p}}\backslash\mathbf{H}^n_{\mathbb{H}}$ and d' is a constant independent of $\mathfrak{p}$.

Proof. Let $\mathbf{G} = \mathbf{U}(V, h_a)$ and set $\Gamma = \mathbf{G}^L(\mathcal{O}_k)$ (see Sections 4.2–4.3). The idea is to construct arithmetic real hyperbolic manifolds which are totally geodesic submanifolds in $\Gamma_{\mathfrak{p}}\backslash\mathbf{H}^n_{\mathbb{H}}$, and to apply [16, Theorem A.1]. The latter proves in particular the case $n = 1$, so that we will assume $n > 1$ hereafter.

We denote by $\{e_0, \dots, e_n\}$ the standard basis of $V = D^{n+1}$; recall that D is a quaternion algebra over k. Let W be the k-vector space generated by $\{e_0, \dots, e_n\}$, and $L' \subset W$ be the $\mathcal{O}_k$-lattice with the same basis. Consider the quadratic form on W given by

$$q(x, y) = -ax_0y_0 + \sum_{i=1}^{n} x_iy_i.$$

This is the restriction to W of the Hermitian form h_a; it is admissible in the sense of [16, Section 2.3]. We consider the k-group $\mathbf{SO}(W, q)$ and its simply connected cover $\mathbf{Spin}(W, q)$. The Lie group $\mathbf{Spin}(W, q)(\mathbb{R})/\{\pm I\}$ is isomorphic to the orientation preserving isometry group of the real hyperbolic n-space $\mathbf{H}^n_{\mathbb{R}}$. Since the basis $\{e_0, \dots, e_n\}$ is common to V and W, we have an inclusion $\mathbf{SO}(W, q) \subset \mathbf{U}(V, h_a)$, and composing with the isogeny we obtain a homomorphism $\mathbf{Spin}(W, q) \to \mathbf{G}$ defined over k.

Let $\Gamma' \subset \mathbf{Spin}(W, q)(k)$ be the stabilizer of L'. Since $k \neq \mathbb{Q}$ it is a cocompact arithmetic lattice in $\mathrm{Spin}(n, 1)$. The map $\mathbf{Spin}(W, q) \to \mathbf{G}$ induces a map $\Gamma' \to \Gamma$, and similarly $\Gamma'_I \to \Gamma_I$ for any ideal $I \subset \mathcal{O}_k$. This induces a totally geodesic embedding

$$T_I \hookrightarrow M_I,$$

where $T_I = \Gamma'_I\backslash\mathbf{H}^n_{\mathbb{R}}$. Therefore

$$\mathrm{sys}_1(M_I) \leqslant \mathrm{sys}_1(T_I).$$

From now on, we will assume that $I = \mathfrak{p}$ is a prime ideal. By [16, Theorem A.1] (see also [13, Theorem B]), there exists a constant d independent of $\mathfrak{p}$ such that

$$\mathrm{sys}_1(T_{\mathfrak{p}}) \leqslant \frac{8}{n(n+1)} \log(\mathrm{vol}(T_{\mathfrak{p}})) + d. \tag{6.1}$$

Following the argument as in [13, Theorem B], there exist constants a_1 and a_2 such that

$$a_1 \leqslant \frac{\mathrm{vol}(T_{\mathfrak{p}})}{\mathrm{N}(\mathfrak{p})^{\frac{n(n+1)}{2}}} \leqslant a_2. \tag{6.2}$$

For $\mathfrak{p}$ of norm large enough, we have that $\mathbf{G}^L(\mathfrak{o}_{\mathfrak{p}})$ is parahoric hyperspecial. By Lemmas 4.3 and 4.4, we obtain

$$[\Gamma : \Gamma_{\mathfrak{p}}] = \mathrm{N}(\mathfrak{p})^{(n+1)(2n+3)} \prod_{j=1}^{n+1} \left(1 - \frac{1}{\mathrm{N}(\mathfrak{p})^{2j}}\right).$$

