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On the Thom Isomorphism for Groupoid-Equivariant Representable K-Theory

Zachary J. Garvey
Dartmouth College, zachary.j.garvey.gr@dartmouth.edu

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Examining Committee:

(Chair) **Erik van Erp**

Dana P. Williams

Jody Trout

Robin Deeley

F. Jon Kull, PhD.
Dean of the Guarini School of Graduate and Advanced Studies

Abstract

This thesis proves a general Thom Isomorphism in groupoid-equivariant KK-theory. Through formalizing a certain pushforward functor, we contextualize the Thom isomorphism to groupoid-equivariant representable K-theory with various support conditions. Additionally, we explicitly verify that a Thom class, determined by pullback of the Bott element via a generalized groupoid homomorphism, coincides with a Thom class defined via equivariant spinor bundles and Clifford multiplication. The tools developed in this thesis are then used to generalize a particularly interesting equivalence of two Thom isomorphisms on TX, for a Riemannian $\mathcal{G}$-manifold X.

Contents

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1 Introduction

The primary objective of this thesis is to carefully prove a Thom isomorphism in groupoid-equivariant KK-theory [11], and contextualize it to groupoid-equivariant representable K-theory [10]. Generally speaking, a Thom isomorphism in a cohomology theory applies to only certain types of vector bundles $E \to X$. For example, in ordinary (e.g., singular) cohomology, a Thom isomorphism only exists for oriented vector bundles; bundles like the Möbius bundle on S^1 do not admit a Thom isomorphism. To an oriented vector bundle $E \to X$, a Thom isomorphism identifies the cohomology of X with the cohomology (of a fiberwise-compactification) of E. For other cohomology theories, the relevant vector bundles are still called orientable, but this can mean very different things. In the context of K-theory, a vector bundle is K-orientable iff it admits a Spin^c-structure. Examples of bundles admitting a Spin^c-structure include almost-complex vector bundles, complex vector bundles, and real Spin-bundles, so each of these types of vector bundles admits a Thom isomorphism in complex K-theory.

This thesis is written in the context of groupoid-equivariant K-theory, which is substantially more complicated than ordinary K-theory. One of the major complications arises from the fact that spaces X, equipped with an action of a groupoid $\mathcal{G}$, must be fibered over the object space of $\mathcal{G}$ (loosely, in that there is a map $\rho : X \to \mathcal{G}^{(0)}$). When trying to define groupoid equivariant K-theory out of vector bundles, there are often too few equivariant vector bundles to get a decent theory. The only definition that seems to give an acceptable generalization of ordinary K-theory passes through K-theory for C^*-algebras. Through the correspondence $X \leftrightarrow C_0(X)$, K-theory for C^*-algebras indeed generalizes K-theory for locally compact Hausdorff spaces. Even more general is Kasparov's KK-theory (e.g., [15]), which is a bivariant K-homology and K-theory hybrid, denoted by $KK(A, B)$ for C^*-algebras A and B. This theory has incredibly rich structure, specifically through the application of a certain cup/cap product. A generalization of KK-theory to the groupoid-equivariant setting was worked out by Pierre-Yves Le Gall [11]. Spaces equipped with a groupoid action correspond to C^*-algebras that are "fibered" over the object space of the groupoid, together with a system of $*$-isomorphisms between fibers, which are continuously parameterized by the morphisms of the groupoid $\mathcal{G}$. These are called $\mathcal{G}$-algebras, and Le Gall's KK-theory is denoted by $KK^{\mathcal{G}}(A, B)$ for

$\mathcal{G}$-algebras A and B. Heath Emerson and Ralf Meyer collaborated on several papers relating to groupoid-equivariant KK-theory (e.g., [8], [9], [7], [10]). In [10], they define the groupoid-equivariant representable K-theory of X with Y-compact support to be

$$RK_{\mathcal{G},Y}(X) := KK^{\mathcal{G}\ltimes Y}(C_0(Y), C_0(X)).$$

This representable K-theory is more general than vector-bundle-defined counterparts, and behaves better as a generalized cohomology theory. Therefore, it is an appropriate generalization of ordinary K-theory to the groupoid-equivariant setting. A groupoid-equivariant version of the Thom isomorphism in representable K-theory is generally assumed to be true, despite a lack of suitable references in the literature. Specifically, Emerson and Meyer use such an isomorphism extensively throughout their collaborations, and it is integral to both their construction of a groupoid-equivariant geometric KK-theory [7], and to their definition of an equivariant topological index formula [9]. For their use of the Thom isomorphism, they implicitly reference Lemma 5.4 of [9], which relies on their definition of orientation ([9], Definition 5.2). In this section, they state that an analogue of the usual Thom class for a Spin^c vector bundle $\pi : E \to X$, of (real) rank-k, will give an orientation class $\tau \in RK^k_{\mathcal{G},X}(E)$. They do not attempt to prove this, nor do they elaborate upon the circumstances for which such a Thom isomorphism should exist. Searching the literature more broadly, only a few papers seemed relevant. There is a paper by Moutuou, [17], which develops a very different version of groupoid equivariant KK-theory than the one introduced by LeGall in [11], and proves the Thom isomorphism and twisted counterparts within that theory. However, the Thom isomorphism from Moutuou's work does not seem to restrict in a simple way to the Thom isomorphism needed by Emerson and Meyer, who work in the context of LeGall's definitions. Another paper which deserves mentioning is [18], which proves a Thom isomorphism specifically for groupoids arising as bundles of compact Lie groups, but does not provide us with a sufficiently general result.

In addition to carefully proving a very general equivariant Thom isomorphism in this thesis, we have included a few other details related to the Thom isomorphism. These additional details are specifically included for the purpose of building the foundation for eventually proving certain K-theoretic index theorems for groupoid-equivariant pseudodifferential operators, specifically through generalizing the approach taken by Erik van Erp and Paul Baum in their papers [2], [3], and [4]. A neat proof of the Atiyah-Singer index theorem (see [1]) is given in [3]. An index theorem for a certain class of non-elliptic operators on contact manifolds is proved in [4], relying on the formalism discussed in [3]. Since our motivation for this thesis project is to eventually generalize these index theorems to the groupoid-equivariant setting, working within the context used by Emerson and Meyer is highly advantageous. In particular, they have already proved equivariant index theorems for Dirac operators (Theorem 6.1 of [9]), which is a necessary step in proving index theorems for more general operators. In fact, the particular statements of index theorems used in Baum and van Erp's papers are often via a commutative diagram involving geometric K-homology (e.g., Theorem 5.04 of [3] for Elliptic operators, or Theorem 5.5.1 of [4] for Heisenberg-elliptic operators on contact manifolds). The appropriate groupoid-equivariant analogue of this commutative diagram naturally involves the equivariant geometric KK-theory developed by Emerson and Meyer in [7]. Before studying groupoid-equivariant index theorems, our original motivation was to see if the index theorem of [4] is generalizable to families of operators (an open question), and the fact that Emerson and Meyer's geometric KK-theory is the only theory that reduces to the families case, we were naturally led to try and understand the full groupoid-equivariant families setting. Although our original motivation was to prove such index theorems, the body of this thesis does not discuss index theory directly, and only focuses on one key aspect of this problem. The Thom isomorphism is a key ingredient of index theory, since it converts the X-compactly supported representable K-theory of TX, a receptacle for the topological data of an equivariant elliptic pseudodifferential operator, to the X-compactly supported representable K theory of TTX. A detailed understanding of this isomorphism, which involves equating two very different Thom isomorphisms for $TTX \to TX$ through a rotation trick, was used in [3] to prove the Atiyah Singer index theorem. We use the formalism of this thesis to appropriately frame their rotation trick within $RK_{\mathcal{G}}$ in subsection 6.2.

In this thesis, we study the equivariant Thom class, denoted τ_E, of a Spin^c-$\mathcal{G}$-bundle $E \to X$, which will be constructed in two very different ways. There is a rather direct construction of τ_E that relies on constructing spinor bundles over E out of a principal Spin^c-bundle, then identifying them outside the zero section of E via Clifford multiplication. We will define τ via this construction. Part of the benefit of a concrete realization of the Thom class via vector bundles lies in the fact that not all representable K theory classes can be expressed in terms of vector bundles. That is, $RK_{\mathcal{G},Y}(X)$ (which is defined to be $KK^{\mathcal{G}\ltimes Y}(C_0(Y), C_0(X)))$ is generally strictly larger than $VK_{\mathcal{G},Y}(X)$ (i.e., a group involving vector bundles, similar to classical K-theory). Of particular importance is the fact that VK is closed with respect to Kasparov products (Theorem 2.7.7), implying that many Kasparov products involving the Thom class can be computed more readily in terms of vector bundle constructions.

To prove the equivariant Thom isomorphism, we adapt a non-equivariant argument used by Le-Gall for compact spaces X ([11], THEORÈME 7.4). Le Gall's proof involves constructing an invertible KK-class via pullback of a model class, namely the Bott-element on $\mathbb{R}^n$, through a generalized groupoid homomorphism. A generalized groupoid homomorphism from $\mathcal{G}$ to $\mathcal{H}$ can be thought of as a diagram of the form

$$\begin{array}{ccc} \mathcal{G}_\Omega & & \\ {\scriptstyle p_*}\downarrow & \searrow{\scriptstyle f} & \\ \mathcal{G} & & \mathcal{H} \end{array}$$

for some open surjection, $p : \Omega \to \mathcal{G}^{(0)}$, and some strict groupoid homomorphism f, whose domain, $\mathcal{G}_\Omega$, is a groupoid formed by taking the object space Ω with morphisms from ω to ω' defined to be the groupoid elements $\gamma : p(\omega) \to p(\omega')$. For a generalized groupoid-homomorphism $\varphi : \mathcal{G} \to \mathcal{H}$, Le Gall works out what it means to pull back classes $x \in KK^{\mathcal{H}}(A, B)$ to classes $\varphi^* x \in KK^{\mathcal{G}}(\varphi^* A, \varphi^* B)$. To prove the non-equivariant Thom isomorphism for rank-k Spin^c-bundles $E \to X$, Le Gall uses a generalized groupoid homomorphism, φ, from X (as a groupoid with no morphisms) to Spin$^c(k)$ to pull back an invertible Spin$^c(k)$-equivariant KK-class, called the Bott element, to an invertible $KK(C_0(X), C_0(E))$-class. We perform the same technique in a more general setting, and since we do not require spaces to be compact, we further contextualize the approach to the representable K-theory with Y-compact supports developed by Emerson and Meyer in [10]. This contextualization requires formalizing something Emerson and Meyer refer to as a forgetful functor, but is really more like a pushforward functor, which takes $\mathcal{G} \ltimes X$ objects (e.g., algebras, modules, KK-classes), and pushes them down to $\mathcal{G} \ltimes Y$ objects (algebras, modules, KK-classes) through a continuous $\mathcal{G}$-map $f : X \to Y$. We denote this functor by $\mathfrak{F}_f$, and work out the details of this functor in Section 5. Intuitively, for a

$\mathcal{G} \ltimes X$-object, A, $\mathfrak{F}_f$ assembles all of the fibers A_x for $x \in f^{-1}(y)$, and treats it as a single fiber over $y \in Y$. Making this precise is not really that straightforward, so we dedicated a whole section to it.

The specific version of the Thom isomorphism proved in this thesis is as follows.

Theorem 1.0.1 (Groupoid-Equivariant Thom Isomorphism)**.** *Assume all topologies are second-countable and locally-compact Hausdorff. Let $\mathcal{G}$ be a topological groupoid, and let X be a $\mathcal{G}$-space. Suppose $\pi : E \to X$ is a continuous* Spin^c*-$\mathcal{G}$-bundle on X of rank k over $\mathbb{R}$.*

Then the Thom class of E, $\tau \in RK^k_{\mathcal{G},X}(E)$, satisfies: for any $\mathcal{G}$-space Y, and continuous $\mathcal{G}$-map $f : X \to Y$, the map

$$(\cdot) \underset{C_0(X)}{\widehat{\otimes}} \mathfrak{F}_f(\tau) : RK^j_{\mathcal{G},Y}(X) \to RK^{j+k}_{\mathcal{G},Y}(E)$$

is an isomorphism.

The body of this thesis is divided into 5 main sections. Section 2 is a terse compilation of definitions and prerequisite knowledge. This section could be skipped and referred back to if the need for clarification arises. Although nothing in this section is particularly new, we provide explicit constructions for KK-products of VK-classes in Theorem 2.7.7.

In Section 3, we define $\mathcal{G}$-equivariant Spin^c-structures for $\mathcal{G}$-bundles $E \to X$. Bundles admitting such structure are the natural candidate for $RK_{\mathcal{G}}$-oriented vector bundles, and we give a concrete definition for the Thom class corresponding to such a bundle. The remainder of this section is dedicated to understanding the Bott generator element used by Kasparov in [15], $\beta_n \in KK^{\mathrm{Spin}^c(n)}(\mathbb{C}, C_0(\mathbb{R}^n)\widehat{\otimes}\mathbb{C}\ell(n))$, so that we can eventually relate Le Gall's pullback construction to the concretely defined Thom class of subsection 3.2.

Section 4 is a recapitulation of the pullback constructions used in [11]. The lengthy examples in subsection 4.1 can mostly be skipped, since the only necessary construction from this subsection takes r-open regular graphs (Ω, r, s) to the associated prehomomorphism (Ω, r, f). However, in later subsections, the examples are entirely related to the Thom isomorphism. In particular, examples 4.2.5 and 4.3.2, are necessary computations. We end this section by stating the main theorem of [11], that the equivariant Kasparov product commutes with pullback.

Section 5 is where we define $\mathfrak{F}_f$, and we end this section by proving that Kasparov product commutes with this functor.

Section 6 combines the work from each of the previous sections to prove that the Thom class of a rank-k Spin^c-$\mathcal{G}$-bundle $E \to X$, defined in Subsection 3.2, is an invertible element of $KK^{\mathcal{G}\ltimes X}(C_0(X), C_0(E))$. We apply the functor $\mathfrak{F}_f$ to this theorem to contextualize it to representable K-theory with Y-compact supports, thereby proving Theorem 1.0.1 stated above. We end this section by formalizing the rotation argument of [3] within representable K-theory.

2 Prerequisite Definitions

2.1 Clifford Algebras

Clifford algebras are used extensively in KK theory. They are generally used to manipulate gradings, and their representation theory is essential to understanding Bott Periodicity and the Thom isomorphism. The primary purpose of this subsection is to establish notation.

Definition 2.1.1. The **Clifford algebra of** $\mathbb{R}^n$, denoted $C\ell(n)$, is the $\mathbb{R}$-tensor algebra of $\mathbb{R}^n$ modulo the ideal generated by elements of the form $x \otimes y + y \otimes x + 2\langle x, y\rangle$, where $\langle x, y\rangle$ is the standard dot product of x and y in $\mathbb{R}^n$.

The **complex Clifford algebra of** $\mathbb{R}^n$ is $\mathbb{C}\ell(n) := C\ell(n) \otimes_{\mathbb{R}} \mathbb{C}$. There is an inclusion of $\mathbb{R}^n$ into $C\ell(n)$, and an inclusion of $C\ell(n)$ into $\mathbb{C}\ell(n)$. We sometimes write $\mathbb{R}^n \otimes 1$ for the elements of $\mathbb{C}\ell(n)$ in the image of these inclusions, and $1 \otimes \mathbb{C}$ for the image of the natural inclusion of $\mathbb{C}$ into $\mathbb{C}\ell(n)$.

Proposition 2.1.2. *The $\mathbb{C}$-algebra $\mathbb{C}\ell(n)$, together with involution and norm, defined below, is a C^*-algebra.*

1. *Define the involution to be the anti-multiplicative map generated by $x^* := -x$, for $x \in \mathbb{R}^n \otimes 1 \subseteq \mathbb{C}\ell(n)$, and conjugation on $1 \otimes \mathbb{C} \subseteq \mathbb{C}\ell(n)$.*

2. *Define the norm on $\mathbb{C}\ell(n)$ by $||z||^2 := z^* z$.*

Because some references (specifically [15]) use a slightly different Clifford algebra in certain constructions, we relate the two via the following proposition:

Proposition 2.1.3. *Let $\widetilde{\mathbb{C}\ell}(n)$ be the (complexification of) the tensor algebra of $\mathbb{R}^n$ modulo the ideal generated by elements of the form $x \otimes y + y \otimes x - 2\langle x, y\rangle$, with involution induced by conjugation and $x^* = x$ on $\mathbb{R}^n \otimes 1$ (and $||z||^2 = z^* z$). Then $\mathbb{C}\ell(n)$ is isometrically $*$-isomorphic to $\widetilde{\mathbb{C}\ell}(n)$.*

Proof. Let f be the $\mathbb{C}$-algebra homomorphism induced by sending $x \otimes 1 \in \mathbb{R}^n \otimes 1 \subseteq \widetilde{\mathbb{C}\ell}(n)$ to $x \otimes i \in \mathbb{C}\ell(n)$.

Since $f(xy \otimes 1) = (x \otimes i)(y \otimes i) = xy \otimes (-1)$, it follows that

$$f((xy + yx) \otimes 1) = (xy + yx) \otimes (-1) = (2\langle x, y\rangle) \otimes 1 = f(2\langle x, y\rangle);$$

hence, f is well defined on equivalence classes. Also, $f((x \otimes 1)^*) = f(-x \otimes 1) = -x \otimes i = (x \otimes i)^* = f(x)^*$. The adjoint and multiplication on each algebra determine their respective norms, so f is an isometric $*$-homomorphism. Since there is an obvious inverse of f, it follows that f is an isometric $*$-isomorphism. □

Note 2.1.4. The map induced by sending $x \otimes 1 \in \widetilde{\mathbb{C}\ell}(n)$ to $x \otimes (-i) \in \mathbb{C}\ell(n)$ is also an isometric $*$-isomorphism, but it will not preserve orientation (defined below).

Proposition 2.1.5. *There exists an element $\varepsilon \in \mathbb{C}\ell(2r)$ (unique up to a sign), such that:*

1. $\varepsilon^2 = 1$,
2. $\varepsilon^* = \varepsilon$, *and*
3. $\varepsilon e_j = -e_j \varepsilon$ *for all* $1 \leqslant j \leqslant 2r$.

Proof. The element $\varepsilon := i^r e_1 e_2 \cdots e_{2r}$ (or $\varepsilon := 1$ if $r = 0$) satisfies the following conditions.

$$\begin{aligned}
\varepsilon^* &= (-1)^r i^r e_{2r}^* e_{2r-1}^* \cdots e_1^* \\
&= (-1)^r i^r (-1)^{2r} e_{2r} \cdots e_1 \\
&= (-1)^r i^r (-e_1 e_2)(-e_3 e_4) \cdots (-e_{2r-1} e_{2r}) \\
&= (-1)^r i^r (-1)^r e_1 e_2 \cdots e_{2r} \\
&= \varepsilon
\end{aligned}$$

$$\begin{aligned}
\varepsilon^2 &= \varepsilon^* \varepsilon \\
&= (-1)^r i^{2r} e_{2r}^* \cdots e_2^* e_1^* e_1 e_2 \cdots e_{2r} \\
&= (-1)^{2r} \\
&= 1
\end{aligned}$$

$$\begin{aligned}
e_j \varepsilon &= i^r e_j e_1 \cdots e_{2r} \\
&= i^r (-1)^{2r-1} e_1 \cdots e_{2r} e_j \\
&= -\varepsilon e_j \text{ for any } 1 \leqslant j \leqslant 2r
\end{aligned}$$

Suppose S is another element satisfying the three conditions above. Then $S\varepsilon e_j = -S e_j \varepsilon = e_j S \varepsilon$ for all $j \leqslant 2r$. So $S\varepsilon$ is in the center of $\mathbb{C}\ell(2r)$ (which is all multiples of 1). Consequently, $(S\varepsilon)^2 = S\varepsilon S\varepsilon = S^2\varepsilon^2 = 1$ and $S\varepsilon = \pm 1$, which implies that $S = \pm\varepsilon^{-1} = \pm\varepsilon$. □

Definition 2.1.6. An element $\varepsilon \in \mathbb{C}\ell(n)$, satisfying the conditions of Proposition 2.1.5, will be called an **internal grading element**.

Proposition 2.1.7. *For any $x \in \mathbb{C}\ell(2r+1)$, the condition $x e_j = -e_j x$ implies that $x = 0$ (i.e., no internal grading element exists).*

Proof. Notice that $\varepsilon := i^r e_1 e_2 \cdots e_{2r+1} = \varepsilon_{2r} e_{2r+1}$ commutes with e_j for all $1 \leqslant j \leqslant 2r+1$, and hence with all of $\mathbb{C}\ell(2r+1)$. Suppose an internal grading element $S \in \mathbb{C}\ell(2r+1)$ exists. Then S commutes with ε, but ε is odd, so S must anti-commute with ε. This would force us to conclude that $S = 0$; however, since $S^2 = 1$, we reach a contradiction. □

Although there is no internal grading element in $\mathbb{C}\ell(2r+1)$, we still want to be able to make certain choices canonically. To do this, we introduce the notion of orientation.

Definition 2.1.8. A **orientation** for a Clifford algebra $\mathbb{C}\ell(n)$ is a homogeneous element $\omega \in \mathbb{C}\ell(n)$ such that $\omega^* = \pm\omega$, $\omega^*\omega = 1$, and for all homogeneous $x \in \mathbb{C}\ell(n)$, $x\omega = (-1)^{\partial x(\partial\omega+1)}\omega x$.

The **standard orientation** on $\mathbb{C}\ell(n)$ is $\omega_n := i^n e_1 e_2 \cdots e_n$; the standard orientation on $\widetilde{\mathbb{C}\ell}(n)$ is $\widetilde{\omega_n} := e_1 e_2 \cdots e_n$.

The internal grading element ε is an orientation for $\mathbb{C}\ell(2r)$, and $i^r\varepsilon$ is the standard orientation.

Proposition 2.1.9. *The isomorphism $f : \widetilde{\mathbb{C}\ell}(n) \to \mathbb{C}\ell(n)$ defined in Proposition 2.1.3 satisfies $f(\widetilde{\omega_n}) = \omega_n$.*

Theorem 2.1.10 ($\mathbb{C}$-Clifford Periodicity)**.** *For any $r \in \mathbb{N}$, there exists graded C^*-algebra isomorphisms:*

$$\mathbb{C}\ell(2r) \cong M_{2^r}(\mathbb{C}),$$

$$\mathbb{C}\ell(2r+1) \cong M_{2^r}(\mathbb{C}) \times M_{2^r}(\mathbb{C}).$$

The grading on $M_{2^r}(\mathbb{C})$ is given by the splitting into the first and second halves: $\mathbb{C}^{2^r} = \mathbb{C}^{2^{r-1}} \oplus \mathbb{C}^{2^{r-1}}$; the grading on $M_{2^r}(\mathbb{C}) \times M_{2^r}(\mathbb{C})$ is given by the grading-operator that switches the two copies of $M_{2^r}(\mathbb{C})$.

Proof. We define maps φ_n recursively (and prove they isomorphisms inductively) as follows. $\mathbb{C}\ell(0) = \mathbb{R} \otimes_{\mathbb{R}} \mathbb{C} \cong M_1(\mathbb{C})$. Denote this isomorphism by φ_0.

$\mathbb{C}\ell(1) = (\mathbb{C} \oplus \mathbb{C}[e_1])/ < e_1^2 + 1 >$. Define $\varphi_1 : \mathbb{C}\ell(1) \to M_1(\mathbb{C}) \times M_1(\mathbb{C})$ by $\varphi_1(e_1) := (i, -i)$.

For the recursive step, assume $\varphi_{2r} : \mathbb{C}\ell(2r) \to M_{2^r}(\mathbb{C})$ and $\varphi_{2r+1} \to M_{2^r}(\mathbb{C}) \times M_{2^r}(\mathbb{C})$ have been defined and are isomorphisms for all $r < R$.

Define $\varphi_{2R} : \mathbb{C}\ell(2R) \to M_{2^R}(\mathbb{C})$ by:[1]

[1]If $R = 1$, then $\varepsilon = 1$, and there are no e_j with $1 \leqslant j \leqslant 2R-2$. Therefore, when constructing φ_2, only use the last two cases in the recursive definition. One can verify the inductive step is a simplification of the argument that follows.

$$\varphi_{2R}(e_j) := \left(\begin{array}{c|c} 0 & \varphi_{2R-2}(e_j) \\ \hline \varphi_{2R-2}(e_j) & 0 \end{array}\right) \text{ for } 1 \leqslant j \leqslant 2R-2$$

$$\varphi_{2R}(e_{2R-1}) := \left(\begin{array}{c|c} 0 & -I \\ \hline I & 0 \end{array}\right) \text{ and } \varphi_{2R}(e_{2R}) := \left(\begin{array}{c|c} 0 & i\varphi_{2R-2}(\varepsilon) \\ \hline i\varphi_{2R-2}(\varepsilon) & 0 \end{array}\right)$$

Because $\varphi_{2R-2}(\varepsilon)$ induces the usual grading on $M_{2^{R-1}}(\mathbb{C})$ (by hypothesis, φ_{2R-2} is a graded isomorphism), we see that

$$S := \varphi_{2R-2}(\varepsilon) = \pm \left(\begin{array}{c|c} I & 0 \\ \hline 0 & -I \end{array}\right), \text{ and } \varphi_{2R}(e_{2R}) = \left(\begin{array}{c|c} 0 & iS \\ \hline iS & 0 \end{array}\right)$$

It is easy to verify that φ_{2R} is a graded $*$-homomorphism. We will show that it is also a $\mathbb{C}$-vector space isomorphism. Let Σ_r^* be the set of all strictly increasing multi-indices from the set $\{1, 2, ..., 2r\}$. Define e_I for $I = (i_1, i_2, ..., i_{|I|}) \in \Sigma_j^*$ to be $e_{i_1} e_{i_2} \cdots e_{i_{|I|}}$. Denote by I^c the strictly increasing multi-index consisting of all integers (up to $2r$) not in I. Take $e_\varnothing = 1$. For the following, let $I \in \Sigma_{R-1}^*$.

$$\varphi_{2R}(e_I) = \begin{pmatrix} 0 & A \\ A & 0 \end{pmatrix} \text{ for } e_I \text{ odd, and } \begin{pmatrix} B & 0 \\ 0 & B \end{pmatrix} \text{ for } e_I \text{ even}$$

Note: A and B are both given by $\varphi_{2R-2}(e_I)$, but for odd and even e_I, respectively. Therefore, A must be odd and B must be even, by inductive hypothesis. In what follows, note that S is also of even-degree, and that iSA and iSB are both given by

$$iS\varphi_{2R-2}(e_I) = i\varphi_{2R-2}(\varepsilon e_I) = \left((-1)^{|I|+\Sigma I}\right) i^R \varphi_{2R-2}(e_{I^c}).$$

$$\varphi_{2R}(e_{2R}e_I) = \begin{pmatrix} iSA & 0 \\ 0 & iSA \end{pmatrix} \text{ for } e_I \text{ odd, and } \begin{pmatrix} 0 & iSB \\ iSB & 0 \end{pmatrix} \text{ for } e_I \text{ even}$$

$$\varphi_{2R}(e_{2R-1}e_I) = \begin{pmatrix} -A & 0 \\ 0 & A \end{pmatrix} \text{ for } e_I \text{ odd, and } \begin{pmatrix} 0 & -B \\ B & 0 \end{pmatrix} \text{ for } e_I \text{ even}$$

$$\varphi_{2R}(e_{2R}e_{2R-1}e_I) = \begin{pmatrix} 0 & iSA \\ -iSA & 0 \end{pmatrix} \text{ for } e_I \text{ odd, and } \begin{pmatrix} iSB & 0 \\ 0 & -iSB \end{pmatrix} \text{ for } e_I \text{ even}$$

These eight possibilities minimally (i.e., the collection is linearly independent) span all combinations of even and odd block pairs, diagnonal and anti-diagonal block-pairs, and alternating sign block-pairs. Together with the inductive hypothesis, this guarantees that $\{\varphi_{2R}(e_I)\}_{I\in\Sigma_R^*}$ is a basis for $M_{2^r}(\mathbb{C})$, concluding the proof that φ_{2R} is a graded algebra isomorphism. It is also a $*$-homomorphism due to the fact that all choices made in the recursive definition are anti-Hermitian.

