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Periodic Auxetics: Structure and Design

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Abstract

Materials science has adopted the term of auxetic behavior for structural deformations where stretching in some direction entails lateral widening, rather than lateral shrinking. Most studies, in the last three decades, have explored repetitive or cellular structures and used the notion of negative Poisson's ratio as the hallmark of auxetic behavior. However, no general auxetic principle has been established from this perspective. In the present paper, we show that a purely geometric approach to periodic auxetics is apt to identify essential characteristics of frameworks with auxetic deformations and can generate a systematic and endless series of periodic auxetic designs. The critical features refer to convexity properties expressed through families of homothetic ellipsoids.

Keywords: auxetic behavior, periodic framework, auxetic design, homothetic ellipsoids.

Introduction

A flexible structure exhibits auxetic behavior when stretching in a given direction involves widening in orthogonal directions. In elasticity theory, this type of behavior amounts to *negative Poisson's ratios* (for orthogonal directions paired with the given tensile direction). Materials and structures with negative Poisson's ratios have attracted increased attention after the 1987 publication of Lakes' results on foams [23]. The term *auxetic* (from *increase,*

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growth) gained currency after [16]. The extensive literature generated in three decades of auxetic studies is reviewed in several recent surveys e.g. [1, 14, 17, 21, 22, 24, 26, 30, 32].

Although the role of the underlying geometry is widely recognized, the approach via Poisson's ratios involves modeling assumptions about and dependence on physical properties of the material under consideration. The catalog of auxetic structures found in this literature is rather confined and no general principles of auxetic design have been formulated.

The main purpose of the present paper is to show that a *purely geometric* approach to periodic auxetics, introduced by the authors in [7], leads to new insights and solves two fundamental problems on structure and design, namely, what geometry is required for auxetic behavior and how to construct periodic frameworks with auxetic deformations.

1 Distinctive constituents of the geometric theory

Our theory applies to *periodic bar-and-joint frameworks*. The dimensions most important for applications are two and three, but the theory works in arbitrary dimension d. Periodic frameworks represent the next of kin to finite linkages.

The foundations of a deformation theory for periodic frameworks were presented in [3], with rigidity aspects developed in [4] and additional perspectives discussed in [5, 10]. In [8], we formulated and proved a periodic analog of Maxwell's theorem on liftings and stresses and used it to obtain a complete characterization of *expansive behavior* in dimension two through the notion of periodic pseudotriangulation [6, 8]. A pseudotriangle is a simple planar polygon with exactly three vertices on its convex hull. *Expansive behavior*, when all distances between pairs of vertices increase or stay the same, clearly satisfies the 'growth' requirements of *auxetic behavior*, that is, *expansive implies auxetic*.

Periodic auxetics. The theory introduced in [7] looks *directly* at deformation trajectories which exhibit auxeticity. The 'lateral widening upon stretching', or rather, in the reverse sense, 'lateral shrinking upon compression' is adequately captured in the notion of *contraction* (norm at most one) for the

linear transformation which takes the periodicity lattice at one moment to the periodicity lattice at a subsequent moment.

This characterization is completely geometric, valid in arbitrary dimension d and focused on the evolution of the periodicity lattice of the framework. After a choice of independent generators, the deformation of the periodicity lattice gives a curve in the space of symmetric $d \times d$ matrices, through the variation of the Gram matrix of these generators. In this language, a one-parameter deformation is auxetic when all velocity vectors along the curve belong to the *positive semidefinite cone* and *strictly* auxetic, when velocity vectors are in the *positive definite cone.*

Remarks. In contrast to the conventional route via Poisson's ratios, this geometric approach is formulated directly and exclusively in terms of structure and periodicity. The mathematical model of periodic frameworks is in agreement with the rigid unit mode theory used for crystalline materials [13]. The bar-and-joint mechanical model is frequently adopted in the literature [15, 18, 28] and is gaining critical importance in *structural design* due to recent advances in digital manufacturing and the quest for mechanical metamaterials [2, 31].

Besides being completely rigorous, the geometric approach brings the following advantages:

(i) a linear *infinitesimal theory* which allows detection of auxetic capabilities via semidefinite programming [7] or other algorithms [9];

(ii) for several degrees of freedom, the notion and description of the *infinitesimal auxetic cone*, which is a *spectrahedral cone* (i.e. a section of the positive semidefinite cone by a linear subspace) or a linear preimage of a spectrahedral cone [7];

(iii) a theorem of isomorphism for auxetic cones under affine transformations of periodic frameworks [11];

(iv) a general method for converting finite linkages with adequate deformations into periodic frameworks with auxetic deformations [11]. This method provides *infinite series* of auxetic designs.

As mentioned earlier, the literature based on the traditional approach has generated only a limited list of sporadic patterns and considers the invention of new auxetic designs a *challenging problem* [26], p. 4792.

2 Outline of the main results

In this paper, we give comprehensive answers to the following central problems:

(1) What structural characteristics are required for auxetic capabilities?
(2) How to design periodic frameworks with auxetic deformations?

We give first a description of key facts and ideas. Technical details are provided in the next section.