Since $\mathrm{vol}(M_{\mathfrak{p}}) = \mathrm{vol}(M)[\Gamma : \Gamma_{\mathfrak{p}}]$, there exist positive constants b_1 and b_2 such that

$$b_1 \leqslant \frac{\mathrm{vol}(M_{\mathfrak{p}})}{\mathrm{N}(\mathfrak{p})^{(n+1)(2n+3)}} \leqslant b_2. \tag{6.3}$$

By plugging the right-hand side of (6.2) in (6.1), and using the left-hand side of (6.3) afterward, we conclude that

$$\begin{aligned}\mathrm{sys}_1(M_{\mathfrak{p}}) &\leqslant \mathrm{sys}(T_{\mathfrak{p}}) \\ &\leqslant \frac{4}{(n+1)(2n+3)} \log(\mathrm{vol}(M_{\mathfrak{p}})) + d',\end{aligned}$$

for some constant d' independent of $\mathfrak{p}$. □

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REFERENCES

1. M. Belolipetsky, *On 2-systoles of hyperbolic 3-manifolds*, Geom. Funct. Anal. **23** (2013), no. 3, 813–827.
2. P. Buser and P. Sarnak, *On the period matrix of a Riemannian surface of large genus (with an appendix by J. Conway and N. Sloan)*, Invent. Math. **117** (1994), 27–56.
3. M. Demazure and P. Gabriel, *Groupes algébriques*. Tome I, North-Holland Publishing Company, Amsterdam, 1970.
4. V. Emery and I. Kim, *Quaternionic hyperbolic lattices of minimal covolume*, arXiv:1802.07776, 2018.
5. M. Gromov, *Systoles and intersystolic inequalities*, Sémin. Congr. **1** (1996), 291–362.
6. L. Guth and A. Lubotzky, *Quantum error correcting codes and 4-dimensional arithmetic hyperbolic manifolds*, J. Math. Phys. **55** (2014), 082202.
7. M. G. Katz, M. Schaps, and U. Vishne, *Logarithmic growth of systole of arithmetic Riemann surfaces along congruence subgroups*, J. Differential Geom. **76** (2007), no. 3, 399–422.
8. I. Kim and J. R. Parker, *Geometry of quaternionic hyperbolic manifolds*, Math. Proc. Cambridge. Philos. Soc. **135** (2003), no. 2, 291–320.
9. I. Kim, *Systole on locally symmetric spaces*, Bull. Lond. Math. Soc. **52** (2020), no. 2, 349–357.
10. S. Lapan, J. S. Meyer, and B. Linowitz, *Systole inequalities for arithmetic locally symmetric spaces*, arXiv:1710.00071v2, 2017.
11. H. C. Lee, *Eigenvalues and canonical forms of matrices with quaternion coefficients*, Proc. Roy. Irish Acad. Sect. A **52** (1949), 253–260.
12. C. Maclachlan and A. W. Reid, *The arithmetic of hyperbolic 3-manifolds,* Grad. Texts Math, vol. 219. Springer, Berlin, 2013.

13. J. S. Meyer, *Totally geodesic spectra of quaternionic hyperbolic orbifolds*, arXiv:1505.03643, 2015.
14. P. G. P. Murillo, *On arithmetic manifolds with large systole*, Ph.D. thesis, IMPA, 2017.
15. P. G. P. Murillo, *On growth of systole along congruence coverings of Hilbert modular varieties*, Algebr. Geom. Topol. **17** (2017), no. 5, 2753–2762.
16. P. G. P. Murillo, *Systole of congruence coverings of arithmetic hyperbolic manifolds (with an appendix by C. Dória and P. G. P. Murillo)*, Groups Geom. Dyn. **13** (2019), 1083–1102.
17. T. Ono, *On algebraic groups and discontinuous groups*, Nagoya Math. J. **27** (1966), 279–322.
18. V. Platonov and A. Rapinchuk, *Algebraic groups and number theory,* Pure Appl. Math, vol. 139, Academic Press Inc, Cambridge, MA, 1994.
19. W. Scharlau, *Quadratic and hermitian forms.,* Grundlehren der mathematischen Wissenschaften, vol. 270, Springer, Berlin, 1985.
20. J. Tits, *Reductive groups over local fields*, Proc. Symp. Pure Math. **33** (1979), Part I, 29–69.