Since we are done with the even case, we will abuse notation slightly, and redefine ε to be the internal grading element for $\mathbb{C}\ell(2R)$; $S := \varphi_{2R}(\varepsilon)$. For $\varphi_{2R+1} : \mathbb{C}\ell(2R+1) \to M_{2^R}(\mathbb{C}) \times M_{2^R}(\mathbb{C})$, we set:

$$\varphi_{2R+1}(e_j) := \left(\varphi_{2R}(e_j), \varphi_{2R}(e_j^*)\right), \quad \text{for } 1 \leqslant j \leqslant 2R$$

$$\varphi_{2R+1}(e_{2R+1}) := (iS, -iS)\,.$$

It is easy to verify that φ_{2R} is indeed a graded $*$-homomorphism. Let $I \in \Sigma_R^*$.

$$\varphi_{2R}(e_I) = (A, -A), \text{ for } e_I \text{ odd, and } (B, B), \text{ for } e_I \text{ even.}$$

Again, A must be odd-degree and B must be even-degree.

$$\varphi_{2R}(e_{2R+1}e_I) = (iSA, iSA), \text{ for } e_I \text{ odd, and } (iSB, -iSB), \text{ for } e_I \text{ even.}$$

These 4 possibilites minimally span all combinations of even and odd pairs, and alternating-sign pairs. Consequently, φ_{2R+1} is an isomorphism. □

Corollary 2.1.11. *Given an (a-priori ungraded) $*$-isomorphism $\varphi : \mathbb{C}\ell(2r) \to End(V)$, the matrix $\varphi(\varepsilon)$ provides a compatible grading on both V and on $End(V)$, and makes φ into a graded $*$-isomorphism.*

Proof. The grading on V is determined by the +1 and -1 eigenspaces of the matrix $S = \varphi(\varepsilon)$. The grading on $End(V)$, induced by S, is given by conjugation by $\widehat{S}(A) := SAS$.

Since ε anti-commutes with odd elements of $\mathbb{C}\ell(2r)$, so does S with the image of odd elements under φ; hence, φ is a graded $*$-isomorphism. $\square$

Theorem 2.1.12. *Any unital $*$-isomorphism $\varphi : \mathbb{C}\ell(2r) \to M_{2^r}(\mathbb{C})$ is unitarily equivalent to the isomorphism φ_{2r} constructed above. That is, there exists $U \in U(2^r)$ such that $\varphi(x) = U\varphi_{2R}(x)U^*$.*

Proof. Proof by induction. There is exactly one unital isomorphism $\mathbb{C}\ell(0) \to \mathbb{C}$. For $\mathbb{C}\ell(2)$, we have that $\varphi_2(e_1) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $\varphi_2(e_2) = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$. Therefore, $\varphi_2(e_1e_2) = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$, and $\varphi_2(\varepsilon) = i\varphi_2(e_1e_2) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. If $\varphi : \mathbb{C}\ell(2) \to M_2(\mathbb{C})$ is any other isomorphism, define $\beta := \{b, \varphi(e_1)b\}$, where b is a $+1$ unit-eigenvector of $\varphi(\varepsilon)$. Notice that $\varphi(\varepsilon)\varphi(e_1)b = -\varphi(e_1)(+b) = -\varphi(e_1)b$ is a -1 eigenvector for the unitary matrix $\varphi(\varepsilon)$. The fact that $\langle \varphi(e_1)b, \varphi(e_1)b\rangle = \langle b, -\varphi(e_1^2)b\rangle = ||b||^2 = 1$, implies that β is an orthonormal basis that diagonalizes $\varphi(\varepsilon)$, and represents $\varphi(e_1)$ as $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Since e_1 and ε generate $\mathbb{C}\ell(2)$ as an algebra, it follows that φ is unitarily equivalent to φ_2 via the unitary change of basis described above.

Assuming the statement is true for isomorphisms $\mathbb{C}\ell(2r) \to M_{2^r}(\mathbb{C})$ for all $0 \leqslant r < R$, we will prove it for $\mathbb{C}\ell(2R)$. Choose an orthonormal basis, $\beta_+ := \{b_1, b_2, ..., b_{2R-2}\}$ for the $+1$ eigenspace of $\varphi(\varepsilon)$. By a similar argument as above, $\beta_- := \varphi(e_{2R-1})\beta_+$ is an orthonotmal basis for the -1 eigenspace for $\varphi(\varepsilon)$, and $\beta := \beta_+ \cup \beta_-$ is an orthonormal basis for $\mathbb{C}^{2^R}$ which represents $\varphi(\varepsilon)$ as $\left(\begin{array}{c|c} I & 0 \\ \hline 0 & -I \end{array}\right)$, and $\varphi(e_{2R-1})$ as $\left(\begin{array}{c|c} 0 & -I \\ \hline I & 0 \end{array}\right)$. Since each generator of $\mathbb{C}\ell(2R)$ permutes the two eigenspaces of $\varphi(\varepsilon)$ we get that:

$$[\varphi(e_j)]_\beta = \begin{pmatrix} 0 & -E_j^* \\ E_j & 0 \end{pmatrix}, \text{ where } E_j \in M_{2^{R-1}}(\mathbb{C}).$$

The coefficients of E_j are given by:

$$\varphi(e_j)b_k = \sum_{\ell=1}^{2^{R-1}} (E_j)_{\ell,k}\varphi(e_{2R-1})b_\ell$$

Multiplying that equation by $-\varphi(e_{2R-1})$ demonstrates that $-E_j^* = E_j$ for all $j \neq$

$2R-1$. Hence,

$$[\varphi(e_j)]_\beta = \begin{pmatrix} 0 & E_j \\ E_j & 0 \end{pmatrix}$$

And furthermore, the set of matrices $\mathcal{E} := \{E_1, E_2, ..., E_{2R-3}, E_{2R-2}, E_{2R}\} \subseteq M_{2^{R-1}}(\mathbb{C})$ satisfies the following properties:

1. $E_j^2 = -I$

2. $E_jE_k = -E_kE_j$ for $j \neq k$.

3. $E_j^* = -E_j$

We can therefore define a $*$-homomorphism $\psi : \mathbb{C}\ell(2R-2) \to M_{2^{R-1}}(\mathbb{C})$ determined by $\psi(e_j) := E_j$ (notice that E_{2R} is left out!). Define $\widehat{\psi} : \mathbb{C}\ell(2R-2) \to M_{2^R}(\mathbb{C})$ by $\widehat{\psi}(e_j) := [\varphi(e_{2R-1}e_j)]_\beta = \begin{pmatrix} -\psi(e_j) & 0 \\ 0 & \psi(e_j) \end{pmatrix}$. This is a composition of the 1-1 map (multiplication by e_{2R-1}) $\mathbb{C}\ell(2R-2) \to \mathbb{C}\ell(2R)$ and the isomorphism φ (in β-coordinates). Therefore, $\widehat{\psi}$ is 1-1, implying that ψ is 1-1.

Since ψ is a one-to-one $*$-homomorphism, counting dimensions ensures that ψ is a $*$-isomorphism. By our inductive hypothesis, we can find a unitary $U \in U(2^{R-1})$ such that $U\psi U^* = \varphi_{2R-2}$. Furthermore, iE_{2R} is actually a ψ-compatible grading element, since it is a self-adjoint unitary that anti-commutes with all other E_j's. By the uniqueness of grading elements (modulo sign), it must be the case that $UE_{2R}U^* = \pm i\varphi_{2R-2}(\varepsilon_{2R-2})$.

Let $\widehat{U} \in U(2R)$ be the matrix with U on the block diagonal, and define $\widehat{\varphi} := \widehat{U}[\varphi]_\beta\widehat{U}^*$. Then, for all $1 \leqslant j \leqslant 2R$,

$$\widehat{\varphi}(e_j) = \begin{pmatrix} U & 0 \\ 0 & U \end{pmatrix} \begin{pmatrix} 0 & -E_j^* \\ E_j & 0 \end{pmatrix} \begin{pmatrix} U^* & 0 \\ 0 & U^* \end{pmatrix} = \begin{pmatrix} 0 & -UE_j^*U^* \\ UE_jU^* & 0 \end{pmatrix}.$$

In particular, for $1 \leqslant j \leqslant 2R-2$,

$$\widehat{\varphi}(e_j) = \begin{pmatrix} 0 & U\psi(e_j)U^* \\ U\psi(e_j)U^* & 0 \end{pmatrix} = \left(\begin{array}{c|c} 0 & \varphi_{2R-2}(e_j) \\ \hline \varphi_{2R-2}(e_j) & 0 \end{array}\right), \text{ and}$$

$$\widehat{\varphi}(e_{2R-1}) = \begin{pmatrix} 0 & U(-I)U^* \\ U(I)U^* & 0 \end{pmatrix} = \left(\begin{array}{c|c} 0 & -I \\ \hline I & 0 \end{array}\right), \text{ and}$$

$$\widehat{\varphi}(e_{2R}) = \begin{pmatrix} 0 & UE_{2R}U^* \\ UE_{2R}U^* & 0 \end{pmatrix} = \left(\begin{array}{c|c} 0 & \pm i\varphi_{2R-2}(\varepsilon) \\ \hline \pm i\varphi_{2R-2}(\varepsilon) & 0 \end{array}\right).$$

With the exception of the sign of $\widehat{\varphi}(e_{2R})$, the theorem is proved. To determine the sign, notice that $\widehat{U}$ being block diagonal implies that:

$$\begin{aligned}
\widehat{\varphi}(\varepsilon_{2R}) &= \left(\begin{array}{c|c} I & 0 \\ \hline 0 & -I \end{array}\right) \\
&= \widehat{\varphi}(i^R e_1 \cdots e_{2R-2}e_{2R-1}e_{2R}) \\
&= \widehat{\varphi}(i\varepsilon e_{2R-1}e_{2R}) \\
&= \begin{pmatrix} i\varphi_{2R-2}(\varepsilon) & 0 \\ 0 & i\varphi_{2R-2}(\varepsilon) \end{pmatrix} \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \begin{pmatrix} 0 & \pm i\varphi_{2R-2}(\varepsilon) \\ \pm i\varphi_{2R-2}(\varepsilon) & 0 \end{pmatrix} \\
&= \begin{pmatrix} i\varphi_{2R-2}(\varepsilon) & 0 \\ 0 & i\varphi_{2R-2}(\varepsilon) \end{pmatrix} \begin{pmatrix} -(\pm i\varphi_{2R-2}(\varepsilon)) & 0 \\ 0 & \pm i\varphi_{2R-2}(\varepsilon) \end{pmatrix} \\
&= \begin{pmatrix} -(\pm i^2\varphi_{2R-2}(\varepsilon^2)) & 0 \\ 0 & \pm i^2\varphi_{2R-2}(\varepsilon^2) \end{pmatrix} \\
&= \begin{pmatrix} \pm I & \\ & -(\pm I) \end{pmatrix}.
\end{aligned}$$

Which implies that $\pm = +1$, and the sign of $\widehat{\varphi}(e_{2R})$ matches φ_{2R}. □

2.2 The Spin^c Group

Definition 2.2.1. We define the **Pin**, **Spin**, and **Spin**^{c} groups to be as follows.

1. The Pin group, denoted $\mathrm{Pin}(n)$, is the subgroup of $C\ell(n)^\times$ generated by unit vectors.

2. The Spin group, denoted $\mathrm{Spin}(n)$, is the subgroup of $\mathrm{Pin}(n)$ consisting only of even-graded elements.

3. The Spin^c group, denoted $\text{Spin}^c(n)$, is the subgroup of $\mathbb{C}\ell(n)^\times$ generated by elements in $\text{Spin}(n) \otimes_{\mathbb{R}} 1$ and $1 \otimes_{\mathbb{R}} U(1)$.

These groups all naturally act on $\mathbb{R}^n$ by conjugation inside $C\ell(n)$ (or $\mathbb{C}\ell(n)$, respectively). If $u \in S^{n-1} \subseteq \mathbb{R}^n \subseteq C\ell(n)$ is a unit vector, and $x \in \mathbb{R}^n$ is any other vector, then ux is equivalent to $-xu - 2\langle x, u\rangle$; hence,

$$\begin{aligned} uxu^{-1} &= uxu^* \\ &= -uxu \\ &= (xu + 2\langle x, u\rangle)u \\ &= xu^2 + 2\langle x, u\rangle u \\ &= -(x - 2\langle x, u\rangle u) \end{aligned}$$

The expression $\langle x, u\rangle u$ is the orthogonal projection of x onto u; therefore, $x - 2\langle x, u\rangle u$ is the reflection of x across the subspace orthogonal to u. We denote this operation by $R_{u^\perp}(x)$. In this notation, $uxu^{-1} = -R_{u^\perp}(x)$.

Therefore, conjugation by an element of $\text{Pin}(n)$ yields an action on $\mathbb{R}^n$ by orthogonal matrices. Because of the minus sign, the determinant of these matrices is difficult to keep track of, so we instead define $\phi : \text{Pin}(n) \to O(n)$ by $\phi(u) = R_{u^\perp}$ for unit vectors u (or equivalently, $\phi(u)(x) = uxu$). Since $O(n)$ is generated by reflections, ϕ must be onto. The next proposition verifies that ϕ is a 2-to-1 map.

Proposition 2.2.2. *The kernel of* ϕ *is* $\{\pm 1\} \subseteq \text{Pin}(n)$.

The proof here is based on [12] (Proposition 2.4).

Proof. At the very least, it is immediate that $\{\pm 1\} \subseteq \ker \phi$. We will exploit gradings here, and it is helpful to notice that, because $\text{Pin}(n)$ is generated by unit vectors (which are homogeneous) under multiplication, every element in $\text{Pin}(n)$ must be homogeneous. Since the determinant is multiplicative, and the determinant of $R_{u^\perp}$ is -1 for all unit vectors u, it follows that, for any element $z \in \text{Pin}(n)$ of degree $\partial z \in \{0, 1\}$, $\det(\phi(z)) = (-1)^{\partial z}$.

Therefore, any element $z \in \ker \phi = \phi^{-1}(\{I\})$ must be homogeneous of degree 0, since $\det I = +1$.

Suppose $z \in \ker \phi$, then z must be of even degree. Write $z = z_1 z_2 \cdots z_{2k}$ for unit vectors $z_1, ..., z_{2k}$. Then

$$\begin{aligned} x &= \phi(z)(x) \\ &= z_1 z_2 \cdots z_{2k} x z_{2k} \cdots z_2 z_1 \\ &= z x z^{-1} \end{aligned}$$

Therefore, $xz = zx$. Since each z_j can be written as a linear combination of $e_1, e_2, ..., e_n$, it follows that z can be expressed as a polynomial in $e_1, e_2, ..., e_n$. In reduced form, each e_j will occur at with multiplicity at most 1 in each term of z; so the distinct terms of z can be labelled by subsets $I \subseteq \{1, 2, ..., n\}$ with $|I|$ even. Since z is even, each term, e_I, must be even-degree. Because distinct basis elements anti-commute, it follows that, for $j \leqslant n$, $e_j e_I = e_I e_j$ if and only if $j \notin I$. Consequently, the polynomial expression for z must have a coefficient of zero for any term containing e_j (for any $1 \leqslant j \leqslant n$). That is, z must be a scalar; hence, $z = \pm 1$. □

Similarly, restricting ϕ to a map $\phi : \mathrm{Spin}(n) \to SO(n)$ is a 2:1 map, and for $n \geqslant 3$, $\mathrm{Spin}(n)$ is simply connected, and therefore, the universal covering space of $SO(n)$.

Lifting ϕ to a map $\phi : \mathrm{Spin}^c(n) \to SO(n)$ by sending everything in $1 \otimes_{\mathbb{R}} U(1)$ to the identity matrix, and anything in $\mathrm{Spin}(n) \otimes_{\mathbb{R}} 1$ to $SO(n)$ via $\phi \otimes 1$. In this case, $\ker \phi \cong U(1)$.

Therefore, $\phi : \mathrm{Spin}^c(n) \to SO(n) \subseteq \mathrm{GL}_n(\mathbb{R})$ is a (real) representation of $\mathrm{Spin}^c(n)$. Among complex representations, of particular importance is the irreducible representation $\varphi_{2r} : \mathbb{C}\ell(2r) \to M_{2^r}(\mathbb{C})$, which restricts to the unitary representation $\varphi_{2r} : \mathrm{Spin}^c(2r) \to U(2^r)$. Since $\varepsilon \in \mathrm{Spin}^c(2r)$ (see Proposition 2.1.5), the representation space $\mathbb{C}^{2^r}$ can be graded by $\varphi_{2r}(\varepsilon)$. In the odd case, the only distinct irreducible unitary $*$-representations of $\mathbb{C}\ell(2r+1)$ are $\pi_0 \circ \varphi_{2r+1}$ and $\pi_1 \circ \varphi_{2r+1}$ (acting on $\mathbb{C}^{2^r}$). Both of these representation spaces necessarily carry a trivial grading on $\mathbb{C}^{2^r}$. Consequently, $\pi_j \circ \varphi_{2r+1} : \mathrm{Spin}^c(2r+1) \to U(2^r)$ are irreducible unitary representations in the odd case.

2.3 Groupoids

Definition 2.3.1. Let X,Y and Z be sets and $f : X \to Z$, $g : Y \to Z$ be functions. We define the set $X \times_{f,g} Y$ (sometimes denoted $X \times_Z Y$, if the maps are understood) to be the subset of $X \times Y$ consisting of points (x, y) satisfying $f(x) = g(y)$. If X, Y, and Z are topological spaces with f, g continuous maps, then $X \times_Z Y$ is topologized as a subset of $X \times Y$.

Proposition 2.3.2. *Let X, Y, and Z be LCH spaces, and let $f : X \to Z$ and $g : Y \to Z$ be continuous. Then $X \times_Z Y$ is a closed subset of $X \times Y$.*

Proof. Let $(x_0, y_0) \in X \times Y$ and suppose that $f(x_0) \neq g(y_0)$. Since Z is Hausdorff, we can choose neighborhoods $U_x \subseteq Z$ of $f(x_0)$ and $U_y \subseteq Z$ of $g(y_0)$ such that $U_x \cap U_y = \varnothing$. Pulling back the open sets, we get that $V := f^{-1}(U_x) \times g^{-1}(U_y)$ is an open subset of $X \times Y$. If $(x, y) \in V$, then $f(x) \in U_x$ and $g(y) \in U_y$, which implies that $f(x) \neq g(y)$. We conclude that $V \cap (X \times_Z Y) = \varnothing$. □

Corollary 2.3.3. *Let X, Y, and Z be LCH spaces, and let $f : X \to Z$ and $g : Y \to Z$ be continuous. Suppose $K \subseteq X$ and $C \subseteq Y$ are compact subsets. Then $K \times_Z C$ is compact.*

Proof. By the previous proposition, $K \times_Z C$ is a closed subset of $K \times C$, which is a compact space. Since $X \times Y$ is Hausdorff, $K \times_Z C$ is compact. □

Definition 2.3.4. A **groupoid** is the set of isomorphisms in a small catergory, equipped with the structure of morphism composition.

Let $\mathcal{G}$ be a groupoid. We denote the set of identity morphisms in $\mathcal{G}$ by Z (or $\mathcal{G}^{(0)}$ if further specificity is needed). We refer to Z as either the **object space** of $\mathcal{G}$ or as the **base space** of $\mathcal{G}$. The notation $\mathcal{G}^{(1)}$ for morphisms will not be used, since we will abuse notation and write $\mathcal{G}$ for the morphism set, identifying $\mathcal{G}^{(0)} \subseteq \mathcal{G}$ via the identity morphisms on each object. For all $\gamma \in \mathcal{G}$, let $s(\gamma)$ be the identity morphism on the domain of γ; $r(\gamma)$, the identity morphism on the codomain of γ. We refer to $s, r : \mathcal{G} \to Z$ as the **source** and **range** maps, respectively.

Definition 2.3.5. A **topological groupoid** is a groupoid equipped with a topology so that composition $\mathcal{G} \times_{r,s} \mathcal{G} \to \mathcal{G}$, inversion $\mathcal{G} \to \mathcal{G}$, and the range and source maps $\mathcal{G} \to Z$ are all continuous. We will not require that the range and source maps are open unless otherwise indicated.

Definition 2.3.6. A groupoid $\mathcal{G}$ is **proper** if the map $(r,s) : \mathcal{G} \to \mathcal{G}^{(0)} \times \mathcal{G}^{(0)}$ is proper.

A proper groupoid is one where the collection of morphisms between any two compact subsets of $\mathcal{G}^{(0)}$ is compact. In particular, proper LCH groupoids have compact automorphism groups.

Definition 2.3.7. A (left) **$\mathcal{G}$-action** on a topological space X consists of an **anchor map** $\rho : X \to Z$, and maps $\alpha_\gamma : \rho^{-1}(s(\gamma)) \to \rho^{-1}(r(\gamma))$, for all $\gamma \in \mathcal{G}$, subject to the conditions:

1. $\alpha_{\gamma\gamma'} = \alpha_\gamma \circ \alpha_{\gamma'}$
2. $\alpha_{z_0}(x) = x$ for all $z_0 \in Z$ and $x \in \rho^{-1}(z_0)$.
3. $\alpha : \mathcal{G} \times_{s,\rho} X \to X$, sending (γ, x) to $\alpha_\gamma(x)$, is a continuous map

Topological spaces equipped with a $\mathcal{G}$-action are called $\mathcal{G}$-spaces.

Definition 2.3.8. A **$\mathcal{G}$-map** between $\mathcal{G}$-spaces X and Y is a continuous function $f : X \to Y$ that is $\mathcal{G}$-equivariant. That is, given anchor maps $\rho_X : X \to Z$ and $\rho_Y : Y \to Z$, the function f must satisfy $\rho_X = \rho_Y \circ f$ and $\gamma.f(x) = f(\gamma.x)$ for all $(\gamma, x) \in \mathcal{G} \times_{s,\rho_X} X$.

Definition 2.3.9. A $\mathcal{G}$-space X is **cocompact** (or **$\mathcal{G}$-compact**) if any cover of X by $\mathcal{G}$-invariant open sets has a finite subcover.

Definition 2.3.10. Let $\mathcal{G}$ and $\mathcal{H}$ be topological groupoids. A **strict groupoid homomorphism** from $\mathcal{G}$ to $\mathcal{H}$ is a covariant functor, $f : \mathcal{G} \to \mathcal{H}$. That is, $(g, g') \in \mathcal{G}^{(2)}$ implies $(f(g), f(g')) \in \mathcal{H}^{(2)}$ and $f(gg') = f(g)f(g')$.

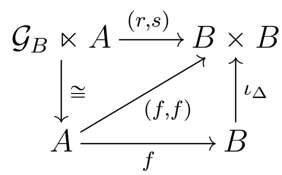
Definition 2.3.11. If $f : A \to B$ is a $\mathcal{G}$-map between $\mathcal{G}$-spaces, then a subset $K \subseteq A$ is B-**compact** if f restricted to K is proper.

Definition 2.3.12. Given a $\mathcal{G}$-space X, the **action groupoid** $\mathcal{G} \ltimes X$ is the groupoid with object space X, and morphisms from x to x' given by the elements $\gamma \in \mathcal{G}$ with $\gamma.x = x'$. This groupoid is identified with the set $\mathcal{G} \times_{s,\rho} X$ together with the operation $(\gamma, x)(\eta, y) = (\gamma\eta, y)$, whenever $x = s(\eta)$. The topology on $\mathcal{G} \ltimes X$ is induced by this identification.

Please note: a right $\mathcal{G}$-action, on a space X with anchor map ρ, will have action groupoid denoted by $X \rtimes \mathcal{G}$. For right actions, an element $\gamma \in \mathcal{G}$ maps $\rho^{-1}(r(\gamma)) \to \rho^{-1}(s(\gamma))$; therefore, $X \rtimes \mathcal{G}$ will denote the space $X \times_{\rho,r} \mathcal{G}$ (*not* $X \times_{\rho,s} \mathcal{G}$), with the composition $(y, \gamma)(y', \gamma') = (y, \gamma\gamma')$.

Definition 2.3.13. Let X be a $\mathcal{G}$-space, then X is a **proper $\mathcal{G}$-space** if $\mathcal{G} \ltimes X$ is a proper groupoid.

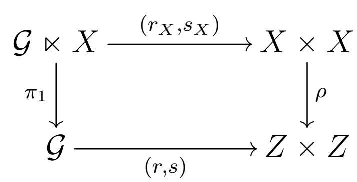
Example 2.3.14. Let $f : A \to B$ be a map between LCH spaces. Let $\mathcal{G}_B$ be the set B as a groupoid (only objects). Then A being B-compact means that f is proper. A being $\mathcal{G}_B$-proper means that $(r, s) : \mathcal{G}_B \ltimes A \to B \times B$ is proper. Since the diagram below commutes, and inclusion by the diagonal (ι_Δ) is proper, the notions of $\mathcal{G}_B$-proper and B-compact are the same.



However, if A is $\mathcal{G}_B$-compact (i.e., A is cocompact as a $\mathcal{G}_B$-space), it follows that A is compact, since every subset of A is $\mathcal{G}_B$-invariant. Consequently, B-compact spaces need not be $\mathcal{G}_B$-compact. Because of this issue, we will prefer to use the terminology cocompact, rather than $\mathcal{G}_B$-compact, to avoid confusion with the concept of B-compactness.

Proposition 2.3.15. *If $\mathcal{G}$ is a proper locally compact Hausdorff groupoid, and X is a LCH $\mathcal{G}$-space with continuous anchor map $\rho : X \to \mathcal{G}^{(0)}$, then X is automatically a proper $\mathcal{G}$-space.*

Proof. Let $K \subseteq X \times X$ be compact, and consider the following commutative diagram:



A diagram chase will verify

$$(r_x, s_x)^{-1}(K) \subseteq \pi_1^{-1}((r, s)^{-1}(\rho(K))).$$

Since $(r_X, s_X)(\gamma, x) = (\gamma.x, x)$, the following is also true:

$$(r_x, s_x)^{-1}(K) \subseteq \mathcal{G} \times_{s,\rho} \pi_2(K).$$

Consequently,

$$\begin{aligned}(r_x, s_x)^{-1}(K) &\subseteq \pi_1^{-1}((r,s)^{-1}(\rho(K))) \cap \mathcal{G} \times_{s,\rho} \pi_2(K) \\ &\subseteq (r,s)^{-1}(\rho(K)) \times_{s,\rho} \pi_2(K).\end{aligned}$$

The set $(r,s)^{-1}(\rho(K))$ is compact from the continuity of ρ, the compactness of K, and the properness of $\mathcal{G}$. The set $\pi_2(K)$ is compact from the continuity of π_2, and lastly $(r,s)^{-1}(\rho(K)) \times_{s,\rho} \pi_2(K)$ is compact from Proposition 2.3.2, and using the fact that closed subsets of compact Hausdorff spaces are compact. Since $(r_X, s_X)^{-1}(K)$ is also a closed subset of a compact Hausdorff space, it is also compact.

□

Definition 2.3.16. We will say that a $\mathcal{G}$-space X is a **smooth $\mathcal{G}$-manifold** if

1. the fibers of $\rho : X \to Z$ are smooth manifolds determined by an atlas for X, consisting of open sets $V \subseteq X$ and homeomorphisms $\varphi : V \to \rho(V) \times \mathbb{R}^n$ satisfying $\pi_1 \circ \varphi = \rho$.

2. the change of coordinate functions are continuous,

3. all fiber-wise derivatives of change of coordinate functions should exist and be continuous

4. elements $\gamma \in \mathcal{G}$ should act as diffeomorphisms $\gamma. : \rho^{-1}(s(\gamma)) \to \rho^{-1}(r(\gamma))$.

Notice that this definition implies that smooth $\mathcal{G}$-manifolds have open anchor maps.

Definition 2.3.17. Let X and Y be smooth $\mathcal{G}$-manifolds with anchor maps ρ_X and ρ_Y, respectively. A **smooth $\mathcal{G}$-map** from X to Y is a $\mathcal{G}$-map $\phi : X \to Y$ such that if $p \in X$, then there exists a coordinate neighborhood U of $\phi(p)$ in Y and a coordinate neighborhood $\widetilde{U}$ of p in $\phi^{-1}(U)$, with coordinate charts $\psi_X : \widetilde{U} \to \rho_X(\widetilde{U}) \times \mathbb{R}^k$ and $\psi_Y : U \to \rho_Y(U) \times \mathbb{R}^\ell$, such that all fiber-wise derivatives of $\psi_Y \circ \phi \circ (\psi_X)^{-1}$ are continuous maps from $\rho_X(\widetilde{U}) \times \mathbb{R}^k \to \rho_Y(U) \times \mathbb{R}^\ell$.

Definition 2.3.18. A **$\mathcal{G}$-bundle** on a second-countable locally compact Hausdorff $\mathcal{G}$-space X will be a vector bundle $\pi : E \to X$, equipped with a fiberwise linear $\mathcal{G}$-action such that π is a $\mathcal{G}$-equivariant continuous map. The $\mathcal{G}$-bundle is smooth if E and X are also $\mathcal{G}$-manifolds and π is a smooth $\mathcal{G}$-map.

Definition 2.3.19. A **smooth section** of a $\mathcal{G}$-vector bundle E, is a continuous section of $\pi : E \to X$, which is a smooth map from X to E (as $\mathcal{G}$-manifolds). We denote smooth sections of E by $\Gamma^\infty(E)$, or $\Gamma^\infty(\pi, X, E)$ if the bundle structure needs specification. Furthermore, if $\eta \in \Gamma^\infty(E)$, then we say that η is **$\mathcal{G}$-equivariant** if $\gamma_\cdot^{-1}(\eta(\gamma.x)) = \eta(x)$ for all $(\gamma, x) \in \mathcal{G} \times_{s,\rho} X$.

Note: We will use the notation Γ_c to refer to sections with compact support, Γ_0 to refer to sections which vanish at infinity, and $L^2(E)$ to refer to L^2-sections of E, with respect to a given $\mathcal{G}$-invariant Hermitian metric on E. These invariant metrics do not always exist.

2.4 $\mathcal{G}$-algebras

Definition 2.4.1. Let X be a locally compact Hausdorff space. A **$C_0(X)$-algebra** is a pair (A, θ) consisting of a C*-algebra A and a homomorphism $\theta : C_0(X) \to \mathcal{ZM}(A)$ such that $\overline{\theta(C_0(X)) \cdot A} = A$.

Note: Explicit reference to θ is dropped if no ambiguity is present. For example, $(\theta(f))(a)$ is written as fa, for $f \in C_0(X)$ and $a \in A$.

Definition 2.4.2. If A is a $C_0(X)$-algebra, the **fiber of A above** $x \in X$ is defined to be $A_x := A/I_x A$, where $I_x := \{f \in C_0(X) : f(x) = 0\}$. The subscripts b and c are reserved for the bounded and compactly supported counterparts of A, respectively. These algebras are defined by $A_b := \{a \in \mathcal{M}(A) : \forall \varphi \in C_0(X), \varphi a \in A\}$, and $A_c := C_c(X) \cdot A$.

Example 2.4.3. Suppose $A = C_0(Y)$ where Y is second-countable LCH and $p : Y \to X$ is a continuous function. Then $\mathcal{M}(A) = C_b(Y) = \mathcal{ZM}(A)$, and $\theta : f \mapsto p^*(f) = (f \circ p)$. For $x \in X$, define $Y_x := p^{-1}(\{x\})$. Then

$$A_x = C_0(Y)/I_x C_0(Y) = C_0(Y)/\{f \in C_0(Y) : f|_{Y_x} \equiv 0\} \cong C_0(Y_x).$$

Additionally,

$$\begin{aligned} A_b &= \{f \in C_b(Y) : \forall h \in C_0(X), f \cdot (h \circ p) \in C_0(Y)\} \\ &= \{f \in C_b(Y) : \forall \varepsilon > 0, \exists X\text{-compact set } A \subseteq Y \text{ such that } \|f|_{A^c}\| < \varepsilon\}. \\ A_c &= C_c(X) \cdot A \\ &= \{f \in C_0(Y) : \exists \text{ compact } K \subseteq X \text{ such that } \mathrm{supp}(f) \subseteq p^{-1}(K)\}. \end{aligned}$$

In the particular case where $Y = X \times X'$ maps to X via the coordinate projection $p : X \times X' \to X$, then

$$A = C_0(X \times X') \cong C_0(X) \otimes C_0(X'),$$

$$A_x \cong C_0(X'),$$

$$A_b \cong C_b(X) \otimes C_0(X'), \text{ and}$$

$$A_c \cong C_c(X) \cdot (C_0(X) \otimes C_0(X'))$$

Note: The “compactly supported” elements can’t really be written as $C_c(X) \otimes C_0(X')$, since $C_c(X)$ isn’t a C*-algebra.

Definition 2.4.4. Let A be a $C_0(X)$-algebra. Assume X is LCH.

1. **Restriction**: If U is an open subset of X, then A_U is defined to be the $C_0(U)$-algebra $C_0(U)A$. If F is a closed subset of X, then A_F is defined to be the $C_0(F)$-algebra A/I_FA, where $I_F \subseteq C_0(X)$ is the ideal of functions vanishing on F. A_U and A_F are called A restricted to U and A restricted to F, respectively.

2. **Product**: Suppose B is also a $C_0(X)$-algebra. Then $A \otimes_{max} B$ is a $C_0(X \times X)$-algebra. By restricting $A \otimes_{max} B$ to the diagonal $\Delta_X \subseteq X \times X$, we get a $C_0(X)$-algebra, which we denote $A \otimes_{C_0(X)} B$ or $A \otimes_X B$. See [5], section 3.2.

3. **Pullback**: Let $p : Y \to X$ be a continuous function between LCH-spaces. Then define p^*A to be the $C_0(Y)$-algebra obtained by restricting the $C_0(X \times Y)$-algebra $A \otimes_{max} C_0(Y)$ to the pullback $X \times_{id,p} Y \subseteq X \times Y$.

The definition of product over $C_0(X)$, given above, will be denoted with a hat, $\underset{X}{\widehat{\otimes}}$, if graded products are being used. If necessary, we will explicitly indicate the $C_0(X)$-structure that is being used for balancing the tensor product. For example: $A \otimes_X B$ is sometimes denoted $A \underset{\theta_A,\theta_B}{\otimes} B$.

Example 2.4.5. Let X_1, X_2, and Y be locally compact Hausdorff spaces, and $f_i : X_i \to Y$ continuous functions. The usual isomorphism $C_0(X_1) \otimes C_0(X_2) \cong C_0(X_1 \times X_2)$ is $C_0(Y \times Y)$-linear. The $C_0(Y \times Y)$ action on $C_0(X_1 \times X_2)$ is by pullback through the map $(\pi_1^* f_1, \pi_2^* f_2) : X_1 \times X_2 \to Y \times Y$ taking $(\pi_1^* f_1, \pi_2^* f_2)(x_1, x_2) := (f_1(x_1), f_2(x_2))$. This function satisfies $(\pi_1^* f_1, \pi_2^* f_2)^{-1}(\Delta_Y) = X_1 \times_Y X_2$; consequently,

$$C_0(X_1) \otimes_Y C_0(X_2) = \frac{C_0(X_1) \otimes C_0(X_2)}{I_{\Delta_Y}(C_0(X_1) \otimes C_0(X_2))} \cong \frac{C_0(X_1 \times X_2)}{I_{\Delta_Y} C_0(X_1 \times X_2)} = C_0(X_1 \times_Y X_2)$$

We will use this fact repeatedly without explicit reference.

Proposition 2.4.6. *(Corollaire 3.16 from [5]) Let A be a $C_0(X)$-algebra; B, a $C_0(Y)$-algebra. Then $(A \otimes_{max} B)_{(x,y)} \cong A_x \otimes_{max} B_y$.*

Corollary 2.4.7. *Let A and B be $C_0(X)$ algebras. Denote by $I_{\Delta_X} \subseteq C_0(X \times X)$ the ideal of functions vanishing on the diagonal $\Delta_X \subseteq X \times X$, and $J_s \subseteq A \otimes_{max} B$ the closed ideal generated by simple tensors of the form $ga \otimes b - a \otimes gb$ for all $a \in A$, $b \in B$, and $g \in C_0(X)$. Then $I_{\Delta_X}(A \otimes_{max} B) = J_s$.*

Proof. This is proved in, for instance, [6], Lemma 2.4. □

We will also use this equivalence without explicit reference.

Maximal tensor products are used in this paper because spacial (minimal) tensor products do not satisfy many desirable properties, such as associativity. See section 3.3. of [5] for some counterexamples pertaining to minimal tensor products.

Proposition 2.4.8. *Let A and B be $C_0(X)$-algebras, and let $p : Y \to X$ and $q : Z \to Y$ be continuous functions.*

1. $p^* A \otimes_Y p^* B \cong p^*(A \otimes_X B)$

2. $q^*(p^* A) = (p \circ q)^* A$

Definition 2.4.9. A **homomorphism of $C_0(X)$-algebras**, $\varphi : A \to B$, is a $*$- homomorphism that is also a morphism of $C_0(X)$-modules.

To every $C_0(X)$-algebra morphism $\varphi : A \to B$, there corresponds a family of $*$-homomorphisms $\varphi_x : A_x \to B_x$.

Definition 2.4.10. Let $\mathcal{G}$ be a locally compact topological groupoid with base space Z, and let A be a $C_0(Z)$-algebra. Then an action of $\mathcal{G}$ on A is a family of $*$-isomorphisms $\alpha_\gamma : A_{s(\gamma)} \to A_{r(\gamma)}$, indexed by $\gamma \in \mathcal{G}$, such that for any composable pair $(\gamma, \gamma') \in \mathcal{G}^{(2)}$, $\alpha_{\gamma\circ\gamma'} = \alpha_\gamma \circ \alpha_{\gamma'}$. The action of $\mathcal{G}$ on A is called continuous if it can be obtained by restricting an isomorphism of $C_0(\mathcal{G})$-algebras, $\alpha : s^*A \to r^*A$, to the fibers above each $\gamma \in \mathcal{G}$. If A is a C^*-algebra with a continuous action of $\mathcal{G}$, then we call A a **$\mathcal{G}$-algebra** (sometimes $\mathcal{G}$-C^*-algebra).

The word "action" will be used to refer exclusively to continuous actions. Any possibly discontinuous action will be clearly indicated as such.

2.5 Hilbert Modules over $\mathcal{G}$-algebras

Definition 2.5.1. Let B be a $C_0(X)$-algebra, and let $\mathcal{E}$ be a Hilbert B-module. **The fiber of $\mathcal{E}$ over** $x \in X$ is the Hilbert B_x-module $\mathcal{E}_x := \mathcal{E} \otimes_B B_x$ (the internal tensor product of Hilbert B-modules).

Identifying $\mathcal{E} = \overline{\mathcal{E}B}$, we can define a homomorphism from $C_0(X)$ to the center of $\mathcal{L}(\mathcal{E})$ by $\psi \in C_0(X) \mapsto [\xi \in \mathcal{E} \mapsto \xi\psi \in \mathcal{E}]$. This homomorphism can be used to equip $\mathcal{K}(\mathcal{E})$ with the structure of a $C_0(X)$-algebra.

Definition 2.5.2. Let A and B be $C_0(X)$-algebras, and $\mathcal{E}$ a Hilbert B-module. A $*$-representation $\pi : A \to \mathcal{L}(\mathcal{E})$ is called a representation of $C_0(X)$-algebras if $\pi(\varphi a)\xi = \pi(a)(\xi\varphi)$ for all $a \in A, \varphi \in C_0(X)$, and $\xi \in \mathcal{E}$.

In this case, for all $x \in X$, $\pi(I_x A) \subseteq \mathcal{E}(I_x B)$. Therefore, a representation of $C_0(X)$-algebras decomposes into a family of $*$-representations $\pi_x : A_x \to \mathcal{L}(\mathcal{E}_x)$.

Proposition 2.5.3. *Let $\mathcal{E}$ be a Hilbert A-module, $\mathcal{F}$ a Hilbert B-module, and $\pi : A \to \mathcal{L}(\mathcal{F})$ a representation of $C_0(X)$-algebras.*

1. *For all $x \in X$, $(\mathcal{E} \otimes_A \mathcal{F})_x = \mathcal{E}_x \otimes_{A_x} \mathcal{F}_x$.*

2. *Suppose $R \in \mathcal{L}(\mathcal{E})$, and $S \in \mathcal{L}(\mathcal{F})$ satisfies $\pi(a)S = S\pi(a)$ for all $a \in A$. Then the operator $R \otimes S : \xi \otimes \eta \mapsto R(\xi) \otimes S(\eta)$ defines an element of $\mathcal{L}(\mathcal{E} \otimes_A \mathcal{F})$. Furthermore, for all $x \in X$, $(R \otimes S)_x = R_x \otimes S_x$.*

Proof. This is proved as Proposition 4.1 of [11]. □

Definition 2.5.4. Let B and D be $C_0(X)$-algebras, $\mathcal{E}$ a Hilbert B-module, and $\mathcal{F}$ a Hilbert D-module. The external tensor product of $\mathcal{E}$ and $\mathcal{F}$ over X is defined to be:

$$\mathcal{E} \otimes_{X,ext} \mathcal{F} := (\mathcal{E} \otimes_{max} \mathcal{F}) \otimes_{B \otimes_{max} D} (B \otimes_X D)$$

In other words, $\mathcal{E} \otimes_{X,ext} \mathcal{F}$ is the completion of the algebraic tensor product (of $C_0(X)$-modules) with respect to the $B \otimes_{max} D$-valued norm: $\langle \xi \otimes \eta, \nu \otimes \zeta \rangle := q(\langle \xi, \nu \rangle \otimes_{max} \langle \eta, \zeta \rangle)$, where $q : B \otimes_{max} D \to B \otimes_X D$ is the quotient map.

Note: The fiber of $\mathcal{E} \otimes_{X,ext} \mathcal{F}$ above $x \in X$ is canonically isomorphic to $\mathcal{E}_x \otimes_{ext} \mathcal{F}_x$ as Hilbert $A_x \otimes B_x$-modules.

Definition 2.5.5. If $p : Y \to X$ is a continuous function, then we define $p^*\mathcal{E}$ to be the Hilbert p^*B-module $\mathcal{E} \otimes_{X,ext} C_0(Y)$.

Equivalently, $p^*\mathcal{E} \cong \mathcal{E} \otimes_B p^*B$.

Definition 2.5.6. Let $\mathcal{G}$ be a locally compact topological groupoid, B a $\mathcal{G}$-algebra, and $\mathcal{E}$ a Hilbert B-module. A **continuous action of $\mathcal{G}$ on $\mathcal{E}$** is a unitary $V \in \mathcal{L}(s^*\mathcal{E}, r^*\mathcal{E})$ such that, for all $(\gamma, \gamma') \in \mathcal{G}^{(2)}$, $V_\gamma V_{\gamma'} = V_{\gamma \circ \gamma'}$ and $V_{\gamma^{-1}} = V_\gamma^*$. If this is the case, we call $\mathcal{E}$ a **Hilbert $\mathcal{G}$-B-module**.

Definition 2.5.7. If $\mathcal{E}$ is a (right) Hilbert G-B-module satisfying the property: for all $\xi \in s^*\mathcal{E}$, and $b \in s^*B$, $V(\xi.b) = V(\xi).\alpha(b)$, then we call $\mathcal{E}$ a **$\mathcal{G}$-equivariant Hilbert B-module**.

Definition 2.5.8. Let B be a $\mathcal{G}$-algebra. A representation, $\pi : A \to \mathcal{L}(\mathcal{E})$, of a $\mathcal{G}$-algebra A on a $\mathcal{G}$-equivariant Hilbert B-module $\mathcal{E}$ is a **$\mathcal{G}$-equivariant representation** if, for all $\gamma \in \mathcal{G}$ and all $a_{s(\gamma)} \in A_{s(\gamma)}$,

$$V_\gamma(\pi_{s(\gamma)}(a_{s(\gamma)}))V_\gamma^* = \pi_{r(\gamma)}(\alpha_\gamma a_{s(\gamma)})$$

2.6 Equivariant Kasparov Groups

Definition 2.6.1. Let A and B be $\mathcal{G}$-algebras. A **$\mathcal{G}$-equivariant A-B-bimodule** is a pair $(\mathcal{E}, \pi)$, where $\mathcal{E}$ is a $\mathbb{Z}/2\mathbb{Z}$-graded and $\mathcal{G}$-equivariant Hilbert B-module and π is a

$\mathcal{G}$-equivariant and degree-preserving representation of A into $\mathcal{L}(\mathcal{E})$ (all $\mathcal{G}$-actions are also degree-preserving).

Definition 2.6.2. Let A and B be graded $\mathcal{G}$-algebras. A **$\mathcal{G}$-equivariant Kasparov A-B-bimodule** (or Kasparov $\mathcal{G}$-A-B-module) is a triple $(\mathcal{E}, \pi, F)$ composed of a $\mathcal{G}$-equivariant A-B-bimodule $(\mathcal{E}, \pi)$ and an odd-degree operator $F \in \mathcal{L}(\mathcal{E})$ such that

1. $\forall a \in A$, $(F^2 - I)\pi(a) \in \mathcal{K}(\mathcal{E})$
2. $\forall a \in A$, $(F - F^*)\pi(a) \in \mathcal{K}(\mathcal{E})$
3. $\forall a \in A$, $[F, \pi(a)] \in \mathcal{K}(\mathcal{E})$
4. $\forall a \in r^*A$, $\pi(a)(V(s^*F)V^* - r^*F) \in r^*\mathcal{K}(\mathcal{E})$.

Definition 2.6.3. Two Kasparov $\mathcal{G}$-A-B-modules, $(\mathcal{E}, \pi, F)$ and $(\mathcal{E}', \pi', F')$, are **unitarily equivalent** if there exists a $\mathcal{G}$-equivariant unitary $U \in \mathcal{L}(\mathcal{E}, \mathcal{E}')$ of degree 0 such that $UFU^* = F'$ and, for all $a \in A$, $U\pi(a)U^* = \pi'(a)$. We denote the set of unitary equivalence classes of Kasparov $\mathcal{G}$-A-B-modules by $E^{\mathcal{G}}(A, B)$.

Definition 2.6.4. We say $(\mathcal{E}_0, \pi_0, F_0), (\mathcal{E}_1, \pi_1, F_1) \in E^{\mathcal{G}}(A, B)$ are **homotopic** if there exists a class $(\mathcal{E}, \pi, F) \in E^{\mathcal{G}}(A, B[0,1])$ such that for $t \in \{0,1\}$, $\mathcal{E} \otimes_{B[0,1],ev_t} B \cong \mathcal{E}_t$, and under these isomorphisms, $\pi \otimes_{B[0,1],ev_t} 1 = \pi_t$, and $F \otimes_{B[0,1],ev_t} 1 = F_t$. The action, V, of $\mathcal{G}$ on $\mathcal{E}$ must restrict to the actions, V^t, of $\mathcal{G}$ on $\mathcal{E}_t$ according to the commutative diagram below:

$$\begin{array}{ccc}
s^*\mathcal{E} & \xrightarrow{\quad V \quad} & r^*\mathcal{E} \\
\downarrow \cong & & \downarrow \cong \\
s^*\mathcal{E} \underset{s^*BI}{\otimes} s^*BI & & r^*\mathcal{E} \underset{r^*BI}{\otimes} r^*BI \\
\downarrow 1\otimes ev_t & & \downarrow 1\otimes ev_t \\
s^*\mathcal{E} \underset{s^*BI,ev_t}{\otimes} s^*B & & r^*\mathcal{E} \underset{r^*BI,ev_t}{\otimes} r^*B \\
\downarrow \cong & & \downarrow \cong \\
s^*(\mathcal{E} \underset{BI,ev_t}{\otimes} B) & & r^*(\mathcal{E} \underset{BI,ev_t}{\otimes} B) \\
\downarrow \cong & & \downarrow \cong \\
s^*(\mathcal{E}_t) & \xrightarrow[\quad V^t \quad]{} & r^*(\mathcal{E}_t)
\end{array}$$

Definition 2.6.5. Homotopy equivalence classes of elements in $E^{\mathcal{G}}(A,B)$ is denoted by $KK^{\mathcal{G}}(A,B)$ or $KK_0^{\mathcal{G}}(A,B)$. The odd KK groups will be defined by $KK_1^{\mathcal{G}}(A,B) := KK^{\mathcal{G}}(A, B\widehat{\otimes}\mathbb{C}\ell(1))$.

In the usual way, $E^{\mathcal{G}}(A,B)$ is an abelian semigroup under the direct sum operation, and $KK^{\mathcal{G}}(A,B)$ happens to be an abelian group.

Definition 2.6.6. Let A and B be $\mathcal{G}$-algebras, define $\breve{E}^{\mathcal{G}}(A,B)$ to be the set of all triples $(\mathcal{E},\pi,F)$ consisting of an ungraded $\mathcal{G}$-equivariant A-B-bimodule, where π and F satisfy the conditions of definition 2.6.2, but are ungraded. Let $\widetilde{KK}^{\mathcal{G}}(A,B)$ be the abelian group of homotopy equivalence classes of elements in $\breve{E}^{\mathcal{G}}(A,B)$.

Proposition 2.6.7. *Let A and B be $\mathcal{G}$-algebras, then there is a well-defined group homomorphism* $\widehat{\cdot} : \widetilde{KK}^{\mathcal{G}}(A,B) \to KK_1^{\mathcal{G}}(A,B)$.

Proof. For now, assume that A and B are trivially graded. Let $(\mathcal{E},\pi,F) \in E_1^{\mathcal{G}}(A,B)$. Through a standard simplification, it is sufficient to assume $F = F^*$. Since this simplification is via a compact perturbation of F, and the almost-equivariance condition (part 4. of definition 2.6.2) does not distinguish between compact perturbations of F, this standard simplification can be done in the equivariant setting. Define a (graded) $\mathcal{G}$-equivariant A-B-bimodule $\widehat{\mathcal{E}} := \mathcal{E}\oplus\mathcal{E}$, graded via the direct sum, and where $\mathcal{G}$ acts diagonally on $\widehat{\mathcal{E}}$. Define a $B\widehat{\otimes}\mathbb{C}\ell(1)$ action on $\widehat{\mathcal{E}}$ via $(\xi,\xi').(b\widehat{\otimes}1) := (\xi.b, \xi'.b)$ and $(\xi,\xi').(b\widehat{\otimes}e) := (-\xi'.b, \xi.b)$. A compatible $B\widehat{\otimes}\mathbb{C}\ell(1)$-valued inner product on $\widehat{\mathcal{E}}$ is given by

$$\langle(\xi,\xi'),(\eta,\eta')\rangle_{\wedge} := (\langle\xi,\eta\rangle + \langle\xi',\eta'\rangle)\otimes 1 + (\langle\xi,\eta'\rangle - \langle\xi',\eta\rangle)\otimes e.$$

This structure allows us to take $\widehat{\mathcal{E}}$ as a right Hilbert $B\widehat{\otimes}\mathbb{C}\ell(1)$-module. Define an odd-degree operator $\widehat{F} := \begin{pmatrix} 0 & iF \\ -iF & 0 \end{pmatrix}$. Notice that $\widehat{F}$ is indeed $B\widehat{\otimes}\mathbb{C}\ell(1)$-linear. Define a representation $\widehat{\pi} : A \to \mathcal{L}(\widehat{\mathcal{E}})$ by diagonal action by π. Observe that $(\widehat{\mathcal{E}},\widehat{\pi},\widehat{F})$ satisfies all of the requirements in definition 2.6.2 (recall that we are assuming $F^* = F$). Thus, $\left(\widehat{\pi},\widehat{\mathcal{E}},\widehat{F}\right) \in E^{\mathcal{G}}(A, B\widehat{\otimes}\mathbb{C}\ell(1))$. Since a homotopy can also suffer this procedure, this construction forms a well-defined map on homotopy equivalence classes. □

Interestingly, the $B\widehat{\otimes}\mathbb{C}\ell(1)$-linearity of operators in previous proof implies that general operators $T \in \mathcal{L}(\widehat{\mathcal{E}})$ look like

$$T = \begin{pmatrix} A & iB \\ -iB & A \end{pmatrix}.$$

Even homogeneous operators, T, have $B = 0$; odd operators have $A = 0$. There is a trade-off between the grading chosen for $\widehat{\mathcal{E}}$ and the induced grading and general form of operators in $\mathcal{L}(\widehat{\mathcal{E}})$. At the end of Section 3, we show this construction is unitarily equivalent to taking $\widehat{\mathcal{E}}$, with a grading induced by swapping copies of $\mathcal{E}$, and where $\mathcal{L}(\widehat{\mathcal{E}})$ only consists of block diagonal operators.

There are likely conditions under which the map $\widehat{\cdot} : \widetilde{KK} \to KK_1$ is an isomorphism, but it is not relevant for this thesis.

In section 6 of [11], LeGall verifies that the Kasparov product construction can be performed equivariantly. We will only state the definition, and direct the reader to [11] for the statements and proofs of the typical properties of the Kasparov product, such as the equivariant technical theorem and the existence and associativity of equivariant products.

Definition 2.6.8. Let $x_1 = (\mathcal{E}_1, \pi_1, F_1) \in E^{\mathcal{G}}(A, B)$ and $x_2 = (\mathcal{E}_2, \pi_2, F_2) \in E^{\mathcal{G}}(B, D)$. Then $x_3 = \left(\mathcal{E}_1 \underset{B}{\widehat{\otimes}} \mathcal{E}_2, \pi_1 \widehat{\otimes} 1, F_3\right) \in E^{\mathcal{G}}(A, D)$ is a **(cap) product** of x_1 and x_2 if

1. (Connexion) T_3 is a T_2**-connexion** for $\mathcal{E}_{1,2} := \mathcal{E}_1 \underset{B}{\widehat{\otimes}} \mathcal{E}_2$. That is, for all $\xi \in \mathcal{E}_1$,

$$\theta_\xi \circ T_2 - (-1)^{\partial x \partial T_2} T_3 \circ \theta_\xi \in \mathcal{K}(\mathcal{E}_2, \mathcal{E}_{1,2});$$

$$T_2 \circ \theta_\xi^* - (-1)^{\partial x \partial T_2} \theta_\xi^* \circ T_3 \in \mathcal{K}(\mathcal{E}_{1,2}, \mathcal{E}_2),$$

   where the operator $\theta_\xi : \mathcal{E}_2 \to \mathcal{E}_{1,2}$ maps η to $\xi \underset{B}{\widehat{\otimes}} \eta$. The adjoint sends a simple tensor $\eta_1 \underset{B}{\widehat{\otimes}} \eta_2$ to $\langle \xi, \eta_1 \rangle_B \cdot \eta_2$.

2. (Positivity) For all $a \in A$, $\pi(a)[T_1 \widehat{\otimes} 1, T_3]\pi(a)^* \geqslant 0$ modulo $\mathcal{K}(\mathcal{E}_{1,2})$.

We sometimes write $F_3 \in F_1 \# F_2$ if F_3 is an operator satisfying this definition. On KK-classes, we write $[x] \underset{B}{\widehat{\otimes}} [y]$.

Definition 2.6.9. Let $\mathcal{G}$ be a second-countable locally compact Hausdorff groupoid with object space X. Suppose A, B, and D are $\mathcal{G}$-algebras, and assume D has a countable approximate unit. Define the D**-tensor operator**

$$\sigma_D : KK^{\mathcal{G}}(A,B) \to KK^{\mathcal{G}}\left(A\widehat{\underset{X}{\otimes}}D, B\widehat{\underset{X}{\otimes}}D\right)$$

by the formula on cycles: $\sigma_D(\mathcal{E},\pi,F) := \left(\mathcal{E}\widehat{\underset{X}{\otimes}}D, \pi\widehat{\otimes}1, T\widehat{\otimes}1\right)$.

Definition 2.6.10. Let $\mathcal{G}$ be a second-countable locally compact Hausdorff groupoid with object space X. Suppose A_j, B_j, and D are $\mathcal{G}$-algebras, and assume A_2, B_1 have countable approximate units. Let $x_1 \in KK^{\mathcal{G}}\left(A_1, B_1\widehat{\underset{X}{\otimes}}D\right)$ and $x_2 \in KK^{\mathcal{G}}\left(D\widehat{\otimes}A_2, B_2\right)$. Then the **(cup/cap) product** of x and y is defined by the cap product:

$$x\widehat{\underset{D}{\otimes}}y := \sigma_{A_2}(x) \underset{B_1\widehat{\otimes}D\widehat{\otimes}A_2}{\widehat{\otimes}} \sigma_{B_1}(y)$$

2.7 The Groups $VK_{\mathcal{G},Y}$ and $RK_{\mathcal{G},Y}$

Definition 2.7.1 (See [10])**.** Let $\mathcal{G}$ be a LCH topological groupoid, and let X, Y be LCH $\mathcal{G}$-spaces with a $\mathcal{G}$-map $f : X \to Y$. Define the **representable K-theory of X with Y-compact support** to be

$$RK_{\mathcal{G},Y}(X) := KK^{\mathcal{G}\ltimes Y}(C_0(Y), C_0(X)),$$

representable K-theory of X (with no support conditions) to be $RK_{\mathcal{G}}(X) := RK_{\mathcal{G},X}(X) = KK^{\mathcal{G}\ltimes X}(C_0(X), C_0(X))$. Suppose $\mathcal{G}$ has a compatible Haar system, then define $K_{\mathcal{G}}(X) := KK(\mathbb{C}, \mathcal{G} \ltimes C_0(X))$.

The following definition is partly motivated by Chapter 5 of [14].

Definition 2.7.2.

1. For a LCH and second-countable $\mathcal{G}$-space X, define the category $\mathcal{RV}ect_{\mathcal{G}}(X)$ to be the category of pairs (E,σ), where $E \to X$ is a complex $\mathbb{Z}/2\mathbb{Z}$-graded $\mathcal{G}$-bundle with a Hermitian metric under which $\mathcal{G}$ acts on the fibers of E through a unitary action, and σ is a self-adjoint degree-1 $\mathcal{G}$-equivariant endomorphism of E. The morphisms, from (E,σ) to (E',σ'), will be given by $\mathcal{G}$-bundle maps $\varphi : E \to E'$ satisfying $\varphi \circ \sigma = \sigma' \circ \varphi$.

2. Denote by $\mathcal{V}ect_{\mathcal{G}}(X)$ the subcategory of pairs $(E,\sigma) \in \mathcal{RV}ect_{\mathcal{G}}(X)$ with $\sigma^2 = 1$ and outside some $\mathcal{G}$-compact subset of X.

3. If Y is a second-countable LCH $\mathcal{G}$-space, and $f : X \to Y$ is a $\mathcal{G}$-map, then denote by $\mathcal{RV}ect_{\mathcal{G},Y}(X)$ the subcategory of pairs $(E, \sigma) \in \mathcal{RV}ect_{\mathcal{G}}(X)$ satisfying $\sigma^2 = 1$ outside a Y-compact subset of X. If ambiguity is present, we will use the notation $\mathcal{RV}ect_{\mathcal{G},f}(X)$ for this category.

This definition could be simplified by restricting to a situation where all $\mathcal{G}$-bundles can be equipped with such a metric. For example, Emerson and Meyer typically work under several assumptions with respect to the groupoid action. See section 2 of [9] for more details on useful conditions for $\mathcal{G}$-bundles to be well-behaved; of specific relevance would be, for example, Definition 2.11, Proposition 2.19 of [9].

Definition 2.7.3. An element $(E, \sigma) \in \mathcal{RV}ect_{\mathcal{G},Y}(X)$ is **degenerate** if σ is globally an automorphism. Given $(E_0, \sigma_0), (E_1, \sigma_1) \in \mathcal{RV}ect_{\mathcal{G}}(X)$ (or $\mathcal{V}ect_{\mathcal{G}}(X)$, or $\mathcal{RV}ect_{\mathcal{G},Y}(X)$), we say that (E_0, σ_0) is **homotopic** to (E_1, σ_1) iff there exists $(E, \Sigma) \in \mathcal{RV}ect_{\mathcal{G}}(X \times [0, 1])$ (respectively, $\mathcal{V}ect_{\mathcal{G}}(X \times [0, 1])$, $\mathcal{RV}ect_{\mathcal{G},Y}(X \times [0, 1])$) with $E|_{X\times\{j\}} \cong E_j$ and $\Sigma|_{X\times\{j\}} \cong \sigma_j$ for $j = 0, 1$. Two elements (E_0, σ_0) and (E_1, σ_1) are **equivalent** if they are homotopic after possibly adding degenerate elements to each. We will write $(E_0, \sigma_0) \sim_h (E_1, \sigma_1)$ for this notion of equivalence.

Definition 2.7.4. The set of $\sim_h$-equivalence classes of elements in $\mathcal{RV}ect_{\mathcal{G}}(X)$, $\mathcal{V}ect_{\mathcal{G}}(X)$, and $\mathcal{RV}ect_{\mathcal{G},Y}(X)$) will be denoted respectively by $VK_{\mathcal{G}}(X)$, $V_{\mathcal{G}}(X)$ and $VK_{\mathcal{G},Y}(X)$.

Proposition 2.7.5. *There is a well-defined map* $\nu_{\mathcal{G},Y,X} : VK_{\mathcal{G},Y}(X) \to RK_{\mathcal{G},Y}(X)$.

Proof. Let $(E, \sigma) \in \mathcal{RV}ect_{\mathcal{G},Y}(X)$, where $\pi : E \to X$ is the projection. Then $\Gamma_0(E)$ is a Hilbert $C_0(X)$-module with respect to its fiber-wise Hermitian product. We will treat $C_0(X)$ as a $\mathcal{G} \ltimes Y$-algebra via $f^* : C_0(Y) \to C_b(X)$ and through the $\mathcal{G} \ltimes Y$ action on X given by $(\gamma, y).(x) := \gamma.x$ for any $(\gamma, y) \in \mathcal{G} \ltimes Y$ and $x \in f^{-1}(y)$. Extend the $\mathcal{G}$ action on E to a $\mathcal{G} \ltimes Y$ action in the same fashion. Because the $\mathcal{G}$-action on E is unitary with respect to the Hermitian product, $\Gamma_0(E)$ has an action of $\mathcal{G} \ltimes Y$ satisfying definition 2.5.6. Since π is $\mathcal{G}$-equivariant, $\Gamma_0(E)$ is a $\mathcal{G} \ltimes Y$-equivariant Hilbert $C_0(X)$-module. The adjointable operators, $\mathcal{L}(\Gamma_0(E))$, is isomorphic to $\Gamma_b(End(E))$, and $\mathcal{K}(\Gamma_0(E)) \cong \Gamma_0(End(E))$. Since $\sigma^2 - 1$ and $\sigma^* - \sigma$ are zero outside of a Y-compact subset, pointwise scalar multiplication with a function f^*g for $g \in C_0(Y)$ will yield an element in $\Gamma_0(End(E))$. Therefore, the triple $\nu_{\mathcal{G},Y,X}(\sigma, E) := (\Gamma_0(E), f^*, \sigma)$ satisfies the conditions of Definition 2.6.2, and determines a class in $KK^{\mathcal{G}\ltimes Y}(C_0(Y), C_0(X)) = RK_{\mathcal{G},Y}(X)$. This construction works on homotopies (definition 2.7.3), and yields homotopies (definition 2.6.4). Adding degenerate cycles is trivial in KK theory, so this construction produces a well-defined map on $\sim_h$-equivalence classes, $\nu_{\mathcal{G},Y,X} : VK_{\mathcal{G},Y}(X) \to RK_{\mathcal{G},Y}(X)$. □

This map need not be surjective, even for relatively nice examples. Some conditions for surjectivity are given in [10]. Even restricting to groups, VK and RK are not necessarily the same. Juliane Sauer gives a nice example of when $VK \neq RK$ for a space equipped with a proper smooth action of a totally disconnected group in [21]. See also [19] and [16].

Definition 2.7.6. We define $VK^1_{\mathcal{G},Y}(X)$ to ungraded-homotopy equivalence classes of pairs (σ, E) satisfying all conditions of Definition 2.7.2 except that E and σ are ungraded.

Combining the construction of Proposition 2.7.5 with the construction of Proposition 2.6.7 yields a map $\nu^1_{\mathcal{G},Y,X} : VK^1_{\mathcal{G},Y}(X) \to KK_1^{\mathcal{G}\ltimes Y}(C_0(Y), C_0(X)) =: RK^1_{\mathcal{G},Y}(X)$.

Theorem 2.7.7. *Let X_1 and X_2 be spaces over Y. Suppose $v_1 \in VK_{\mathcal{G},Y}(X_1)$ and $v_2 \in VK_{\mathcal{G},Y}(X_2)$, then there exists a class $z \in VK_{\mathcal{G},Y}(X_1 \times_Y X_2)$ satisfying*

$$\nu(z) = \nu(x) \underset{C_0(Y)}{\widehat{\otimes}} \nu(y)$$

Proof. Let $x = [(\sigma, E)] \in VK_{\mathcal{G},Y}(X_1)$ and $y = [(\eta, F)] \in VK_{\mathcal{G},Y}(X_2)$. Consider the projection maps $\pi_j : X_1 \times_Y X_2 \to X_j$, and define the operator $\sigma \boxtimes \eta \in End(\pi_1^* E \widehat{\otimes} \pi_2^* F)$ by

$$\sigma \boxtimes \eta := \frac{1}{\sqrt{2}} \left(\pi_1^* \sigma \widehat{\otimes} 1 + 1 \widehat{\otimes} \pi_2^* \eta \right)$$

then $(\sigma \boxtimes \eta, \pi_1^* E \widehat{\otimes} \pi_2^* F) \in \mathcal{RV}ect_{\mathcal{G},Y}(X_1 \times_Y X_2)$, since

$$(\sigma \boxtimes \eta)^2 = \frac{1}{2} \left(\pi_1^* \sigma^2 \widehat{\otimes} 1 + (1 + (-1)^{\partial\sigma\partial\eta})(\pi_1^* \sigma \widehat{\otimes} \pi_2^* \eta) + 1 \widehat{\otimes} \pi_2^* \eta^2 \right),$$

which outside of some Y-compact neighborhood $K_1 \times K_2$, is $(1/2)(1\widehat{\otimes}1 + 1\widehat{\otimes}1) = 1$. We will now verify that $\nu(\sigma \boxtimes \eta, \pi_1^* E \widehat{\otimes} \pi_2^* F) \in RK_{\mathcal{G},Y}(X_1 \times_Y X_2)$ from Proposition 2.7.5 is a homotopic to a product of $\nu(\sigma, E)$ and $\nu(\eta, F)$ (Definition 2.6.8). The homotopy is

$$h_t := \left(tM + \frac{(1-t)}{2}\right)^{1/2} \pi_1^*\sigma \widehat{\otimes} 1 + \left(tN + \frac{(1-t)}{2}\right)^{1/2} 1 \widehat{\otimes} \pi_2^*\eta,$$

where $M, N = (1 - M) \in \Gamma_b(End(\pi_1^* E \widehat{\otimes} \pi_2^* F))$ are chosen by the Kasparov Technical lemma so that $M^{1/2}\pi_1^*\sigma\widehat{\otimes}1 + N^{1/2}1\widehat{\otimes}\pi_2^*\eta \in \sigma\#\eta$, such as in THEOREMÈ 6.2 of [11]. Such M and N must exist since all spaces are second-countable locally compact Hausdorff. $\square$

Although we will not be using this theorem in any integral way in this thesis, we prove it here to demonstrate why having representatives in $VK_{\mathcal{G},Y}$ is nice for computing KK-products. Since we will show that the Thom class is an element $\tau_E \in VK_{\mathcal{G},X}(E)$, this means that computations involving Kasparov product with τ_E will be often be simpler to carry out explicitly.

3 Bott Periodicity

3.1 Spin^c-$\mathcal{G}$-Bundles

Throughout, $\mathcal{G}$ will denote a second-countable LCH groupoid. The object space of $\mathcal{G}$ will be denoted by Z (or $\mathcal{G}^{(0)}$ if specificity is required). Let X be a second-countable LCH $\mathcal{G}$-space, and suppose $\pi : E \to X$ is a $\mathcal{G}$-bundle on X of (real) rank k.

Definition 3.1.1. The bundle E is said to be a **Spin^c-$\mathcal{G}$-bundle** if there exists a principal $\mathrm{Spin}^c(k)$-bundle[2] $p : P \to X$ and bundle map $\eta : P \to \mathcal{F}(E)$ (where $\mathcal{F}(E)$ is the principal $\mathrm{GL}(k, \mathbb{R})$-bundle of frames on E) satisfying:

1. commutativity of the diagram:

$$\begin{array}{ccc} P \times \mathrm{Spin}^c(k) & \longrightarrow & P \\ \Big\downarrow{\scriptstyle \eta\times\phi} & & \Big\downarrow{\scriptstyle \eta} \\ \mathcal{F}(E) \times \mathrm{GL}(k, \mathbb{R}) & \longrightarrow & \mathcal{F}(E) \end{array}$$

   where $\phi : \mathrm{Spin}^c(k) \to \mathrm{GL}(k, \mathbb{R})$ is the usual lift of the covering map $\mathrm{Spin}(k) \to \mathrm{SO}(k)$, and the horizontal arrows represent group actions.

2. There is a (left) $\mathcal{G}$-action on P (commuting with the action of $\mathrm{Spin}^c(k)$), such that $p : P \to X$ and $\eta : P \to \mathcal{F}(E)$ are both $\mathcal{G}$-equivariant.

Note: $\mathcal{G}$ acts on E, which lifts to an action of $\mathcal{G}$ on $\mathcal{F}(E)$. This lifted action automatically commutes with the action of $\mathrm{GL}(k, \mathbb{R})$.

If r is defined by $k = 2r$ or $k = 2r + 1$, then there are interesting representations of $\mathrm{Spin}^c(k)$ on both $\mathbb{R}^k$ (via $\phi : \mathrm{Spin}^c(k) \to \mathrm{SO}(k)$), and $\mathbb{C}^{2^r}$. The action of $\mathrm{Spin}^c(k)$ on $\mathbb{C}^{2^r}$ is induced by one of the following:

$$\mathrm{Spin}^c(2r) \subseteq \mathbb{C}\ell(2r) \cong M_{2^r}(\mathbb{C})$$

$$\mathrm{Spin}^c(2r+1) \subseteq \mathbb{C}\ell(2r+1) \cong M_{2^r}(\mathbb{C} \times \mathbb{C}) \xrightarrow{\pi_1 \text{ or } \pi_2} M_{2^r}(\mathbb{C})$$

As in 2.1.5, the grading on $\mathbb{C}\ell(2r)$ is recovered by the element $\varepsilon := i^r e_1 e_2 \cdots e_{2r} \in \mathrm{Spin}^c(2r) \subseteq \mathbb{C}\ell(2r)$. However, no such element exists in $\mathbb{C}\ell(2r+1)$ (2.1.7). Therefore, a

[2] Assumed to be locally trivial. In later sections, groupoid principal bundles are defined in such a way that they are not necessarily locally trivial.

representation φ of $\mathrm{Spin}^c(2r)$ on $\mathbb{C}^{2^r}$ is automatically graded. That is, $\varphi(\varepsilon)$ grades $\mathbb{C}^{2^r}$ so that $\mathrm{Spin}^c(2r)$ acts by grading-preserving linear maps. However, for any representation of $\mathrm{Spin}^c(2r+1)$ on $\mathbb{C}^{2^r}$, there is no grading on $\mathbb{C}^{2^r}$ by which $\mathrm{Spin}^c(2r+1)$ acts by grading-preserving linear maps.

Fixing representations $\phi : \mathrm{Spin}^c(k) \to SO(k)$ and $\psi_k : \mathrm{Spin}^c(k) \to \mathrm{GL}(\mathbb{C}^{2^r})$ ($k = 2r$ or $k = 2r+1$), we can form bundles on X using the $\mathcal{G}$-$\mathrm{Spin}^c(k)$-datum on a $\mathcal{G}$-bundle E.

$$E \cong P \underset{\mathrm{Spin}^c(k)}{\times} \mathbb{R}^k$$

$$\$:= P \underset{\mathrm{Spin}^c(k)}{\times} \mathbb{C}^{2^r}$$

If $k = 2r$, then $\$$ is actually a graded $\mathbb{C}$-$\mathcal{G}$-bundle via the bundle map $(p, v) \mapsto (p, \varphi(\varepsilon)(v)) = (p.\varepsilon, v)$. Additionally, through these identifications we can guarantee the existence of invariant metrics. This is rather important, since the existence of invariant metrics on general $\mathcal{G}$-bundles is not usually guaranteed (see, section 2 of [9]).

Theorem 3.1.2. *If $\pi : E \to X$ is a real Spin^c-$\mathcal{G}$-bundle, then there exists a $\mathcal{G}$-invariant inner product on the fibers of E, and a $\mathcal{G}$-invariant Hermitian product on the fibers of the associated spinor bundle $\$$ such that $\mathcal{G}$ acts on the fibers of $\$$ through unitary maps.*

3.2 The Thom Class of a Spin^c-$\mathcal{G}$-Bundle

If $F \to X$ is any (real) rank k $\mathcal{G}$-bundle with an invariant metric, we denote by $\mathbb{C}\ell(F)$ the $\mathcal{G}$-bundle of Clifford algebras $\mathcal{FO}(F) \times_{O(k)} \mathbb{C}\ell(k)$, where $O(k)$ acts on $\mathbb{C}\ell(k)$ by orthonormal change of basis on $\mathbb{R}^k$.

If $E \to X$ is a Spin^c-$\mathcal{G}$-bundle with Spin^c-Datum (P, η), then

$$\mathbb{C}\ell(E) \cong P \underset{\mathrm{Spin}^c(k)}{\times} \mathbb{C}\ell(k) \cong P \underset{\mathrm{Spin}^c(k)}{\times} M_{2^r}\left(\mathbb{C}^{(k-2r+1)}\right)$$

where $\mathrm{Spin}^c(k)$ acts on $\mathbb{C}\ell(k)$ by conjugation (i.e., through η), and on $M_{2^r}(\mathbb{C})$ (or $M_{2^r}(\mathbb{C}\times\mathbb{C})$) similarly under the isomorphism φ_k.

In the case where k is even, then $\mathbb{C}\ell(E) \cong End(\$)$. Otherwise, $\mathbb{C}\ell(E) \cong End(\$)\widehat{\otimes}(\mathbb{C}\times\mathbb{C}, S_{swap})$.

We can always define a map $E \to \mathbb{C}\ell(E)$ by inclusion of $\mathbb{R}^k$ into $\mathbb{C}\ell(k)$ (this is $O(k)$-equivariant, and therefore $\mathrm{Spin}^c(k)$-equivariant). If k is even, this determines a map

$c : E \to End(\$)$, whose image is specifically contained in the odd-elements of $End(\$)$.

However, if k is odd, then the inclusion defines a map $E \to \mathbb{C}\ell(E) \cong P \underset{\mathrm{Spin}^c(k)}{\times} M_{2^r}(\mathbb{C} \times \mathbb{C})$. Composing with a projection to the diagonal in $\mathbb{C} \times \mathbb{C}$ would result in the zero map, since the image of E in $\mathbb{C}\ell(E)$ is odd, so we just project to the first component of $\mathbb{C} \times \mathbb{C}$ instead, yielding a map $c : E \to \left(P \underset{\mathrm{Spin}^c(k)}{\times} M_{2^r}(\mathbb{C})\right) \cong End(\$)$. There is no grading on $\$$ here, and projecting via π_1 will often give an inequivalent class.

Roughly, we want to make $(c, \pi^*\$)$ into a $VK^k_{\mathcal{G},X}(E)$ class, but c needs to be modified to satisfy the requirements of definition 2.7.2. Define $m : E \to \mathbb{R}$ by

$$m(v) := \begin{cases} 1 & if\ ||v|| < 1 \\ ||v||^{-1} & if\ ||v|| \geqslant 1 \end{cases}$$

Using m and $i \in \mathbb{C}$, we can define $\lambda_E := [(imc, \pi^*\$)] \in VK^k_{\mathcal{G},X}(E)$.

Definition 3.2.1. We define the **Thom class** of the Spin^c-$\mathcal{G}$-bundle of E to be the class

$$\tau_E := \overline{\lambda_E} := \left[\left(imc, \overline{\pi^*\$}\right)\right] \in VK_{\mathcal{G},X}(E) \subseteq KK^{\mathcal{G} \ltimes X}(C_0(X), C_0(E)).$$

Here, $\overline{\pi^*\$}$ is the complex conjugate of $\pi^*\$$. That is, if $\flat : \pi^*\$ \to \overline{\pi^*\$}$ is the identity map, then for all $v \in E$, $(imc)(v) = im(v)c(v) : \overline{\pi^*\$}_v \to \overline{\pi^*\$}_v$ acts by $im(v)c(v)\flat(s) := i\flat(m(v)c(v)s)$.

3.3 Bott Periodicity in KK

In this section, we follow the construction in section 5 of [15] to define the Bott element, and the inverse Bott element

$$\beta_n \in KK^{\mathrm{Spin}^c(n)}(\mathbb{C}, C_0(\mathbb{R}^n)\widehat{\otimes}\mathbb{C}\ell(n)),$$

$$\alpha_n \in KK^{\mathrm{Spin}^c(n)}_n(C_0(\mathbb{R}^n)\widehat{\otimes}\mathbb{C}\ell(n), \mathbb{C}).$$

The Bott-Periodicity theorem can be stated in KK as follows:

Theorem 3.3.1 (Bott-Periodicity)**.** *The elements α_n and β_n are inverses in KK^{Spin^c}. That is:*

1. $\alpha_n \underset{\mathbb{C}}{\widehat{\otimes}} \beta_n = 1_{C_0(\mathbb{R}^n)} \in KK^{\mathrm{Spin}^c}(C_0(\mathbb{R}^n), C_0(\mathbb{R}^n))$

2. $\beta_n \underset{C_0(\mathbb{R}^n)}{\widehat{\otimes}} \alpha_n = 1_{\mathbb{C}} \in KK^{\mathrm{Spin}^c}(\mathbb{C}, \mathbb{C})$.

Compare this statement with Theorem 7 of [15].

3.4 Kasparov's Construction of α_n and β_n

In this subsection, we contextualize constructions made in [15] to our slightly different definitions.

Define a function $f : \mathbb{R}^n \to \mathbb{C}\ell(n)$ by

$$f(x) := \frac{ix}{(1 + ||x||^2)^{1/2}}.$$

The function f is bounded ($||f(x)||^2 = ||x||^2 / (1 + ||x||^2) < 1$; therefore, can be viewed as an element of $\mathcal{M}(C_0(\mathbb{R}^n) \otimes \mathbb{C}\ell(n)) \cong \mathcal{L}(C_0(\mathbb{R}^n) \otimes \mathbb{C}\ell(n))$. It satisfies the following two conditions:

$$1 - f^2(x) = 1 - \frac{-xx}{1 + ||x||^2} = \frac{(1 + ||x||^2) - ||x||^2}{1 + ||x||^2} = \frac{1}{1 + ||x||^2} \in \mathcal{K}(C_0(\mathbb{R}^n) \otimes \mathbb{C}\ell(n));$$

$$f^*(x) = \frac{(-i)(-x)}{(1 + ||x||^2)^{1/2}} = f(x).$$

Furthermore, f is $\mathrm{Spin}^c(n)$-equivariant (taking $\mathrm{Spin}^c(n)$ to act on $\mathbb{C}\ell(n)$ by conjugation).

Consequently, $(C_0(\mathbb{R}^n) \otimes \mathbb{C}\ell(n), f)$ forms a class in $KK^{\mathrm{Spin}^c(n)}(\mathbb{C}, C_0(\mathbb{R}^n) \otimes \mathbb{C}\ell(n))$, where $C_0(\mathbb{R}^n) \otimes \mathbb{C}\ell(n)$ is graded in the usual way. We denote this KK-class by β_n. In Kasparov's paper [15], he defines $KK^{-n}(A, B) := KK(A, B \widehat{\otimes} \mathbb{C}\ell(n))$, but we want to relate these classes back to KK_0 and KK_1. This identificaiton is made explicit in the next subsection.

3.5 Equivalent Representatives for α_n and β_n

We inevitably wish to prove that a Thom class generated via generalized groupoid-homomorphism pullback of β_n is the same (perhaps with a sign difference) as the Thom class of Definition 3.2.1.

Let $n = 2r > 0$, and recall from Theorem 2.1.10 that $\varphi_n : \mathbb{C}\ell(n) \to End(\mathbb{C}^{2^r})$ is a graded $*$-isomorphism with respect to the induced grading on $\mathbb{C}^{2^r}$ from $\varphi_{2r}(\varepsilon)$. The element $f \in \mathcal{L}(C_0(\mathbb{R}^n)\otimes\mathbb{C}\ell(n))$ defining β_n determines an element in $\varphi_n(f) \in \mathcal{L}(C_0(\mathbb{R}^n)\otimes \mathbb{C}^{2^r})$. Since the image of f is contained within $\mathbb{C}\ell(n)^{(1)}$, it follows that the image of $\varphi_n(f)$ acts on $\mathbb{C}^{2^r}$ by an odd automorphism. Taking $\mathrm{Spin}^c(n)$ to act on $\mathbb{C}^{2^r}$ through multiplication under φ_n, the function $\varphi_n(f) : \mathbb{R}^n \to End(\mathbb{C}^{2^r})$ is $\mathrm{Spin}^c(n)$-equivariant, and therefore, $\widetilde{\beta_n} := [(C_0(\mathbb{R}^n) \otimes \mathbb{C}^{2^r}, \varphi_n(f))] \in KK^{\mathrm{Spin}^c(n)}(\mathbb{C}, C_0(\mathbb{R}^n))$. To simplify some notation, let $G := \mathrm{Spin}^c(n)$, and let $S := \overline{\mathbb{C}^{2^r}}$.

Lemma 3.5.1. *With the notation in the previous paragraph, there is an invertible class* $\mu_{2r} := [(S, 0)] \in KK^G(\mathbb{C}, \mathbb{C}\ell(2r))$.

Proof. One can view S as a $\mathbb{Z}/2\mathbb{Z}$-graded imprimitivity $\mathbb{C}$-$\mathbb{C}\ell(n)$-bimodule, with the following structure. For clarity, let $\flat : \mathbb{C}^{2^r} \to \overline{\mathbb{C}^{2^r}} = S$ be the identity map (which is an even-graded conjugate-linear isomorphism). The right action of $x \in \mathbb{C}\ell(n)$ on $\flat(v) \in S$ is given by $\flat(v).x := \flat(\varphi_n(x)^* v)$. The $\mathbb{C}\ell(n)$-valued inner product will be given by $\langle x, y\rangle_{\mathbb{C}\ell n} := \varphi_{2r}^{-1}(\Theta_{x,y})$, where $\Theta_{x,y}$ is the matrix representation of the linear map (from $\mathbb{C}^{2^r}$ to $\mathbb{C}^{2^r}$) given by $(z) \mapsto x\langle y, z\rangle_{\mathbb{C}}$. With these definitions, we check the only non-obvious condition (in our case) for an imprimitivity bimodule:

$$\begin{aligned}
\flat(x).\langle \flat(y), \flat(z)\rangle_{\mathbb{C}\ell(n)} &= \flat(x).\varphi_n^{-1}(\Theta_{y,z}) \\
&= \flat(\varphi_n(\varphi_n^{-1}(\Theta_{y,z}^*))x) \\
&= \flat(\Theta_{z,y}x) \\
&= \flat(z\langle y, x\rangle_{\mathbb{C}}) \\
&= \langle x, y\rangle_{\mathbb{C}}\,\flat(z) \\
&= {}_{\mathbb{C}}\langle \flat(x), \flat(y)\rangle\,\flat(z)
\end{aligned}$$

The action of $g \in \mathrm{Spin}^c(n)$ on $\mathbb{C}$ is trivial, on $A \in \mathbb{C}\ell(n)$ is through conjugation ($g.A = gAg^{-1}$), and on $v \in S$ is via $g.\flat(v) := \flat(\varphi_{2n}(g^{-1}).v)$ (note that this action is through unitaries, since $g^{-1} = g^*$ for $g \in \mathrm{Spin}^c(n)$). We check that S is indeed a

$\mathrm{Spin}^c(n)$-equivariant Hilbert $\mathbb{C}\ell(n)$-module (see definition 2.5.7):

$$\begin{aligned} g.(\flat(v).A) &= g.\flat(\varphi_n(A)^* v) \\ &= \flat(\varphi_n(g^* A^*)v) \\ &= \flat(\varphi_n(gAg^*)^* \varphi_n(g^*)v) \\ &= \flat(\varphi_n(g^*)v).(g.A) \\ &= (g.\flat(v)).(g.A) \end{aligned}$$

The inverse of μ_{2r} is the class $\mu_{2r}^{-1} := [(\mathbb{C}^{2^r}, \varphi_{2r}, 0)] \in KK^{\mathcal{G}}(\mathbb{C}\ell(2r), \mathbb{C})$. This is an inverse since the map $S \underset{\mathbb{C}\ell(2r)}{\hat{\otimes}} \mathbb{C}^{2^r} \to \mathbb{C}$ defined by $\flat(x) \underset{\mathbb{C}\ell(2r)}{\hat{\otimes}} y \mapsto \langle x, y \rangle_{\mathbb{C}}$ is an isomorphism. An intuitive way to see this is to think of $\overline{\mathbb{C}^{2^r}} \underset{\mathbb{C}\ell(2r)}{\hat{\otimes}} \mathbb{C}^{2^r}$ as the space of $\mathbb{C}\ell(2r)$-linear maps (i.e, $M_{2^r}(\mathbb{C})$-linear maps) on $\mathbb{C}^{2^r}$, which can only consist of multiples of the identity. The induced action of $\mathrm{Spin}^c(2r)$ on $\mathbb{C}$ is trivial. Hence, $\mu_{2r} \underset{\mathbb{C}\ell(2r)}{\hat{\otimes}} \mu_{2r}^{-1} = 1_{\mathbb{C}}$. The product in the opposite order gives the class $[(\mathbb{C}\ell(2r), 0)] \in KK^G(\mathbb{C}\ell(2r), \mathbb{C}\ell(2r))$, which is clearly $1_{\mathbb{C}\ell(2r)} = \sigma^G_{\mathbb{C}\ell(2r)}(1_{\mathbb{C}})$. □

Lemma 3.5.2. *Let $n = 2r$. Using the cup/cap product,*

$$\beta_{2r} = \widetilde{\beta}_{2r} \underset{\mathbb{C}}{\hat{\otimes}} \mu_{2r} \in KK^G(\mathbb{C}, C_0(\mathbb{R}^n) \hat{\otimes} \mathbb{C}\ell(2r)).$$

Proof. By definition 2.6.10,

$$\widetilde{\beta}_{2r} \underset{\mathbb{C}}{\hat{\otimes}} \mu_{2r} := \widetilde{\beta}_{2r} \underset{C_0(\mathbb{R}^n)}{\hat{\otimes}} \sigma_{C_0(\mathbb{R}^n)}(\mu_{2r})$$

The class $\sigma_{C_0(\mathbb{R}^n)}(\mu_{2r})$ is given by $[(0, S \hat{\otimes} C_0(\mathbb{R}^n))] = [(0, C_0(\mathbb{R}^n, S)]$ with the obvious $C_0(\mathbb{R}^n)$-module action. The Hilbert $C_0(\mathbb{R}^n) \otimes \mathbb{C}\ell(n)$-module associated with the product of $\widetilde{\beta}_{2r}$ and μ_{2r} is

$$C_0(\mathbb{R}^n, \mathbb{C}^{2^r}) \underset{C_0(\mathbb{R}^n)}{\hat{\otimes}} C_0(\mathbb{R}^n, S) \cong C_0(\mathbb{R}^n, \mathbb{C}^{2^r} \hat{\otimes} \overline{\mathbb{C}^{2^r}}) \cong C_0(\mathbb{R}^n, End(\mathbb{C}^{2^r})) \cong C_0(\mathbb{R}^n, \mathbb{C}\ell(n)).$$

The operator $\varphi_{2r}(f) \hat{\otimes} 1 \in \varphi_{2r}(f) \# 0$, under these identifications, is equivalent to pointwise multiplication by f in $C_0(\mathbb{R}^n, \mathbb{C}\ell(n))$. Similarly, the somewhat awkward right module action of $\mathbb{C}\ell(n)$ on S recovers the right-module action of $C_0(\mathbb{R}^n)\widehat{\otimes}\mathbb{C}\ell(n)$ on itself by right multiplication. This is because the natural isomorphism (i.e., through the Riesz representation) between S and $(\mathbb{C}^{2^r})^*$ relates left multiplication by $\varphi_{2r}(A)^*$ to pre-composition with $\varphi_{2r}(A)$. Consequently,

$$\widetilde{\beta}_{2r}\underset{\mathbb{C}}{\widehat{\otimes}}\mu_{2r} = [(C_0(\mathbb{R}^n, \mathbb{C}\ell(n)), f)] = \beta_{2r}.$$

□

Let $b_n := [(\mathbb{R}^n \times \mathbb{C}^{2^r}, imc)] \in VK^n_{\mathrm{Spin}^c(n)}(\mathbb{R}^n)$ (see the discussion preceeding definition 3.2.1) , then $\nu(b_{2r}) = \widetilde{\beta}_{2r} \in KK^{\mathrm{Spin}^c(2r)}(\mathbb{C}, C_0(\mathbb{R}^{2r}))$ via a homotopy between the following two functions from $\mathbb{R}^n$ to $\mathbb{R}^n$: $[v \mapsto v \cdot m(v)]$ and $[v \mapsto f(v)]$. Since $b_{2r+1} \in VK^1_{\mathrm{Spin}^c(2r+1)}(\mathbb{R}^{2r+1})$ is ungraded, we define

$$\widetilde{\beta}_{2r+1} := \widehat{\nu(b_{2r+1})} \in KK_1^{\mathrm{Spin}^c(2r+1)}(\mathbb{C}, C_0(\mathbb{R}^{2r+1})) := KK^{\mathrm{Spin}^c(2r+1)}(\mathbb{C}, C_0(\mathbb{R}^{2r+1})\widehat{\otimes}\mathbb{C}\ell(1)).$$

Theorem 3.5.3. *Let $n = 2r$ or $n = 2r + 1$. Then*

$$\widetilde{\beta_n}\underset{\mathbb{C}}{\widehat{\otimes}}\mu_{2r} = \beta_n \in KK^{\mathrm{Spin}^c(n)}(\mathbb{C}, C_0(\mathbb{R}^n)\widehat{\otimes}\mathbb{C}\ell(n))$$

Proof. The even case has been proved in Lemma 3.5.2. Let $n = 2r + 1$, and consider $b_n \in VK^1_{\mathrm{Spin}^c(n),\cdot}(\mathbb{R}^n)$ given above. Suppressing the left action by scalar multiplication and denoting $\psi := \pi_1 \circ \varphi_{2r+1}$,

$$\nu(b_n) = [(C_0(\mathbb{R}^n, \mathbb{C}^{2^r}), imc)] = [(C_0(\mathbb{R}^n, \mathbb{C}^{2^r}), \psi(f))] \in \widetilde{KK}^{\mathrm{Spin}^c(n)}(\mathbb{C}, C_0(\mathbb{R}^n))$$

is an ungraded KK-class. Using the construction from proposition 2.6.7, we pass to the odd KK-class:

$$\widetilde{\beta_n} := \widehat{\nu(b_n)} \in KK_1^{\mathrm{Spin}^c(n)}(\mathbb{C}, C_0(\mathbb{R}^n)) := KK^{\mathrm{Spin}^c(n)}(\mathbb{C}, C_0(\mathbb{R}^n)\widehat{\otimes}\mathbb{C}\ell(1)).$$

Represented by

$$\widetilde{\beta}_n = \left[\left(C_0(\mathbb{R}^n, \mathbb{C}^{2^r} \oplus \mathbb{C}^{2^r}), \begin{pmatrix} 0 & i\psi(f) \\ -i\psi(f) & 0 \end{pmatrix}\right)\right]$$

Denote the Hilbert module $\mathcal{E}_1 := C_0(\mathbb{R}^n, \mathbb{C}^{2^r} \oplus \mathbb{C}^{2^r})$, which is graded with respect to the grading operator $S_1(x) :\equiv \begin{pmatrix} I_{2^r} & 0 \\ 0 & -I_{2^r} \end{pmatrix}$. The right action of $e \in \mathbb{C}\ell(1)$ on $\mathcal{E}_1$ is by left multiplication with the operator $[e]_1 = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$. Consider $\mathcal{E}_2 := \mathcal{E}_1$, but with grading operator $S_2(x) :\equiv \begin{pmatrix} 0 & I_{2^r} \\ I_{2^r} & 0 \end{pmatrix}$ and matrix representation of e given by $[e]_2 = \begin{pmatrix} iI & 0 \\ 0 & -iI \end{pmatrix}$. These Hilbert $\mathbb{C}_0(\mathbb{R}^n)\widehat{\otimes}\mathbb{C}\ell(1)$-modules are unitarily equivalent through the unitary:

$$U := \frac{1}{\sqrt{2}} \begin{pmatrix} iI & I \\ I & iI \end{pmatrix} : \mathcal{E}_2 \to \mathcal{E}_1$$

That is, $U^* S_1 U = S_2$ and $U^*[e]_1 U = [e]_2$. Furthermore,

$$U^* \begin{pmatrix} 0 & i\psi(f) \\ -i\psi(f) & 0 \end{pmatrix} U = \begin{pmatrix} \psi(f) & 0 \\ 0 & -\psi(f) \end{pmatrix}$$

So, again suppressing the obvious representation of $\mathbb{C}$ on $\mathcal{E}_2$, an equivalent representative of β_n can be given by

$$\widetilde{\beta}_n = \left[\left(\mathcal{E}_2, \begin{pmatrix} \psi(f) & 0 \\ 0 & -\psi(f) \end{pmatrix} \right) \right] = [(\mathcal{E}_2, \varphi_{2r+1}(f))] \,.$$

The last equality holds because $\psi = \pi_1 \circ \varphi_{2r+1}$ and $\pi_2 \circ \varphi_{2r+1}$ are negatives of each other on odd-homogeneous elements of $\mathbb{C}\ell(2r+1)$, and consequently φ_{2r+1} and $(\psi, -\psi)$ coincide on odd-homogeneous elements. Using the Hilbert $\mathbb{C}_0(\mathbb{R}^n)\widehat{\otimes}\mathbb{C}\ell(1)$-module $\mathcal{E}_2$ instead of $\mathcal{E}_1$ is nice in that all operators $T \in \mathcal{L}(\mathcal{E}_2)$ must be block diagonal (in order to commute with $[e]_2$). In this picture, $(\varphi_1^{-1})_* : \mathcal{E}_2 \xrightarrow{\cong} C_0(\mathbb{R}^n, C^{2^r}\widehat{\otimes}\mathbb{C}\ell(1))$, where $\mathbb{C}^{2^r}$ is treated as a trivially graded vector-space.

Now consider

$$\sigma_{C_0(\mathbb{R}^n, \mathbb{C}\ell_1)}(\mu_{2r}) = [(C_0(\mathbb{R}^n, S\widehat{\otimes}\mathbb{C}\ell_1), 0)] \in KK^{\mathrm{Spin}^c(n)}(C_0(\mathbb{R}^n)\widehat{\otimes}\mathbb{C}\ell_1, C_0(\mathbb{R}^n)\widehat{\otimes}\mathbb{C}\ell_n).$$

Fix graded isomorphisms

$$\begin{aligned}
\left(\mathbb{C}^{2^r}\oplus\mathbb{C}^{2^r}, \text{grading} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right)\underset{\mathbb{C}\ell(1)}{\widehat{\otimes}}(S\widehat{\otimes}\mathbb{C}\ell(1)) &\cong (\mathbb{C}^{2^r}\widehat{\otimes}\mathbb{C}\ell(1))\underset{\mathbb{C}\ell(1)}{\widehat{\otimes}}(S\widehat{\otimes}\mathbb{C}\ell(1)) \\
&\cong (\mathbb{C}^{2^r}\widehat{\otimes}S)\widehat{\otimes}\mathbb{C}\ell(1) \\
&\cong End(\mathbb{C}^{2^r})\widehat{\otimes}\mathbb{C}\ell(1) \\
&\cong \mathbb{C}\ell(2r)\widehat{\otimes}\mathbb{C}\ell(1) \\
&\cong \mathbb{C}\ell(2r+1)
\end{aligned}$$

Notice that, through this composition of isomorphisms, acting by $\varphi_{2r+1}(f(x))\underset{\mathbb{C}\ell(1)}{\widehat{\otimes}}1$, for some $x\in\mathbb{R}^n$, becomes multiplication by $f(x)$. Consequently,

$$\widetilde{\beta}_n\underset{\mathbb{C}}{\widehat{\otimes}}\mu_{2r} = [(C_0(\mathbb{R}^n, \mathbb{C}\ell(n)), f)] = \beta_n \in KK^{\mathrm{Spin}^c(n)}(\mathbb{C}, C_0(\mathbb{R}^n)\widehat{\otimes}\mathbb{C}\ell(n))$$

□

The purpose of this theorem is to express β_n as a class in either KK_0 or KK_1 rather than a class in KK_n.

Definition 3.5.4. For $n = 2r$ or $n = 2r+1$, Let $\alpha_n \in KK^{\mathrm{Spin}^c(n)}(\mathbb{C}\ell(n)\widehat{\otimes}C_0(\mathbb{R}^n), \mathbb{C})$ be the class defined in section 5 of [15]. In the odd-case, fix some orientation preserving identification $\mathbb{C}\ell(2r+1)\cong\mathbb{C}\ell(2r)\widehat{\otimes}\mathbb{C}\ell(1)$, and define $\widetilde{\alpha_n} := \mu_{2r}^{-1}\underset{\mathbb{C}\ell(2r)}{\widehat{\otimes}}\alpha_n$.

We won't go into too much detail for α_n, and we will use Theorem 3.3.1, proved in [15], applied to $\widetilde{\alpha_n}$ and $\widetilde{\beta_n}$. In the odd cases, we will suppress explicit mention of the identifications $KK(\mathbb{C}\ell(1)\underset{A}{\widehat{\otimes}}, B\widehat{\otimes}\mathbb{C}\ell(1)) \cong KK(A,B)$ and $KK(\mathbb{C}\ell(1)\underset{A}{\widehat{\otimes}}, B)\cong KK(A, B\widehat{\otimes}\mathbb{C}\ell(1))$, as was done in the statement of Theorem 3.3.1.

4 Groupoid Homomorphisms and Pullback Constructions

This section on generalized groupoid homomorphisms is broken up into four subsections. The first subsection will recall the notion of generalized groupoid homomorphism and work through some important examples. These generalized homomorphisms are often called Hilsum-Skandalis morphisms. The next three sections recount the constructions outlined in LeGall's work [11], for pulling back $\mathcal{H}$-algebras to $\mathcal{G}$-algebras, $\mathcal{H}$-Hilbert modules to $\mathcal{G}$-Hilbert modules, and $KK^{\mathcal{H}}$-classes to $KK^{\mathcal{G}}$-classes via a generalized groupoid homomorphism from $\mathcal{G}$ to $\mathcal{H}$.

4.1 Generalized Groupoid Homomorphisms

This section takes definitions from [11] and expands on some examples and arguments. The examples are somewhat detailed, and only serve as examples of manipulating the somewhat awkward definitions. The essential content of this subsection, for the Thom isomorphism, is entirely contained in the material preceding Example 4.1.10.

Definition 4.1.1. Let Y be a LCH space and let Ω be a LCH (right) $\mathcal{H}$-space, where $\mathcal{H}$ is a LCH topological groupoid. We say that $f : \Omega \to Y$ is a **principal $\mathcal{H}$-bundle** if

1. $f : \Omega \to Y$ is an $\mathcal{H}$-invariant continuous surjection,

2. the action of $\mathcal{H}$ on Ω is free and proper,

3. whenever $f(\omega) = f(\eta)$, there exists (an implicitly unique) $\gamma \in \mathcal{H}$ with $\omega.\gamma = \eta$ (f-fiberwise transitive).

Notice that this definition of principal bundle does not automatically guarantee that $f : \Omega \to Y$ admits local sections, nor does it guarantee local triviality. Worse yet, the fibers over f might not be homeomorphic to all of $\mathcal{H}$, but merely a sub-groupoid of $\mathcal{H}$. More specifically, if $\mathcal{H}$ is not a transitive groupoid, then a single fiber of f is necessarily homeomorphic to a transitive sub-groupoid of $\mathcal{H}$.

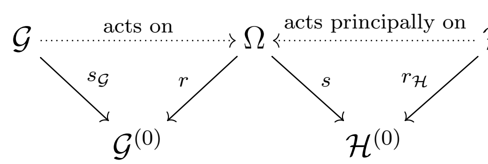
Example 4.1.2. Let $\mathcal{H}$ be a LCH groupoid, and let $f : \Omega \to Y$ be a principal $\mathcal{H}$-bundle. For another LCH groupoid $\mathcal{G}$, define $\widehat{\mathcal{H}} := \mathcal{H} \coprod \mathcal{G}$, whose object space consists of the two disjoint object spaces of $\widehat{\mathcal{H}}$. Let the anchor map $\rho : \Omega \to \widehat{\mathcal{H}}^{(0)}$ be the composition of the original anchor map to $\mathcal{H}^{(0)}$ followed by inclusion into $\widehat{\mathcal{H}}^{(0)} = \mathcal{H}^{(0)} \coprod \mathcal{G}^{(0)}$. Then $\mathcal{G}$ simply does not act on Ω at all. Therefore, freeness, properness, and transitivity are not affected by $\mathcal{G}$, and Ω is a principal $\widehat{\mathcal{H}}$-bundle simply because it is a principal $\mathcal{H}$-bundle.

Definition 4.1.3. If $\mathcal{G}$ and $\mathcal{H}$ are LCH topological groupoids, then a **graph** from $\mathcal{G}$ to $\mathcal{H}$ is topological space Ω together with continuous maps $r : \Omega \to \mathcal{G}^{(0)}$ and $s : \Omega \to \mathcal{H}^{(0)}$ such that:

1. $\mathcal{G}$ acts on Ω on the left so that $\omega \in \Omega$ and $\gamma \in \mathcal{G}$ are composable whenever $s_{\mathcal{G}}(\gamma) = r(\omega)$,

2. $\mathcal{H}$ acts on Ω on the right so that $\omega \in \Omega$ and $\gamma \in \mathcal{H}$ are composable whenever $s(\omega) = r_{\mathcal{H}}(\gamma)$,

3. the right $\mathcal{H}$-action equips $r : \Omega \to \mathcal{G}^{(0)}$ with the structure of a principal $\mathcal{H}$-bundle.



Definition 4.1.4. A graph Ω from $\mathcal{G}$ to $\mathcal{H}$ is called **regular** if there exists a LCH space Z and a surjective, open, and continuous map $q : Z \to \mathcal{G}^{(0)}$ that admits a lift via r to a map $\widetilde{q} : Z \to \Omega$.

Proposition 4.1.5. *Let $\mathcal{G}$ be a LCH topological groupoid, and $\pi : P \to X$ be a locally-trivial principal $\mathcal{G}$-bundle on the second-countable and LCH space X. If the anchor map for the (right) $\mathcal{G}$ action on P is denoted by ρ, then the triple (P, π, ρ) is a* regular *graph from X to $\mathcal{G}$.*

Proof. Since it is immediate that (P, π, ρ) is a graph from X to $\mathcal{G}$, it suffices to show regularity. In fact, we show that π itself is open, as a consequence of local triviality: Assume $U \subseteq P$ is an open set, and $x \in \pi(U)$. Then for any $y \in \pi^{-1}(x) \cap U$, there exists, under a local trivialization (V, ψ) of P above $x \in V \subseteq X$, a product neighborhood $W_b \times W_f \subseteq V \times \mathcal{G}$ with $y \in \psi^{-1}(W_b \times W_f) \subseteq U$ and therefore $x \in \pi\left(\psi^{-1}(W_b \times W_f)\right) = W_b$, which is an open neighborhood of x and a subset of $\pi_P(U)$. $\square$

Definition 4.1.6. Two graphs Ω and Ω' from $\mathcal{G}$ to $\mathcal{H}$ are **equivalent** if there is a homeomorphism between them that is both $\mathcal{G}$ and $\mathcal{H}$ equivariant.

Definition 4.1.7. A **generalized groupoid homomorphism from $\mathcal{G}$ to $\mathcal{H}$** is an equivalence class of regular graphs $[\Omega]$ from $\mathcal{G}$ to $\mathcal{H}$.

Definition 4.1.8. Let $\mathcal{G}$ be a LCH topological groupoid and Y a LCH space. If $p : Y \to \mathcal{G}^{(0)}$ is a continuous function, we define

$$\mathcal{G}_Y := \left(Y \underset{p,r}{\times} \mathcal{G} \underset{s,p}{\times} Y \right) = \{(y, \gamma, y') \in Y \times \mathcal{G} \times Y : p(y) = r(\gamma); p(y') = s(\gamma)\}.$$

The source and range maps on $\mathcal{G}_Y$ are given by $s(y, \gamma, y') = y'$ and $r(y, \gamma, y') = y$. Composition is given by $(y, \gamma, y')(y', \gamma', y'') := (y, \gamma \circ \gamma', y'')$

Notice that $\mathcal{G}_Y$ makes sense for spaces Y that are not equipped with a $\mathcal{G}$-action. In the case where Y is equipped with a $\mathcal{G}$-action, $\mathcal{G} \ltimes Y$ can be naturally identified with a proper (w.r.t. containment) subgroupoid of $\mathcal{G}_Y$.

Definition 4.1.9. A **pre-morphism of groupoids** from $\mathcal{G}$ to $\mathcal{H}$ is a triple (Y, p, f) consisting of a LCH space Y, a continuous open surjection $p : Y \to \mathcal{G}^{(0)}$ and a strict groupoid homomorphism $f : \mathcal{G}_Y \to \mathcal{H}$.

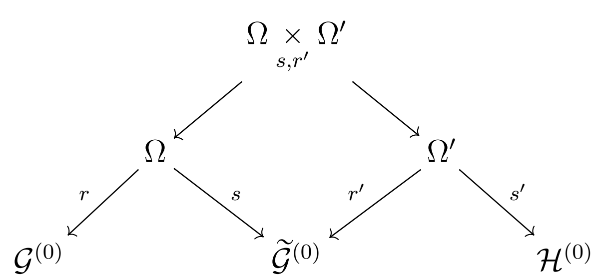
Let (Y, p, f) be a pre-homomorphism from $\mathcal{G}$ to $\mathcal{H}$, and denote by $\iota : Y \to \mathcal{G}_Y$ the inclusion defined by $\iota(y) := (y, p(y), y) \in \mathcal{G}_Y$. We will construct a graph corresponding to (Y, p, f) as follows. Let $\widetilde{\Omega} := Y \underset{f \circ \iota, r_{\mathcal{H}}}{\times} \mathcal{H}$. Let $\Omega_{(Y,p,f)}$ be the quotient of $\widetilde{\Omega}$ by the equivalence relation $(y, h) \sim (y', h')$ if and only if $p(y) = p(y')$ and $h' = f(y', p(y), y)h$. Define $r([(y, h)]) := p(y) \in \mathcal{G}^{(0)}$ and $s([(y, h)]) := r_{\mathcal{H}}(h) \in \mathcal{H}^{(0)}$. On the left, $\gamma \in \mathcal{G}$ acts by: $\gamma.[(y, h)] := [(y', f(y', \gamma, y)h)]$, where y' is any element of $p^{-1}(r_{\mathcal{G}}(\gamma))$ (the choice of y' does not matter). On the right, $\alpha \in \mathcal{H}$ acts by: $[(y, h)].\alpha := [(y, h\alpha)]$. With these definitions, one can show that $\Omega_{(Y,p,f)}$ is a regular graph with p open.

Conversely, if (Ω, r, s) is a graph from $\mathcal{G}$ to $\mathcal{H}$, then we can form a strict homomorphism $f : \mathcal{G}_\Omega \to \mathcal{H}$ as follows. Given $(z, \gamma, y) \in \mathcal{G}_\Omega$, we have that $r(z) = r_{\mathcal{G}}(\gamma) = r(\gamma.y)$; therefore, (by principality) there exists a unique $h \in \mathcal{H}$ with $zh = \gamma.y$. Define $f(z, \gamma, y) := h$. If Ω is a regular graph, then there exists Z and $q : Z \to \mathcal{G}^{(0)}$, a continuous open surjection that lifts to $\widetilde{q} : Z \to \Omega$; therefore, $(Z, q, f \circ \widetilde{q})$ is a pre-homomorphism, where $f : \mathcal{G}_\Omega \to \mathcal{H}$ is defined above.

Example 4.1.10. Suppose (Ω, r, s) is a regular graph from $\mathcal{G}$ to $\mathcal{H}$ with r an open map. Then the corresponding pre-homomorphism is (Ω, r, f), where $f : \mathcal{G}_\Omega \to \mathcal{H}$ is defined by $f(\omega, \gamma, \omega') = h$ for the unique $h \in \mathcal{H}$ satisfying $\omega h = \gamma\omega'$. Consider the construction of the graph corresponding to this pre-homomorphism. Since $f(\omega, r(\omega), \omega) = h$ must satisfy $\omega h = r(\omega)\omega = \omega$, it follows that $h = s(\omega)$. Therefore, $f \circ \iota = s$, and consequently $\Omega \underset{f\circ\iota, r_{\mathcal{H}}}{\times} \mathcal{H} = \Omega \underset{s, r_{\mathcal{H}}}{\times} \mathcal{H}$. Consider the function $\psi : \Omega \underset{s, r_{\mathcal{H}}}{\times} \mathcal{H} \to \Omega$ by $\psi(\omega, h) := \omega.h$. The equality $\psi(\omega, h) = \psi(\omega', h')$ holds if and only if $\omega h = \omega' h'$. Denote $\eta := f(\omega', r(\omega), \omega)$. Applying the definition of f, $(\omega)h = (\omega'\eta)h$. So $\omega h = \omega' h'$ if and only if $\eta h = h'$ (and $r(\omega) = r(\omega')$). In summary, $\psi(\omega, h) = \psi(\omega', h')$ if and only if $(\omega, h) \sim (\omega', h')$. Consequently, ψ passes to an injective function on the $\sim$ equivalence classes in $\Omega \times_{\mathcal{H}^0} \mathcal{H}$. It is clearly surjective, since $\psi(\omega, s(\omega)) = \omega$. Therefore, the regular graph corresponding to the pre-homomorphism (Ω, r, f) is equivalent to (Ω, r, s). That is, the composition of constructions (r-open regular graphs) $\to$ (pre-homomorphisms) $\to$ (r-open regular graphs) is the identity modulo graph equivalences.

Definition 4.1.11. Let (Y, p, f) be a pre-homomorphism from $\mathcal{G}$ to $\widetilde{\mathcal{G}}$; (Y', p', f'), a pre-homomorphism from $\widetilde{\mathcal{G}}$ to $\mathcal{H}$. Recall the inclusion $\iota : Y \to \mathcal{G}_Y$. **The composition of pre-homomorphisms** is given by $(Z, p, f' \circ \widetilde{f})$, where $Z := Y \underset{f\circ\iota,\, p'}{\times} Y'$ and $\widetilde{f} : \mathcal{G}_Z \to \mathcal{H}_{Y'}$ takes $((y, y'), \gamma, (z, z'))$ to $(y', f((y, \gamma, z)), z')$.

Example 4.1.12. Suppose that (Ω, r, s) is a regular graph from $\mathcal{G}$ to $\widetilde{\mathcal{G}}$ such that r an open map (i.e., $r : \Omega \to \mathcal{G}^{(0)}$ is a continuous open surjection lifting to the identity map). Let (Ω', r', s') be a regular graph from $\widetilde{\mathcal{G}}$ to $\mathcal{H}$, similarly assuming that r' is an open map. The composition of (Ω, r, f) and (Ω', r', f') is given by the pre-homomorphism $\left(\Omega \underset{s,r'}{\times} \Omega', r, f' \circ \widetilde{f}\right)$ from $\mathcal{G}$ to $\mathcal{H}$. Recall that $f : \mathcal{G}_\Omega \to \widetilde{\mathcal{G}}$ sends $(\omega, \gamma, \omega') \in \mathcal{G}_\Omega$ to the unique element $\eta \in \widetilde{\mathcal{G}}$ satisfying $\omega\eta = \gamma\omega'$. The strict groupoid homomorphism $f' : \widetilde{\mathcal{G}}_{\Omega'} \to \mathcal{H}$ is defined similarly. A natural choice for the composite graph might look like



However, the space $\Omega \underset{s,r'}{\times} \Omega'$ ***is not a graph from*** $\mathcal{G}$ ***to*** $\mathcal{H}$. However, modding out by the relation $(\omega\widetilde{\gamma}, \omega') \sim' (\omega, \widetilde{\gamma}\omega')$ for any $\widetilde{\gamma} \in \widetilde{\mathcal{G}}$ from source $r'(\omega')$ to range $s(\omega)$ is actually a regular graph from $\mathcal{G}$ to $\mathcal{H}$. The range and source maps for this $\widetilde{\mathcal{G}}$-balanced Cartesian product (denoted by r and s', respectively), along with the $\mathcal{G}$ and $\mathcal{H}$ actions, are the obvious choices. We will now show that

$$\left(\Omega \underset{s,r'}{\times} \Omega' / \sim'\right) \cong \left(\left(\Omega \underset{s,r'}{\times} \Omega'\right) \underset{s',r_{\mathcal{H}}}{\times} \mathcal{H} / \sim\right).$$

Define $\psi : \Omega \underset{s,r'}{\times} \Omega' \to \Omega \underset{s,r'}{\times} \Omega' \underset{s',r_{\mathcal{H}}}{\times} \mathcal{H} / \sim$ by

$$\psi(\omega, \omega') := [(\omega, \omega', s'(\omega'))]$$

The equality $\psi(\omega, \omega') = \psi(\eta, \eta')$ holds if and only if $(\omega, \omega', s'(\omega')) \sim (\eta, \eta', s'(\eta'))$. By definition of $\sim$, this occurs if and only if

$$r(\omega) = r(\eta) \;\text{ and }\; s'(\omega') = (f' \circ \widetilde{f})((\omega, \omega'), r(\eta), (\eta, \eta'))s'(\eta').$$

Define $\gamma := f(\omega, r(\eta), \eta) \in \widetilde{\mathcal{G}}$, and by definition of f, $\omega\gamma = \eta$. Similiarly, define h to be the following element:

$$\begin{aligned}(f' \circ \widetilde{f})((\omega, \omega'), r(\eta), (\eta, \eta')) &= f'(\omega', f(\omega, r(\eta), \eta), \eta') \\ &= f'(\omega', \gamma, \eta') \\ &=: h \in \mathcal{H}\end{aligned}$$

Based on these definitions, $\omega' h = \gamma\eta'$ and $s'(\omega') = hs'(\eta')$. The second equality is equivalent to $h = s'(\omega')$, and the the first equality translates to $\omega' = \gamma\eta'$. In summary, $\psi(\omega, \omega') = \psi(\eta, \eta')$ if and only if $(\eta, \eta') = (\omega\gamma, \gamma^{-1}\omega')$ (i.e., $(\eta, \eta') \sim' (\omega, \omega')$). So ψ defines an injective function on $\sim'$-equivalence classes. For surjectivity, notice that $(\omega, \omega', h) \sim (\omega, \omega' h, s_{\mathcal{H}}(h))$, since

$$
\begin{aligned}
f'(\omega' h, f(\omega, r(\omega), \omega), \omega')h &= f'(\omega' h, s(\omega), \omega')h \\
&= f'(\omega' h, r'(\omega'), \omega')h \\
&= h^{-1}h \\
&= s_{\mathcal{H}}(h).
\end{aligned}
$$

The conclusion of this example is that the composition of pre-homomorphisms, corresponding to (r-open) regular graphs, corresponds to the $\widetilde{\mathcal{G}}$-balanced Cartesian product of the composite graphs.

4.2 Pullback of Algebras

Consider a strict homomorphism between LCH groupoids, $f : \mathcal{G} \to \mathcal{H}$, and an $\mathcal{H}$-algebra D with action $\alpha : s^*D \to r^*D$. Define f^*D to be the $C_0(\mathcal{G}^{(0)})$-algebra gotten by pullback of D (as a $C_0(\mathcal{H}^{(0)})$-algebra) by $f|_{\mathcal{G}^{(0)}}$, equipped with the $\mathcal{G}$ action $f^*\alpha : f^*(s^*D) \to f^*(r^*D)$ defined by $(f^*\alpha)_\gamma := \alpha_{f(\gamma)}$.

If $\varphi = (Y, p, f)$ is a pre-homomorphism from $\mathcal{G}$ to $\mathcal{H}$, we want to define a $\mathcal{G}$-algebra φ^*D, whenever D an $\mathcal{H}$-algebra. Taking f^*D yields a $\mathcal{G}_Y$-algebra; however, we need to use $p : Y \to \mathcal{G}^{(0)}$ to push this down to a $\mathcal{G}$-algebra. The fact that p is an open surjection will be required in order for the propositions in this section to hold.

Let B be a $C_0(Z)$-algebra; $p : Y \to Z$, a continuous open surjection. Fix a continuous family of measures with compact support (alternatively, proper support), $\{\mu_z\}_{z\in Z}$, on the fibers $p^{-1}(\{z\})$ of p. For any element $\delta = \{\delta_y\}_{y\in Y} \in (p^*B)_b$, we can construct an element $\{(\widetilde{\delta})_z\}_{z\in Z} = \widetilde{\delta} \in D_c$ (alternatively, in D_b) by integration:

$$
(\widetilde{\delta})_z := \int_{y\in p^{-1}(z)} \delta_y d\mu_z(y)
$$

Again, we want to be able to push a $\mathcal{G}_Y$-algebra, D, down to a $\mathcal{G}$-algebra, and this requires something like averaging elements over the induced $Y \times_p Y$-action on D. This induced action is given by restricting the action of $\mathcal{G}_Y$ to just those elements (y, γ, y') where $\gamma \in \mathcal{G}^{(0)}$.

Denote by $\alpha : s^*D \to r^*D$ the action of $Y \times_p Y$ on D, and suppose $x \in D_b$. Then we consider the family of elements $(\mu * x)_y \in D_y$ given by

$$(\mu * x)_y := \int_{y' \in Y_{p(y)}} \alpha_{(y,y')}(x_{y'}) d\mu_{p(y)}(y')$$

Lemma 4.2.1. *Consider the C^*-algebra of $Y \times_p Y$-invariant elements of D_b given by $D^{Y \times_p Y} := \{d \in D_b : \forall (y, y') \in Y \times_p Y, \alpha_{(y,y')}(d_{y'}) = d_y\}$. Then the family $\{(\mu * x)_y\}_{y \in Y}$, defined above, forms an element $\mu * x \in D^{Y \times_p Y}$, whenever $x \in D_b$.*

The proof of this Lemma is in [11].

Since $\mu * x$ is $Y \times_p Y$-invariant, we could define, for any $z \in Z$, $(\mu * x)_z := (\mu * x)_y$, for an arbitrary $y \in p^{-1}(z)$. For similar reasons, $C_0(Y)^{Y \times_p Y} \cong C_b(Z)$, so the algebra $D^{Y \times_p Y}$ has a natural action of $C_0(Z)$.

Definition 4.2.2. If $p : Y \to Z$ is a continuous open surjection between LCH and σ-compact spaces Y and Z, and D is a $Y \times_p Y$-algebra, then denote by $p_! D$ the $C_0(Z)$-algebra $C_0(Z) \cdot D^{Y \times_p Y}$.

Proposition 4.2.3 (See [11] Proposition 3.2)**.** *With the situation of Definition 4.2.2,*

(a) D is canonically isomorphic to $p^* p_! D$ as $Y \times_p Y$-algebras

(b) If $\varphi : D \to D'$ is a $Y \times_p Y$-algebra homomorphism, then there exists $p_! \varphi : p_! D \to p_! D'$ such that $p^* p_! \varphi = \varphi$.

Considering the relevant case where D is a $\mathcal{G}_Y$-algebra and $Z = \mathcal{G}^{(0)}$, we consider $\alpha_{(y,\gamma,y')} : D \to D$ for $(y, \gamma, y') \in \mathcal{G}_Y$. Since $p_! \alpha_{(y,\gamma,y')}$ does not depend on the particular choice of $y \in p^{-1}(r(\gamma))$ nor $y' \in p^{-1}(s(\gamma))$, we can define a $\mathcal{G}$-action on $p_! D$, denoted by $p_! \alpha$.

Definition 4.2.4. Let $\varphi = (Y, p, f)$ be a pre-homomorphism from $\mathcal{G}$ to $\mathcal{H}$. If D is an $\mathcal{H}$-algebra, then $\varphi^* D$ is defined to be the $\mathcal{G}$-algebra $p_! f^* D$.

Example 4.2.5. Suppose E is a vector bundle, of real rank k, on a second-countable LCH space X. The frame bundle of E, denoted $\mathcal{F}(E)$, is a fibre bundle on X, whose fiber above a point $x \in X$ is the space of all ordered vector-space bases for the vector space E_x.

One can identify $\mathcal{F}(E)$ with a sub-fibre-bundle of $\bigoplus_{\ell=1}^{k} E$. Suppose $\beta = \{b_1, b_2, ..., b_k\}$ is an element of $\mathcal{F}(E)$, and $A = (a_{i,j})_{i,j} \in \mathrm{GL}(\mathbb{R}^k)$. The right-action of A on β is given by:

$$\beta.A := \left\{ \sum_{\ell=1}^{k} b_\ell a_{\ell,1}, \sum_{\ell=1}^{k} b_\ell a_{\ell,2}, ..., \sum_{\ell=1}^{k} b_\ell a_{\ell,k} \right\}.$$

If $\beta.A = \beta'$, then the j^{th} column of A returns the β-coordinates of $b'_j \in \beta'$. Therefore, A must be the matrix which changes β'-coordinate vectors to β-coordinate vectors (i.e., $A[x]_{\beta'} = [x]_\beta$), which we will denote by either $[I]_\beta^{\beta'}$ or $Q_\beta^{\beta'}$.

With these definitions, $r : \mathcal{F}(E) \to X$ forms a regular graph from X to $\mathrm{GL}_k(\mathbb{R})$ (see Proposition 4.1.5). Denote by $\varphi := (\mathcal{F}(E), r, f)$ the pre-homomorphism associated to $\mathcal{F}(E)$. In this case,

$$\mathcal{G}_Y = X_{\mathcal{F}(E)} = \{(\beta, x, \beta') \in \mathcal{F}(E) \times X \times \mathcal{F}(E) : \beta, \beta' \text{ bases for } E_x\}$$

Since $x \in X$ acts trivially on bases, the map $f : X_{\mathcal{F}(E)} \to \mathrm{GL}_k(\mathbb{R})$ sends (β, x, β') to $[I]_\beta^{\beta'}$. Consider $C_0(\mathbb{R}^k)$ as a $\mathrm{GL}_k(\mathbb{R})$-algebra with $\alpha_A(\phi) := \phi \circ A^{-1}$ for any $\phi \in C_0(\mathbb{R}^k)$ and $A \in \mathrm{GL}_k(\mathbb{R})$. Then

$$f^* C_0(\mathbb{R}^k) = C_0(\mathbb{R}^k) \otimes_{\mathbb{C}} C_0(\mathcal{F}(E)) \cong C_0(\mathbb{R}^k \times \mathcal{F}(E))$$

as an $X_{\mathcal{F}(E)}$-algebra. At β', the fiber is given by

$$(f^* C_0(\mathbb{R}^k))_{\beta'} = C_0(\mathbb{R}^k) \otimes (C_0(\mathcal{F}(E))/I_{\beta'}) \cong C_0(\mathbb{R}^k) \otimes \mathbb{C} \cong C_0(\mathbb{R}^k).$$

Under this identification, the action of $X_{\mathcal{F}(E)}$ on fibers is given by

$$(f^*\alpha)_{(\beta,x,\beta')} = \alpha_{f(\beta,x,\beta')} = \alpha_{[I]_\beta^{\beta'}}.$$

Now we need to push down $f^*C_0(\mathbb{R}^k)$ by $r : \mathcal{F}(E) \to X$:

$$\begin{aligned}\varphi^*C_0(\mathbb{R}^k) &:= r_! f^* C_0(\mathbb{R}^k)\\ &\cong r_! C_0(\mathcal{F}(E) \times \mathbb{R}^k)\\ &:= C_0(X) \cdot C_0(\mathcal{F}(E) \times \mathbb{R}^k)^{\mathcal{F}(E) \times_r \mathcal{F}(E)}\end{aligned}$$

Since $\mathcal{F}(E) \times_r \mathcal{F}(E)$ acts on $\mathcal{F}(E)$ by $(\beta, \beta').\beta' = \beta = \beta'.[I]^{\beta}_{\beta'}$, and on $\mathbb{R}^k$ by $(\beta, \beta').v = f(\beta, r(\beta'), \beta').v = [I]^{\beta'}_{\beta}.v$, it follows that $h \in (C_0(\mathcal{F}(E) \times \mathbb{R}^k))_b$ ($\neq C_b(\mathcal{F}(E) \times \mathbb{R}^k)$, see Definition 2.4.2) is $\mathcal{F}(E) \times_r \mathcal{F}(E)$-invariant if and only if $h(\beta', v) = h(\beta'.A^{-1}, A.v)$ for all $\beta' \in \mathcal{F}(E)$, $v \in \mathbb{R}^k$, and $A \in \mathrm{GL}_k(\mathbb{R})$. Consequently,

$$\begin{aligned}\varphi^*C_0(\mathbb{R}^k) &\cong C_0(X) \cdot C_0(\mathcal{F}(E) \times \mathbb{R}^k)^{\mathcal{F}(E) \times_r \mathcal{F}(E)}\\ &\cong C_0\left(\mathcal{F}(E) \underset{\mathrm{GL}_k(\mathbb{R})}{\times} \mathbb{R}^k\right)\\ &\cong C_0(E).\end{aligned}$$

A similar, but slightly easier computation yields

$$\varphi^*\mathbb{C} \cong C_0(\mathcal{F}(E)/\operatorname{GL}_k(\mathbb{R})) \cong C_0(X).$$

Additionally, if (P, η) is a Spin^c-structure for E, and the associated pre-homomophism is denoted $\varphi_P : X \to \mathrm{Spin}^c(k)$, then a similar computation demonstrates

$$\begin{aligned}\varphi_P^* C_0(\mathbb{R}^k) \cong C_0\left(P \underset{\mathrm{Spin}^c(k)}{\times} \mathbb{R}^k\right) \cong C_0(E);\\ \varphi_P^*\mathbb{C} \cong C_0(X)\end{aligned}$$

4.3 Pullback of Hilbert Modules

Let $\mathcal{G}$ and $\mathcal{H}$ be LCH groupoids, and $\varphi = (Y, p, f)$ a pre-homomorphism from $\mathcal{G}$ to $\mathcal{H}$. Given an $\mathcal{H}$-algebra, B, and an $\mathcal{H}$-equivariant Hilbert B-module $\mathcal{E}$, this subsection outlines the construction of $\varphi^*\mathcal{E}$ given in [11].

If $f:\mathcal{G}\to\mathcal{H}$ is a strict groupoid homomorphism, then $f\circ s_{\mathcal{G}}=s_{\mathcal{H}}\circ f$ and $f\circ r_{\mathcal{G}}=r_{\mathcal{H}}\circ f$. The pullback of $\mathcal{E}$ is defined to be $f^*\mathcal{E} := \mathcal{E} \underset{C_0(\mathcal{H}^{(0)}),ext}{\otimes} C_0(\mathcal{G}^{(0)})$, equipped with the continuous action by unitaries defined by the diagram below:

$$\begin{array}{ccc} s^*_{\mathcal{G}}(f^*\mathcal{E}) & \overset{f^*V}{\dashrightarrow} & r^*_{\mathcal{G}}(f^*(\mathcal{E})) \\ \cong\downarrow & & \downarrow\cong \\ f^*(s^*_{\mathcal{H}}\mathcal{E}) & \underset{V\otimes 1}{\longrightarrow} & f^*(r^*_{\mathcal{H}}\mathcal{E}) \end{array}$$

Definition 4.3.1. Let $p:Y\to\mathcal{G}^{(0)}$ be a continuous map, and let $\mathcal{E}$ be a $\mathcal{G}_Y$-equivariant Hilbert D-module. Then define the following:

$$\mathcal{E}_b := \{R\in\mathcal{L}(D,\mathcal{E}) | R^*R\in D_b\}, \text{ as a } \mathcal{G}_Y\text{-equivariant Hilbert } D_b\text{-module;}$$

$$\mathcal{E}^{Y\times_p Y} := \{\xi\in\mathcal{E}_b | \forall (y',y)\in Y\times_p Y, V_{(y',p(y),y)}(\xi_y)=\xi_{y'}\};$$

$$p_!\mathcal{E} := \mathcal{E}^{Y\times_p Y} p_! D.$$

The Hilbert $p_!D$-module, $p_!\mathcal{E}$, is equipped with an action of $\mathcal{G}$ through $(p_!V)_\gamma \cong V_{(y',\gamma,y)}$, where $y,y'\in Y$ are any points satisfying $p(y)=s(\gamma)$ and $p(y')=r(\gamma)$.

Example 4.3.2. Let $k=2r$ or $k=2r+1$. Consider the $\mathrm{Spin}^c(k)$-equivariant Hilbert $C_0(\mathbb{R}^k)$-module of functions $\mathcal{E}:=C_0(\mathbb{R}^k,\mathbb{C}^{2^r})$, where $\mathrm{Spin}^c(k)$ acts on $\mathbb{R}^k$ via the covering map $\phi:\mathrm{Spin}^c(k)\to SO(k)$, and on $\mathbb{C}^{2^r}$ via the composition $\psi:\mathrm{Spin}^c(k)\subseteq\mathbb{C}\ell(k)\cong M_{2^r}(\mathbb{C}^{(k-2r)+1})\to M_{2^r}(\mathbb{C})\cong\mathrm{End}(\mathbb{C}^{2^r})$, and on $\xi\in\mathcal{E}$ by $V_g(\xi):=\psi(g)(\xi\circ\phi(g^{-1}))$. Suppose $\pi_E:E\to X$ is a rank-k vector bundle with Spin^c-structure $(\pi:P\to X,\eta)$, and denote by $\varphi_P:=(P,\pi,f)$ the associated pre-homomophism from X to $\mathrm{Spin}^c(k)$. That is, $f:X_P\to\mathrm{Spin}^c(k)$ maps $(p_2,x,p_1)\in X_P$ to the unique element $h\in\mathrm{Spin}^c(k)$ satisfying $p_2.h=p_1$. Then

$$\begin{aligned} f^*\mathcal{E} &:= \mathcal{E}\otimes C_0(P) \\ &= C_0(\mathbb{R}^k,\mathbb{C}^{2^r})\otimes C_0(P) \\ &\cong C_0(P\times\mathbb{R}^k,\mathbb{C}^{2^r}) \end{aligned}$$

The X_P-action on $f^*\mathcal{E}$ is given by the commutative diagram:

$$\begin{array}{ccc} C_0(P \times \mathbb{R}^k, \mathbb{C}^{2^r})_{p_1} & \xrightarrow{(f^*V)_{(p_2,x,p_1)}} & C_0(P \times \mathbb{R}^k, \mathbb{C}^{2^r})_{p_2} \\ \cong \downarrow & & \downarrow \cong \\ C_0(\mathbb{R}^k, \mathbb{C}^{2^r}) & \xrightarrow[V_{f(p_2,x,p_1)}]{} & C_0(\mathbb{R}^k, \mathbb{C}^{2^r}) \end{array}$$

Similar to Example 4.2.5, $f^*C_0(\mathbb{R}^k) \cong C_0(P \times \mathbb{R}^k)$ as a $C_0(P)$-algebra, which acts on $f^*\mathcal{E}$ in the obvious way. Now we need to push down $\mathcal{E}$ via π:

$$\begin{aligned} (f^*\mathcal{E})_b &:= \{R \in \mathcal{L}(C_0(P \times \mathbb{R}^k), C_0(P \times \mathbb{R}^k, \mathbb{C}^{2^r})) : R^*R \in C_0(P \times \mathbb{R}^k)_b\} \\ &\cong \{R \in \mathcal{L}(C_0(\mathbb{R}^k) \otimes C_0(P), \mathcal{E} \otimes C_0(P)) : R^*R \in C_0(\mathbb{R}^k) \otimes C_b(P)\} \\ &\cong \{R \in \mathcal{L}(C_0(\mathbb{R}^k), \mathcal{E}) \otimes \mathcal{L}(C_0(P)) : R^*R \in C_0(\mathbb{R}^k) \otimes C_b(P)\} \\ &\cong \mathcal{E} \otimes C_b(P) \end{aligned}$$

Therefore,

$$\begin{aligned} (f^*\mathcal{E})^{P \times_\pi P} &:= \{\xi \in (f^*\mathcal{E})_b : (f^*V)_{(p_2,\pi(p_1),p_1)}\xi_{p_1} = \xi_{p_2}\} \\ &\cong \{\xi \in C_0(P \times \mathbb{R}^k, \mathbb{C}^{2^r})_b | \forall p_1, p_2 \in P,\ \forall v \in \mathbb{R}^k, (h = f(p_2, \pi(p_1), p_1)) \implies \\ &\implies \psi(h)(\xi(p_1, \phi(h^{-1})v)) = \xi(p_2, v)\} \end{aligned}$$

To simplify this further, notice that functions $\xi : P \times \mathbb{R}^k \to \mathbb{C}^{2^r}$ can be lifted uniquely to a continuous map $\widehat{\xi} : P \times \mathbb{R}^k \to P \times \mathbb{C}^{2^r}$, satisfying $\pi_1 \circ \widehat{\xi}(p, v) = p$, via $\widehat{\xi}(p, v) := (p, \xi(p, v))$. Let $g \in \mathrm{Spin}^c(k)$ act on $P \times \mathbb{R}^k$ via $g.(p, v) := (p.g^{-1}, g.v)$, and similarly for $P \times \mathbb{C}^{2^r}$. From above, the ξ, corresponding to elements of $(f^*\mathcal{E})^{P \times_\pi P}$, satisfy $\xi(p.g^{-1}, g.v) = g.\xi(p, v)$ for all $g \in \mathrm{Spin}^c(k)$. Therefore, the $\widehat{\xi}$, corresponding to elements of $(f^*\mathcal{E})^{P \times_\pi P}$, are $\mathrm{Spin}^c(k)$-equivariant; hence, we can view $\widehat{\xi}$ as a continuous map on the quotients:

$$\widetilde{\xi} : E \cong P \underset{\mathrm{Spin}^c(k)}{\times} \mathbb{R}^k \to P \underset{\mathrm{Spin}^c(k)}{\times} \mathbb{C}^{2^r} \cong \$,$$

which fibers over X. The continuous maps $\widetilde{\xi}$, corresponding to elements of $(f^*\mathcal{E})^{P \times_\pi P}$, can therefore be viewed as bounded sections of the bundle $\pi_E^*\$ \to E$. In conclusion,

$$\begin{aligned}\varphi^*\mathcal{E} &:= \pi_! f^*\mathcal{E}\\ &:= (f^*\mathcal{E})^{P\times_\pi P}\cdot \pi_! f^*(C_0(\mathbb{R}^k))\\ &\cong (f^*\mathcal{E})^{P\times_\pi P}\cdot C_0(E)\\ &\cong \Gamma_0(\pi_E^*\$),\end{aligned}$$

where Γ_0 is used to denote sections of a bundle vanishing at infinity.

4.4 Pullbacks in $KK_*^{\mathcal{G}}$

Given an $\mathcal{H}$-equivariant Kasparov A-B-bimodule $(\mathcal{E},\pi,F)\in E^{\mathcal{H}}(A,B)$ (2.6.2), and a generalized homomorphism $\varphi:\mathcal{G}\to\mathcal{H}$, Le Gall defines $\varphi^*(\mathcal{E},\pi,F):=(\varphi^*\mathcal{E},\varphi^*\pi,\varphi^*F)\in E^{\mathcal{G}}(\varphi^*A,\varphi^*B)$ ([11]). The main result in Le Gall's work is the following theorem:

Theorem 4.4.1 ([11] THEORÈME 7.2)**.** *Let $\mathcal{G}$ and $\mathcal{H}$ be topological groupoids that are second-countable and LCH. Suppose φ is a generalized groupoid homomorphism from $\mathcal{G}$ to $\mathcal{H}$.*

1. *For any $\mathcal{H}$-algebras A (separable), B, D and for any $x\in KK^{\mathcal{H}}(A,B)$ and $y\in KK^{\mathcal{H}}(B,D)$,*

$$\varphi^*x\ \widehat{\otimes}_{\varphi^*B}\ \varphi^*y=\varphi^*(x\widehat{\otimes}_B y)\in KK^{\mathcal{G}}(\varphi^*A,\varphi^*D).$$

2. *For any $\mathcal{H}$-algebra A, $\varphi^*(1_A)=1_{\varphi^*A}$.*

3. *If $\psi:\widetilde{\mathcal{G}}\to\mathcal{G}$ is another generalized groupoid homomorphism ($\widetilde{\mathcal{G}}$ second countable, LCH), then*
$$\psi^*\varphi^*(x)=(\varphi\circ\psi)^*(x)\in KK^{\widetilde{\mathcal{G}}}(\psi^*\varphi^*A,\psi^*\varphi^*B).$$

Le Gall uses this theorem to prove a non-equivariant Thom isomorphism for Spin^c bundles on compact spaces, X. We are going to combine this approach with the results of the next section to get an equivariant version that works with various types of support conditions.

5 The $\mathfrak{F}_f$ Functor

Emerson and Meyer refer to a particular type of forgetful functor throughout their work, but such a functor does not appear to be defined anywhere in the literature. However, this functor is evidently non-trivial, since the structure being forgotten is integral to other structures. Essentially, given a map $f : X \to Y$, we want to take objects fibered over X and combine all of the fibers together over $f^{-1}(y)$, for every $y \in Y$, to get objects fibered over Y. In this sense, it might be better to refer to this functor as a pushforward functor. More specifically, for a continuous $\mathcal{G}$-map between second-countable LCH spaces $f : X \to Y$, we use this section to define functors, all denoted $\mathfrak{F}_f$, which take $\mathcal{G} \ltimes X$-algebras into $\mathcal{G} \ltimes Y$-algebras, $\mathcal{G} \ltimes X$-equivariant Hilbert B-modules into $\mathcal{G} \ltimes Y$-equivariant Hilbert $\mathfrak{F}_f(B)$-modules, and classes $x \in KK^{\mathcal{G}\ltimes X}(A, B)$ into classes $\mathfrak{F}_f(x) \in KK^{\mathcal{G}\ltimes Y}(\mathfrak{F}_f(A), \mathfrak{F}_f(B))$. We also discuss the relationship between pullback and forgetful functors. As it is necessary to prove the Thom isomorphism, the primary objective of this subsection is to verify that the Kasparov product commutes with forgetful functors. Understanding this functor is additionally beneficial in formalizing the exact relationship between the two different Thom classes used in Lemma 4.0.2 of [3]. This relationship is explained in Subsection 6.2.

5.1 $\mathfrak{F}_f$ for Algebras

Definition 5.1.1. Given a continuous $\mathcal{G}$-map $f : X \to Y$, we define a strict groupoid homomorphism, $\mathcal{G} \ltimes f : \mathcal{G} \ltimes X \to \mathcal{G} \ltimes Y$, via

$$(\mathcal{G} \ltimes f)(\gamma, x) := (\gamma, f(x)).$$

Lemma 5.1.2. *Let A be a $C_0(X)$-algebra. Then $A \cong A \underset{C_0(X)}{\otimes} C_0(X)$.*

More generally, if A is a $\mathcal{G}$-algebra, then $A \cong A \underset{C_0(\mathcal{G}^{(0)})}{\otimes} C_0(\mathcal{G}^{(0)})$ as $\mathcal{G}$-algebras.

Proof. Let $X := \mathcal{G}^{(0)}$. The proof follows from the requirement that $\theta(C_0(X)) \cdot A$ is dense in A. Observe that the function $\psi : A \underset{C_0(X)}{\odot} C_0(X) \to \theta(C_0(X)) \cdot A$, defined by a lift of the bilinear map $(a, g) \mapsto \theta(g)a$, identifies the algebraic balanced tensor product $A \underset{C_0(X)}{\odot} C_0(X)$ isometrically with $\theta(C_0(X)) \cdot A$. A sequence $(x_n)_{n\in N} \in A \underset{C_0(X)}{\odot} C_0(X)$ is

Cauchy if and only if $(\psi(x_n))_{n\in\mathbb{N}}$ is Cauchy in $\theta(C_0(X)) \cdot A$. Mapping the limit of (x_n), in the completed tensor product, to the limit of $(\psi(x_n))$ in A, extends ψ to an isometric isomorphism $\widehat{\psi} : A \underset{C_0(X)}{\otimes} C_0(X) \to A$.

By 2.4.6, $(A \otimes_X C_0(X))_x \cong A_x \otimes \mathbb{C} \cong A_x$, so $\mathcal{G}$ acts on $A \otimes_X C_0(X)$ through this isomorphism. □

Definition 5.1.3. Let $f : X \to Y$ be a continuous function between LCH spaces. Let A be a $C_0(X)$-algebra. We define $\mathfrak{F}_f(A)$ to be the $C_0(Y)$-algebra with the following structure:

1. As a C*-algebra, $\mathfrak{F}_f(A) := A$.

2. Define $\mathfrak{F}_f(\theta) : C_0(Y) \to \mathcal{Z}(\mathcal{M}(\mathfrak{F}_f(A)))$ via the identification $A \cong A \underset{C_0(X)}{\otimes} C_0(X)$, and letting $h \in C_0(Y)$ act, on $A \otimes_X C_0(X)$, by multiplication with $1_{\mathcal{M}(A)} \otimes (h \circ f)$.

Notice that $\overline{\mathfrak{F}_f(\theta)(C_0(Y))(\mathfrak{F}_f(A))} = \mathfrak{F}_f(A)$, since $\overline{f^*(C_0(Y)) \cdot C_0(X)} = C_0(X)$. The definition we give here is one of many possible equivalent definitions. One can view $\mathcal{Z}(\mathcal{M}(A))$ as $C_b(\mathrm{Prim}(A))$ via the Dauns-Hofmann theorem. Denoting the isomorphism by $\psi : \mathcal{Z}(\mathcal{M}(A)) \to C_b(\mathrm{Prim}(A))$, the $*$-homomorphism $\psi \circ \theta : C_0(X) \to C_b(\mathrm{Prim}(A))$ can be viewed as pullback via a continuous function $\sigma_A : \mathrm{Prim}(A) \to X$; i.e., $\theta(g) = \psi^{-1}(g \circ \sigma_A)$ for any $g \in C_0(X)$. Then, for $h \in C_0(Y)$, $\mathfrak{F}_f(\theta)(h) = \psi^{-1}(h \circ f \circ \sigma_A)$. This could be used as an equivalent definition. Another equivalent approach would be using something like Proposition 2.50 of [20], to uniquely lift $\theta : C_0(X) \to M(A)$ to a compatible map $\overline{\theta} : C_b(X) \to \mathcal{M}(A)$. Then $\mathfrak{F}_f(\theta) = \overline{\theta} \circ f^* : C_0(Y) \to C_b(X) \to \mathcal{M}(A)$. We will not rely on these equivalent definitions in this thesis.

In the following proposition, we extend the definition of $\mathfrak{F}_f$ to a functor, from $C_0(X)$-algebras to $C_0(Y)$-algebras, by setting $\mathfrak{F}_f(\psi) :\equiv \psi$ for $C_0(X)$-morphisms ψ.

Proposition 5.1.4. *If $\psi : A \to B$ is a morphism of $C_0(X)$-algebras, then taking $\mathfrak{F}_f(\psi) : \mathfrak{F}_f(A) \to \mathfrak{F}_f(B)$ to be the same set-function as ψ, yields a $C_0(Y)$-algebra morphism.*

Proof. Let $(a_j) \subseteq A$ and $(h_j) \subseteq C_0(X)$. Since ψ is $C_0(X)$-linear, it has a well-defined counterpart:

$$\psi \otimes 1 : A \underset{X}{\otimes} C_0(X) \to B \underset{X}{\otimes} C_0(X).$$

Given any $g \in C_0(Y)$,

$$\begin{aligned}\mathfrak{F}_f(\theta_B)(g)\left((\psi\otimes 1)\left(\sum_j a_j \underset{X}{\otimes} h_j\right)\right) &= \mathfrak{F}_f(\theta_B)(g)\left(\sum_j \psi(a_j)\underset{X}{\otimes} h_j\right)\\ &= \sum_j \psi(a_j)\underset{X}{\otimes}(h_j\cdot(g\circ f))\\ &= (\psi\otimes 1)\left(\sum_j a_j\underset{X}{\otimes}(h_j\cdot(g\circ f))\right)\\ &= (\psi\otimes 1)\left(\mathfrak{F}_f(\theta_A)(g)\left(\sum_j a_j\underset{X}{\otimes} h_j\right)\right)\end{aligned}$$

□

The reason that the $\mathfrak{F}_f$-functors are not completely trivial to define is that, for an action, α, of $\mathcal{G} \ltimes X$ on A, $\alpha_{(\gamma,x)} : A_x \to A_{\gamma.x}$ could be a completely different map than $\alpha_{(\gamma,x')} : A_{x'} \to A_{\gamma.x'}$, even if $f(x) = f(x')$ in Y. The fiber of $\mathfrak{F}_f(A)$ over $y \in Y$ is $A|_{f^{-1}(y)}$. So how exactly does one define the isomorphism $\mathfrak{F}_f(\alpha)_{(\gamma,y)} : A|_{f^{-1}(y)} \to A|_{f^{-1}(\gamma.y)}$, when "$\gamma$" acts differently on A_x than $A_{x'}$? Perhaps, one could argue that $A|_{f^{-1}(y)} \subseteq \prod\limits_{f(x)=y} A_x$, and we could define $\mathfrak{F}_f(\alpha)_{(\gamma,y)}$ by applying $\alpha_{(\gamma,x)}$ component-wise:

$$A|_{f^{-1}(y)} \subseteq \prod_{f(x)=y} A_x \xrightarrow{\prod \alpha_{(\gamma,x)}} \prod_{f(x)=y} A_{\gamma.x} \supseteq A|_{f^{-1}(\gamma.y)}.$$

The most pressing issue is that the image of $A|_{f^{-1}(y)}$ under this component-wise map needs to lie entirely within $A|_{f^{-1}(\gamma.y)}$, otherwise this construction does not make sense. One can view $C_0(X)$-algebras as algebras of upper semicontinuous sections (see, for example, Lemma 2.1 of [13]). Continuity of the $\mathcal{G} \ltimes X$ action would be required for this map to be well-defined. Interestingly, discontinuous actions do not generally allow for the application of a forgetful functor. This interpretation of $C_0(X)$-algebras is only used here for motivation, and this thesis will continue to avoid using this interpretation explicitly. Instead, we will characterize the source and range pullbacks of $\mathfrak{F}_f(A)$ in a convenient way:

Proposition 5.1.5. *Let $\mathcal{G}$ be a locally compact Hausdorff groupoid, and let A be a $\mathcal{G} \ltimes X$-algebra. Denote the continuous $(\mathcal{G} \ltimes X)$-action by $\alpha : s_X^* A \to r_X^* A$, and consider a continuous $\mathcal{G}$-map, $f : X \to Y$, between locally compact Hausdorff $\mathcal{G}$-spaces. There exist*

$C_0(\mathcal{G} \ltimes Y)$*-algebra isomorphisms:*

$$s_Y^*(\mathfrak{F}_f(A)) \cong \mathfrak{F}_{\mathcal{G}\ltimes f}(s_X^* A)$$

$$r_Y^*(\mathfrak{F}_f(A)) \cong \mathfrak{F}_{\mathcal{G}\ltimes f}(r_X^* A).$$

Proof. Firstly, by a routine diagram chase, the following diagram (in the category of topological spaces) commutes:

$$\begin{array}{ccc}
X & = & X \\
\uparrow{\scriptstyle \pi_1} & & \uparrow{\scriptstyle s_X} \\
X \underset{f,s_Y}{\times} (\mathcal{G} \ltimes Y) & \xrightarrow{\sim} & \mathcal{G} \ltimes X \\
\downarrow{\scriptstyle \pi_2} & & \downarrow{\scriptstyle (\mathcal{G}\ltimes f)} \\
\mathcal{G} \ltimes Y & = & \mathcal{G} \ltimes Y
\end{array}$$

The middle homeomorphism in the above diagram yields an isomorphism of C^*-algebras:

$$\begin{aligned}
C_0(X) \underset{f^*,s_Y^*}{\otimes} C_0(\mathcal{G} \ltimes Y) &\cong C_0\left(X \underset{f,s_Y}{\times} (\mathcal{G} \ltimes Y)\right) \\
&\cong C_0(\mathcal{G} \ltimes X).
\end{aligned}$$

Furthermore, the $C_0(X)$ and $C_0(\mathcal{G} \ltimes Y)$ structural homomorphisms, for each C^*-algebra, are given by pullbacks of the corresponding vertical arrows in the commutative diagram above. Tracking the structure carefully, we verify the claim:

$$\begin{aligned}
s_Y^*(\mathfrak{F}_f(A)) &:= \left(\mathfrak{F}_f(A) \underset{\mathfrak{F}_f(\theta),s_Y^*}{\otimes} C_0(\mathcal{G} \ltimes Y), 1 \otimes id_{\mathcal{G}\ltimes Y}^*\right) \\
&:= \left(\left(A \underset{\theta,id_X^*}{\otimes} C_0(X)\right) \underset{1\otimes f^*,s_Y^*}{\otimes} C_0(\mathcal{G} \ltimes Y), 1 \otimes 1 \otimes id_{\mathcal{G}\ltimes Y}^*\right) \\
&\cong \left(A \underset{\theta,id_X^*\otimes 1}{\otimes} \left(C_0(X) \underset{f^*,s_Y^*}{\otimes} C_0(\mathcal{G} \ltimes Y)\right), 1 \otimes 1 \otimes id_{\mathcal{G}\ltimes Y}^*\right) \\
&\cong \left(A \underset{\theta,s_X^*}{\otimes} C_0(\mathcal{G} \ltimes X), 1 \otimes (\mathcal{G} \ltimes f)^*\right) \\
&\cong \mathfrak{F}_{(\mathcal{G}\ltimes f)}(s_X^* A)
\end{aligned}$$

The associativity used here can be verified carefully, as is done in Lemma 2.4 of [13]. Their lemma is proved for separable A and surjective f, but their argument does not rely on these hypotheses.

A similar argument, with the diagram below, gives the other equivalence.

$$\begin{array}{ccc}
X & = & X \\
{\scriptstyle \pi_1}\uparrow & & \uparrow{\scriptstyle r_X} \\
X \underset{f,r_Y}{\times} (\mathcal{G} \ltimes Y) & \xrightarrow{\sim} & \mathcal{G} \ltimes X \\
{\scriptstyle \pi_2}\downarrow & & \downarrow{\scriptstyle (\mathcal{G} \ltimes f)} \\
\mathcal{G} \ltimes Y & = & \mathcal{G} \ltimes Y
\end{array}$$

Note: the homeomorphism in the middle of this diagram maps $(x, (\gamma, y))$, satisfying $f(x) = \gamma.y$, to $(\gamma, \gamma^{-1}.x) \in \mathcal{G} \ltimes X$. □

Definition 5.1.6. Let A be a $(\mathcal{G} \ltimes X)$-algebra, with a continuous $\mathcal{G} \ltimes X$ action denoted by $\alpha : s_X^* A \to r_X^* A$. Given a continuous $\mathcal{G}$-map, $f : X \to Y$, between locally compact Hausdorff spaces, we define $\mathfrak{F}_f(A)$ to be the $(\mathcal{G} \ltimes Y)$-algebra, whose underlying $C_0(Y)$-structure is given by $\mathfrak{F}_f(\theta)$, and whose continuous action, denoted $\mathfrak{F}_f(\alpha)$, is determined by the diagram:

$$\begin{array}{ccc}
s_Y^*(\mathfrak{F}_f(A)) & \xrightarrow{\mathfrak{F}_f(\alpha)} & r_Y^*(\mathfrak{F}_f(A)) \\
\downarrow{\scriptstyle \cong} & & {\scriptstyle \cong}\downarrow \\
\mathfrak{F}_{(\mathcal{G} \ltimes f)}(s_X^* A) & \xrightarrow{\mathfrak{F}_{(\mathcal{G} \ltimes f)}(\alpha)} & \mathfrak{F}_{(\mathcal{G} \ltimes f)}(r_X^* A)
\end{array}$$

Note 1: the bottom horizontal arrow is the forgetful functor applied to the $C_0(\mathcal{G} \ltimes X)$-algebra homomorphism α (see Proposition 5.1.4). The vertical isomorphisms are the specific identifications used in the proof of Proposition 5.1.5.

Note 2: the definition of $\mathfrak{F}_f$ can be extended to a functor from the category of $\mathcal{G} \ltimes X$-algebras to the category of $\mathcal{G} \ltimes Y$-algebras by defining $\mathfrak{F}_f(\psi)$ to be the same set function as $\psi : A \to B$, for any $\mathcal{G} \ltimes X$-equivariant homomorphism. This definition is justified by the fact that $\mathfrak{F}_{\mathcal{G} \ltimes f}(\alpha) \equiv \alpha$, as a set-function, and ψ "commutes" with the action α.

Proposition 5.1.7. *Let $f : X \to Y$ be a continuous function between LCH spaces.*

1. *Let A be a $C_0(X)$-algebra. Then*

$$f^* \mathfrak{F}_f(A) \cong \left(A \underset{\theta, \pi_1^*}{\otimes} C_0\left(X \underset{Y}{\times} X \right), 1 \otimes \pi_2^* \right),$$

where $\pi_i^*: C_0(X) \to C_b(X \times_Y X)$ *is the pullback of functions via the* i^{th}*-coordinate projection* $\pi_i : X \times_Y X \to X$.

2. *Let* B *be a* $C_0(Y)$*-algebra. Then*

$$\mathfrak{F}_f(f^*B) \cong \left(B \underset{C_0(Y)}{\otimes} C_0(X), 1 \otimes f^*\right).$$

Proof. Applying definitions and carefully tracking the relevant actions,

$$\begin{aligned}
f^*\mathfrak{F}_f(A) &\cong \left(\left(A \underset{X}{\otimes} C_0(X)\right), 1 \otimes f^*\right) \underset{Y,f^*}{\otimes} (C_0(X), id_X^*) \\
&\cong \left((A \otimes_X C_0(X)) \underset{1\otimes f^*, f^*}{\otimes} C_0(X), 1 \otimes id_X^*\right) \\
&= \left(A \underset{\theta, id_X^* \otimes 1}{\otimes} \left(C_0(X) \underset{C_0(Y)}{\otimes} C_0(X)\right), 1 \otimes (1 \otimes id_X^*)\right) \\
&= \left(A \underset{\theta, \pi_1^*}{\otimes} C_0\left(X \underset{Y}{\times} X\right), 1 \otimes \pi_2^*\right)
\end{aligned}$$

In the other order,

$$\begin{aligned}
\mathfrak{F}_f(f^*(B)) &\cong \left((B \otimes_Y C_0(X)) \underset{1\otimes id_X^*, id_X^*}{\otimes} C_0(X), (1 \otimes 1) \otimes f^*\right) \\
&= \left(B \underset{\theta_B, f^* \otimes 1}{\otimes} (C_0(X) \otimes_X C_0(X)), 1 \otimes (1 \otimes f^*)\right) \\
&= \left(B \underset{\theta_B, \pi_1^* f^*}{\otimes} C_0(X \times_X X), 1 \otimes \pi_2^* f^*\right) \\
&= \left(B \underset{\theta_B, f^*}{\otimes} C_0(X), 1 \otimes f^*\right)
\end{aligned}$$

□

5.2 $\mathfrak{F}_f$ for Hilbert Modules

Let $\mathcal{E}$ be a (right) Hilbert B-module, where B is a $C_0(X)$-algebra. The fibers of $\mathcal{E}$ are given by $\mathcal{E}_x := \mathcal{E} \underset{B}{\otimes} B_x$, but there isn't really any additional $C_0(X)$-structure for $\mathcal{E}$ that needs to be specified. Since the underlying C^*-algebra of B and $\mathfrak{F}_f(B)$ are the same, there really isn't much difference between $\mathcal{E}$ as a Hilbert B-module vs. $\mathcal{E}$ as a Hilbert $\mathfrak{F}_f(B)$-module, except that the relevant fibers, after forgetting f, are $\mathcal{E}_y := \mathcal{E} \underset{\mathfrak{F}_f(B)}{\otimes} \mathfrak{F}_f(B)_y \cong \mathcal{E} \underset{B}{\otimes} B|_{f^{-1}(y)}$. Consequently, for any adjointable operator, ϕ, between Hilbert B-modules, $\mathfrak{F}_f(\phi) := \phi$ is also the appropriate definition.

If $\mathcal{E}$ has the additional structure of a continuous $\mathcal{G} \ltimes X$-action by unitaries, $V : s_X^*\mathcal{E} \to r_X^*\mathcal{E}$, we want to define $\mathfrak{F}_f(\mathcal{E}) := \mathcal{E}$ and $\mathfrak{F}_f(V) : s_Y^*\mathfrak{F}_f(\mathcal{E}) \to r_Y^*\mathfrak{F}_f(\mathcal{E})$ in a way similar to Definition 5.1.6. If $\mathcal{E}$ is $\mathcal{G} \ltimes X$-equivariant, then from section 4.2 of [11], one can view continuous actions of $\mathcal{E}$, in an equivalent way, as continuous actions of $\mathcal{K}(\mathcal{E} \oplus B)$ as a $C_0(X)$-algebra. Consequently, we could define $\mathfrak{F}_f(V) : s_Y^*\mathfrak{F}_f(\mathcal{E}) \to r_Y^*\mathfrak{F}_f(\mathcal{E})$ via $\mathfrak{F}_f(\alpha) : s_Y^*\mathfrak{F}_f(\mathcal{K}(\mathcal{E} \oplus B)) \to r_Y^*\mathfrak{F}_f(\mathcal{K}(\mathcal{E} \oplus B))$ for those Hilbert Modules. Alternatively, we could try to define $\mathfrak{F}_f(V)$ directly. Unsurprisingly, these approaches are equivalent.

Proposition 5.2.1. *Let $\mathcal{E}$ be a right Hilbert B-module, where B is a $\mathcal{G} \ltimes X$-algebra. If $f : X \to Y$ is a $\mathcal{G}$-map between LCH, $\mathcal{G}$-spaces, then*

$$s_Y^*\mathfrak{F}_f(\mathcal{E}) \cong \mathfrak{F}_{\mathcal{G}\ltimes f}(s_X^*(\mathcal{E}));$$

$$r_Y^*\mathfrak{F}_f(\mathcal{E}) \cong \mathfrak{F}_{\mathcal{G}\ltimes f}(r_X^*(\mathcal{E})).$$

Proof. By definition of pullback, and applying Proposition 5.1.5, we have that

$$\begin{aligned} s_Y^*\mathfrak{F}_f(\mathcal{E}) &= \mathfrak{F}_f(\mathcal{E}) \underset{\mathfrak{F}_f(A)}{\otimes} s_Y^*(\mathfrak{F}_f(A)) \\ &\cong \mathfrak{F}_f(\mathcal{E}) \underset{\mathfrak{F}_f(A)}{\otimes} \mathfrak{F}_{\mathcal{G}\ltimes f}(s_X^*A) \\ &= \mathcal{E} \underset{A}{\otimes} \mathfrak{F}_{\mathcal{G}\ltimes f}(s_X^*A) \\ &\cong \mathfrak{F}_{\mathcal{G}\ltimes f}(s_X^*\mathcal{E}) \end{aligned}$$

The verification for range map pullbacks is identical. □

Intuitively, propositions 5.1.5 and 5.2.1 are saying that pulling back from $C_0(Y)$ to $C_0(\mathcal{G} \ltimes Y)$ is the same operation as pulling back from $C_0(X)$ to $C_0(\mathcal{G} \ltimes X)$.

In the same way that we defined $\mathfrak{F}_f(\alpha)$ for $\mathcal{G} \ltimes X$ actions α on $C_0(X)$-algebras, we can now define $\mathfrak{F}_f(V)$ for Hilbert B-modules:

Definition 5.2.2. Let $\mathcal{E}$ be a right Hilbert B-module, where B is a $\mathcal{G} \ltimes X$-algebra. If $f : X \to Y$ is a $\mathcal{G}$-map between LCH, $\mathcal{G}$-spaces, then we define $\mathfrak{F}_f(V)$ to be the map given by the diagram below:

$$\begin{array}{ccc} s_Y^*(\mathfrak{F}_f(\mathcal{E})) & \xrightarrow{\mathfrak{F}_f(V)} & r_Y^*(\mathfrak{F}_f(\mathcal{E})) \\ \downarrow \cong & & \cong \downarrow \\ \mathfrak{F}_{(\mathcal{G}\ltimes f)}(s_X^*\mathcal{E}) & \xrightarrow[V]{\mathfrak{F}_{(\mathcal{G}\ltimes f)}(V)} & \mathfrak{F}_{(\mathcal{G}\ltimes f)}(r_X^*\mathcal{E}) \end{array}$$

The bottom map is $\mathfrak{F}_{\mathcal{G}\ltimes f}(V)$, treating V as an adjointable operator, which is the same set function as V. The vertical isomorphisms are the specific isomorphisms described in Proposition 5.2.1.

Proposition 5.2.3. *Let B be a $\mathcal{G} \ltimes X$-algebra. If $(\mathcal{E}, V)$ is a $\mathcal{G} \ltimes X$-equivariant Hilbert B-module, then $(\mathfrak{F}_f(\mathcal{E}), \mathfrak{F}_f(V))$ is a $\mathcal{G} \ltimes Y$-equivariant Hilbert $\mathfrak{F}_f(B)$-module.*

Proof. Let $\xi \in s_Y^*(\mathfrak{F}_f(\mathcal{E}))$ correspond to $\xi' \in \mathfrak{F}_{\mathcal{G}\ltimes f}(s_X^*\mathcal{E})$ as in Proposition 5.2.1, and let $b \in s_Y^*(\mathfrak{F}_f(B))$ correspond to $b' \in \mathfrak{F}_{\mathcal{G}\ltimes f}(s_X^* B)$ as in Proposition 5.1.5. Then $\mathfrak{F}_f(V)(\xi.b) \in r_Y^*(\mathfrak{F}_f(\mathcal{E}))$ corresponds to $V(\xi'.b') = V(\xi').\alpha(b') \in \mathfrak{F}_{\mathcal{G}\ltimes f}(r_X^*\mathcal{E})$, which corresponds to $\mathfrak{F}_f(V)(\xi).\mathfrak{F}_f(\alpha)(b)$.

□

Proposition 5.2.4. *If α is an action on $\mathcal{K}(\mathcal{E}\oplus B)$ that makes the first diagram commute,*

$$\begin{array}{ccc} s_X^*\mathcal{E} & \xrightarrow{V} & r_X^*\mathcal{E} \\ \big\downarrow \iota_{s_X^*\mathcal{E}} & & \big\downarrow \iota_{r_X^*\mathcal{E}} \\ s_X^*\mathcal{K}(\mathcal{E}\oplus B) & \xrightarrow[\alpha]{} & r_X^*\mathcal{K}(\mathcal{E}\oplus B) \end{array}$$

then $\mathfrak{F}_f(\alpha)$ is an action on $\mathfrak{F}_f(\mathcal{K}(\mathcal{E}\oplus B))$ that makes the following diagram commute:

$$\begin{array}{ccc} s_Y^*\mathfrak{F}_f(\mathcal{E}) & \xrightarrow{\mathfrak{F}_f(V)} & r_Y^*\mathfrak{F}_f(\mathcal{E}) \\ \big\downarrow \iota_{s_Y^*\mathfrak{F}_f(\mathcal{E})} & & \big\downarrow \iota_{r_Y^*\mathfrak{F}_f(\mathcal{E})} \\ s_Y^*\mathfrak{F}_f(\mathcal{K}(\mathcal{E}\oplus B)) & \xrightarrow[\mathfrak{F}_f(\alpha)]{} & r_Y^*\mathfrak{F}_f(\mathcal{K}(\mathcal{E}\oplus B)) \end{array}$$

Proof. Apply the $\mathfrak{F}_{\mathcal{G}\ltimes f}$ functor to the first diagram, and use propositions 5.1.5 and 5.2.1. Note that the map $\iota_{s_Y^*\mathfrak{F}_f(\mathcal{E})}$ makes sense because $\mathfrak{F}_f(\mathcal{K}(\mathcal{E}\oplus B)) = \mathcal{K}(\mathfrak{F}_f(\mathcal{E})\oplus\mathfrak{F}_f(B))$. □

5.3 $\mathfrak{F}_f$ for KK-classes

Recall that elements $x \in \mathbb{E}^{\mathcal{G}\ltimes X}(A, B)$ are triples $x = (\mathcal{E}, \varphi, F)$, where $\mathcal{E}$ is a $\mathbb{Z}/2\mathbb{Z}$-graded $\mathcal{G}\ltimes X$-equivariant right Hilbert B-module, φ is a $\mathcal{G}\ltimes X$ equivariant (even-degree) representation of A on $\mathcal{L}(\mathcal{E})$, and F is an (odd-degree) adjointable operator on $\mathcal{E}$ satisfying the four conditions of Definition 2.6.2. We define $\mathfrak{F}_f(x) := (\mathfrak{F}_f(\mathcal{E}), \mathfrak{F}_f(\varphi), \mathfrak{F}_f(F))$.

Lemma 5.3.1. *Let A and B be $\mathcal{G} \ltimes X$-algebras, where $\mathcal{G}$ is a LCH groupoid acting on LCH spaces X and Y. If $f : X \to Y$ is a $\mathcal{G}$-map, and $x \in \mathbb{E}^{\mathcal{G}\ltimes X}(A, B)$, then $\mathfrak{F}_f(x) \in \mathbb{E}^{\mathcal{G}\ltimes Y}(\mathfrak{F}_f(A), \mathfrak{F}_f(B))$.*

Proof. It suffices to show that $\mathfrak{F}_f(F)$ satisfies the usual axioms. The Kasparov axioms require that, for all $a \in A$, $[F, \varphi(a)], (F^2-I)\varphi(a), (F-F^*)\varphi(a) \in \mathcal{K}(\mathcal{E})$. Since $\mathcal{K}(\mathfrak{F}_f(\mathcal{E})) = \mathcal{K}(\mathcal{E})$, and $\mathfrak{F}_f(F)$ and $\mathfrak{F}_f(\varphi)$ are the same set functions as F and φ (respectively), these three conditions are automatic. The only condition that remains to be verified is the almost $\mathcal{G} \ltimes Y$-equivariance of $\mathfrak{F}_f(F)$. The almost equivariance condition for $\mathbb{E}^{\mathcal{G}\ltimes X}(A, B)$ is the following: for any $a \in r_X^* A$,

$$r_X^*(\varphi)(a)(V s_X^*(F)V^* - r_X^* F) \in r_X^*\mathcal{K}(\mathcal{E}).$$

To make identifications more explicit, we will use the following notation:

$$\begin{aligned} \psi_r : r_Y^*\mathfrak{F}_f(A) &\xrightarrow{\sim} \mathfrak{F}_{\mathcal{G}\ltimes f}(r_X^* A), \qquad \psi_s : s_Y^*\mathfrak{F}_f(A) \xrightarrow{\sim} \mathfrak{F}_{\mathcal{G}\ltimes f}(s_X^* A) \\ \widehat{\psi_r} : r_Y^*\mathfrak{F}_f(\mathcal{E}) &\xrightarrow{\sim} \mathfrak{F}_{\mathcal{G}\ltimes f}(r_X^*\mathcal{E}), \qquad \widehat{\psi_s} : s_Y^*\mathfrak{F}_f(\mathcal{E}) \xrightarrow{\sim} \mathfrak{F}_{\mathcal{G}\ltimes f}(s_X^*\mathcal{E}) \end{aligned}$$

We will also use $\phi_V := V$ when we want to treat V as a module homomorphism rather than an action. In this notation, $\mathfrak{F}_f(V) = \widehat{\psi_r}^{-1} \circ \mathfrak{F}_{\mathcal{G}\ltimes f}(\phi_V) \circ \widehat{\psi_s}$. Additionally, $s_Y^*(\mathfrak{F}_f(F)) = \widehat{\psi_s}^{-1} \circ \mathfrak{F}_{\mathcal{G}\ltimes f}(s_X^* F) \circ \widehat{\psi_s}$. Therefore,

$$\begin{aligned} &\mathfrak{F}_f(V) s_Y^*\mathfrak{F}_f(F)\mathfrak{F}_f(V^*) = \\ &= \left(\widehat{\psi_r}^{-1} \circ \mathfrak{F}_{\mathcal{G}\ltimes f}(\phi_V) \circ \widehat{\psi_s}\right) \circ \left(\widehat{\psi_s}^{-1} \circ \mathfrak{F}_{\mathcal{G}\ltimes f}(s_X^* F) \circ \widehat{\psi_s}\right) \circ \left(\widehat{\psi_s}^{-1} \circ \mathfrak{F}_{\mathcal{G}\ltimes f}(\phi_{V^*}) \circ \widehat{\psi_r}\right) \\ &= \widehat{\psi_r}^{-1} \circ \mathfrak{F}_{\mathcal{G}\ltimes f}\left(\phi_V \circ s_X^* F \circ \phi_{V^*}\right) \circ \widehat{\psi_r} \\ &= \widehat{\psi_r}^{-1} \circ \mathfrak{F}_{\mathcal{G}\ltimes f}\left(V(s_X^* F)V^*\right) \circ \widehat{\psi_r} \end{aligned}$$

Furthermore, for any $a \in r_Y^*\mathfrak{F}_f(A)$,

$$(r_Y^*\mathfrak{F}_f(\varphi))(a) = \widehat{\psi_r}^{-1} \circ \mathfrak{F}_{\mathcal{G}\ltimes f}\Big(r_X^*\varphi(\psi_r(a))\Big) \circ \widehat{\psi_r}$$

as an adjointable operator on $r_Y^*\mathfrak{F}_f(\mathcal{E})$. Applying the almost $\mathcal{G} \ltimes X$ equivariance, and combining the identifications made so far, we conclude the proof by verifying the almost $\mathcal{G} \ltimes Y$ equivariance of $\mathfrak{F}_f(F)$ below:

$$\begin{aligned}
&r_Y^*\mathfrak{F}_f(\varphi)(a)\Big(\mathfrak{F}_f(V)s_Y^*\mathfrak{F}_f(F)\mathfrak{F}_f(V^*) - r_Y^*\mathfrak{F}_f(F)\Big) = \\
&= \widehat{\psi_r}^{-1} \circ \mathfrak{F}_{\mathcal{G}\ltimes f}\Big(r_X^*\varphi(\psi_r(a))\Big) \circ \widehat{\psi_r}\left(\widehat{\psi_r}^{-1} \circ \mathfrak{F}_{\mathcal{G}\ltimes f}\Big(V(s_X^*F)V^* - r_X^*F\Big) \circ \widehat{\psi_r}\right) \\
&= \widehat{\psi_r}^{-1} \circ \mathfrak{F}_{\mathcal{G}\ltimes f}\left(\Big(r_X^*\varphi(\psi_r(a))\Big)\Big(V(s_X^*F)V^* - r_X^*F\Big)\right) \circ \widehat{\psi_r} \\
&\in \mathcal{K}(r_Y^*\mathfrak{F}_f(\mathcal{E})) \cong r_Y^*\mathfrak{F}_f(\mathcal{K}(\mathcal{E}))
\end{aligned}$$

□

Theorem 5.3.2. *Let A, B, and D be $\mathcal{G} \ltimes X$-algebras, and suppose $f : X \to Y$ is a $\mathcal{G}$-map between LCH $\mathcal{G}$-spaces. Then if $x_1 \in KK^{\mathcal{G}\ltimes X}(A, B)$ and $x_2 \in KK^{\mathcal{G}\ltimes X}(B, D)$, admit a Kasparov product, then so do $\mathfrak{F}_f(x_1)$ and $\mathfrak{F}_f(x_2)$, and*

$$\mathfrak{F}_f(x_1 \underset{B}{\widehat{\otimes}} x_2) = \mathfrak{F}_f(x_1) \underset{\mathfrak{F}_f(B)}{\widehat{\otimes}} \mathfrak{F}_f(x_2).$$

Proof. If $x_j = [(\mathcal{E}_j, \varphi_j, F_j)]$, and $x_1 \underset{B}{\widehat{\otimes}} x_2 = [(\mathcal{E}_{1,2}, \varphi_1 \otimes 1, F_{1,2})]$. Recall that $\mathcal{E}_{1,2} := \mathcal{E}_1 \otimes_{B,\varphi_2} \mathcal{E}_2$ and $F_{1,2} \in F_1 \# F_2$, i.e., $F_{1,2}$ is an F_2 connexion for $\mathcal{E}_{1,2}$ and for any $a \in A$, $\varphi_1(a)[F_1 \otimes 1, F_{1,2}]\varphi(a^*) \geqslant 0$ modulo $\mathcal{K}(\mathcal{E}_{1,2})$. From Lemma 5.3.1, $(\mathfrak{F}_f(\mathcal{E}_{1,2}), \mathfrak{F}_f(\varphi_1 \otimes 1), \mathfrak{F}_f(F_{1,2})) \in \mathbb{E}^{\mathcal{G}\ltimes Y}(A, D)$, and since being a connexion and satisfying the positivity requirement have nothing to do with the $C_0(X)$-structure, $\mathfrak{F}_f(F_{1,2}) \in \mathfrak{F}_f(F_1)\#\mathfrak{F}_f(F_2)$. □

6 The $\mathcal{G}$-Equivariant Thom Isomorphism

6.1 Thom Isomorphism

The following theorem constructs an invertible element of $KK_k^{\mathcal{G}\ltimes X}(C_0(X), C_0(E))$ for a Spin^c-$\mathcal{G}$-bundle $E \to X$, and proves that this class is the same as the one introduced in Definition 3.2.1. For convenience of notation, let β_n denote the conjugate of the class $\widetilde{\beta}_n$ defined in subsection 3.5, and let α_n be the conjugate of $\widetilde{\alpha}_n$ of definition 3.5.4.

Theorem 6.1.1. *Let $\mathcal{G}$ be a second-countable LCH groupoid, and let $\pi : E \to X$ be a (real) rank-k Spin^c-$\mathcal{G}$-bundle on a second-countable, LCH $\mathcal{G}$-space X. Then any Spin^c-data (P, η) for E forms a generalized groupoid homomorphism φ_P, from $\mathcal{G} \ltimes X$ to $\mathrm{Spin}^c(k)$, satisfying:*

1. $\varphi_P^* C_0(\mathbb{R}^k) \cong C_0(E)$,
2. $\varphi_P^* \mathbb{C} \cong C_0(X)$, *and*
3. $\varphi_P^* \beta_k \in KK_k^{\mathcal{G}\ltimes X}(C_0(X), C_0(E))$ *and* $\varphi_P^* \alpha_k \in KK_k^{\mathcal{G}\ltimes X}(C_0(E), C_0(X))$ *are inverse elements in* $KK^{\mathcal{G}\ltimes X}$ *(see Section 3.3).*
4. $\varphi_P^* \beta_k = \tau_E$ *(see Definition 3.2.1).*

Proof. The principal $\mathrm{Spin}^c(k)$-bundle P is equipped with a left $\mathcal{G}$ action commuting with the projection $P \to X$; hence, induces a left action of $\mathcal{G} \ltimes X$ commuting with the right action of $\mathrm{Spin}^c(k)$ on P. Therefore, P determines a graph from $\mathcal{G} \ltimes X$ to $\mathrm{Spin}^c(k)$. Since P is locally trivializable, Proposition 4.1.5 guarantees that the projection map $\pi : P \to X$ is open; therefore, P determines a regular graph from $\mathcal{G} \ltimes X$ to $\mathrm{Spin}^c(k)$. We define φ_P to be the corresponding generalized groupoid homomorphism. Statements 1 and 2 follow from Example 4.2.5. Statement 3 is a consequence of Theorems 3.3.1 and 4.4.1, since

$$\begin{aligned} \varphi_P^* \beta_k \underset{C_0(E)}{\widehat{\otimes}} \varphi_P^* \alpha_k &= \varphi_P^*(\beta_k \underset{C_0(\mathbb{R}^k)}{\widehat{\otimes}} \alpha_k) \\ &= \varphi_P^*(1_{\mathbb{C}}) \\ &= 1_{C_0(X)} \in KK^{\mathcal{G}\ltimes X}(C_0(X), C_0(X)), \end{aligned}$$

and similarly, $\varphi_P^*\alpha_k \underset{C_0(X)}{\widehat{\otimes}} \varphi_P^*\beta_k = 1_{C_0(E)} \in KK^{\mathcal{G}\ltimes X}(C_0(E), C_0(E))$. Statement 4 is a consequence of Subsection 3.5 together with Example 4.3.2. □

We now restate and prove the version of the Thom isomorphism from the introduction:

Theorem 1.0.1 *Assume all topologies are second-countable and locally-compact Hausdorff. Let $\mathcal{G}$ be a topological groupoid, and let X be a $\mathcal{G}$-space. Suppose $\pi : E \to X$ is a continuous* Spin^c*-$\mathcal{G}$-bundle on X of rank k over $\mathbb{R}$.*

Then the Thom class of E, $\tau \in RK^k_{\mathcal{G},X}(E)$, satisfies: for any $\mathcal{G}$-space Y, and continuous $\mathcal{G}$-map $f : X \to Y$, the map

$$(\cdot) \underset{C_0(X)}{\widehat{\otimes}} \mathfrak{F}_f(\tau) : RK^j_{\mathcal{G},Y}(X) \to RK^{j+k}_{\mathcal{G},Y}(E)$$

is an isomorphism.

Proof of Theorem 1.0.1. From Theorem 6.1.1, we have that $\tau = \varphi_P^*\beta_k \in RK^k_{\mathcal{G},X}(E)$ is invertible. From Theorem 5.3.2, $\mathfrak{F}_f(\tau)$ and $\mathfrak{F}_f(\varphi_P^*\alpha_k)$ are inverses in $KK^{\mathcal{G}\ltimes Y}$; hence the map (from $RK^*_{\mathcal{G},Y}(X)$ to $RK^{*+k}_{\mathcal{G},Y}(E)$):

$$\xi \mapsto \xi \underset{C_0(X)}{\widehat{\otimes}} \mathfrak{F}_f(\tau)$$

is an isomorphism.

6.2 A Rotation between two Thom Classes

In this subsection, we contextualize the rotation trick, used by Erik van Erp and Paul Baum in [3], to equivariant representable K-theory.

Let $\mathcal{G}$ be a second-countable LCH groupoid. Let X be a $\mathcal{G}$-manifold equipped with a $\mathcal{G}$-invariant Riemannnian metric. Then TTX can be equipped with the structure of a Spin^c-$\mathcal{G}$-bundle in two different ways. Let $\pi : TX \to X$ and $\pi_0 : TTX \to TX$ be the usual projections, and let $\pi_1 := d\pi : TTX \to TX$. Using the invariant Riemannian metric, we can equip $\pi_1 : TTX \to TX$ with the structure of a Spin^c-$\mathcal{G}$-bundle via an equivariant almost complex structure. Denote the $C_0(TX)$-algebra $(C_0(TTX), \pi_i^*)$ by B_i. Since $\pi \circ \pi_0 = \pi \circ \pi_1$, it follows that $\mathfrak{F}_{\pi_0}(B_0) = \mathfrak{F}_{\pi_1}(B_1)$. Let $\tau_i \in KK^{\mathcal{G}\ltimes TX}(C_0(TX), B_i)$ be the Thom class for each bundle.

To prove that π_0 and π_1 induce the same Thom isomorphism (in the non-equivariant setting), Baum and van Erp construct a diffeomorphism, $\rho : TTX \to TTX$, in section 4 of [3], satisfying $\pi_0 \circ \rho = \pi_1$, and then they construct an explicit homotopy from the identity map to ρ, denoted by ρ_t. However, $\pi_0 \circ \rho_t \neq \pi_1$, so we cannot form a homotopy in $KK^{\mathcal{G} \ltimes TX}$ between τ_0 and τ_1 via ρ_t^*. However, $\pi \circ \pi_0 \circ \rho_t = \pi \circ \pi_1$, so we can deduce the following result:

Proposition 6.2.1. *With the notation and assumptions of the preceding paragraph,*

$$\mathfrak{F}_{\pi_0}(\tau_0) = \mathfrak{F}_{\pi_1}(\tau_1) \in KK^{\mathcal{G} \ltimes X}(C_0(TX), C_0(TTX))$$

Proof. From Theorem 6.1.1, 4., the Thom class τ_i can be represented as an element in $VK^0_{\mathcal{G},TX}(TTX)$. The representative of this class is exactly the same as described in [3]. The proof of Lemma 4.0.2. in [3] will automatically pass to our equivariant setting, since the homotopy preserves fibers over X, and all $\mathcal{G}$ actions on these bundles are induced by differentiating the $\mathcal{G}$-action on X. This argument contextualised to our case will explicitly demonstrate that our specified representative of $\mathfrak{F}_{\pi_0}(\tau_0)$ is homotopic to our specified representative of $\mathfrak{F}_{\pi_1}(\tau_1)$ via $e^{i\pi t/2}\rho_t^*$. □

This type of rotation was used in [3] as a key ingredient for a proof of the K-theoretic index theorem for elliptic pseudodifferential operators. The version stated here is one step in generalizing Baum and van Erp's argument to a groupoid-equivariant setting.

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