(1) Structural characteristics. The rather stringent structural constraints which must be satisfied by a periodic framework with auxetic deformations become visible when using *lattice coordinates* i.e. coordinates relative to the periodicity lattice, with a chosen basis of generators [5]. As detailed below, these coordinates implicate directly the Gram matrix of the generators and the existence of a strictly auxetic infinitesimal deformation has a *geometric formulation* which brings forth a *family of homothetic ellipsoids*. This family is indexed by all pairs of vertex orbits with at least one edge orbit connection. All edge vector representatives emanating from a vertex v_i to the orbit of, say, v_j must have their origin and endpoint on the associated ellipsoid. In particular, these points must be in *strictly convex position*, a fairly restrictive necessary condition (which, in dimension three, is visually easy to assess). There are also *cycle vanishing relations* involving the centers of these ellipsoids.

The essential fact is that a strictly auxetic infinitesimal deformation is completely expressed through a *geometric diagram with homothetic ellipsoids*.

(2) Auxetic design. The design methodology is based on the possibility to satisfy the requirements identified above in a systematic way. One condition needs particular attention, namely, that vectors which must be periods should generate (over $\mathbb{Z}$) a lattice. In Figure 1, we illustrate a case with two vertex orbits, respecting this condition. In general, the lattice condition will be satisfied when we construct our designs using *rational points* as detailed below in Section 3.

The design procedure starts with a given quotient graph and can generate an *infinite series* of periodic frameworks (with that quotient). By construction, the generated frameworks will have a strictly auxetic infinitesimal deformation. A guaranteed one-parameter deformation (extending the infinitesimal one) is obtained whenever the local deformation space is smooth.
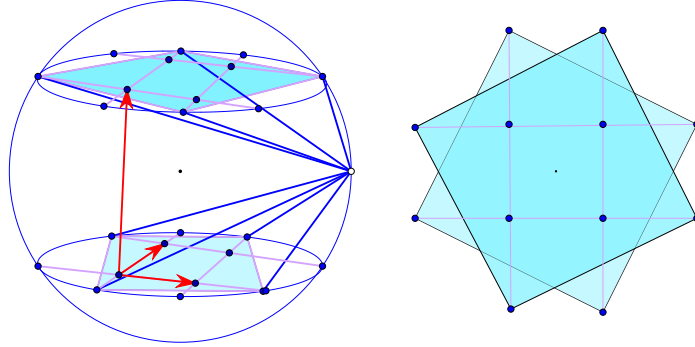
Figure 1: A blueprint for a three-dimensional auxetic periodic framework with one degree of freedom. Periodicity generators are shown as red arrows. The framework has two vertex orbits and eight edge orbits. The white vertex and the endpoints of the eight bars emanating from it are on a sphere. The vertical projection explains why all vectors between vertices in solid blue are in the lattice spanned by the generators.

In this respect, our results in [4] can be used for selecting quotient graphs with generic liftings which do have smooth local deformation spaces. The number of degrees of freedom can be predetermined as well.

Thus, our design method can generate an *infinite virtual catalog* of periodic frameworks with auxetic deformations. While our paper [11] has already demonstrated that infinite series of auxetic designs are possible in any dimension $d \geq 2$, the new method is *comprehensive* and amenable to algorithmic treatment.

3   Technical details, statements and proofs

Periodic graphs and (reduced) quotient graphs. A *d-periodic graph* is a pair (G, Γ), where $G = (V, E)$ is a simple infinite graph with vertices V, edges E and finite degree at every vertex, and $\Gamma \subset Aut(G)$ is a free Abelian group of automorphisms which has rank d, acts without fixed points and has a finite number of vertex (and hence, also edge) orbits. We assume G to be connected. The group Γ is thus isomorphic to $\mathbb{Z}^d$ and is called the *periodicity group* of the periodic graph G. Its elements $\gamma \in \Gamma \simeq \mathbb{Z}^d$ are referred to as *periods* of G. We denote by $G/\Gamma = (V/\Gamma, E/\Gamma)$ the *quotient (multi)graph*,

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with $n = |V/\Gamma|$ the number of vertex orbits and $m = |E/\Gamma|$ the number of edge orbits. We also make use of an associated *reduced* quotient graph, where multiple edges have been replaced with simple edges.

Periodic frameworks and deformations. A *periodic placement* (or simply placement) of a d-periodic graph (G, Γ) in $\mathbb{R}^d$ is defined by two functions: $p : V \rightarrow \mathbb{R}^d$ and $\pi : \Gamma \hookrightarrow \mathcal{T}(\mathbb{R}^d)$, where p assigns points in $\mathbb{R}^d$ to the vertices V of G and π is a faithful representation of the periodicity group Γ, that is, an injective homomorphism of Γ into the group $\mathcal{T}(\mathbb{R}^d)$ of translations in the Euclidean space $\mathbb{R}^d$, with $\pi(\Gamma)$ being a lattice of rank d. These two functions must satisfy the natural compatibility condition $p(\gamma v) = \pi(\gamma)(p(v))$. The translation group $\mathcal{T}(\mathbb{R}^d)$ can be identified with the additive group of vectors in $\mathbb{R}^d$.

A *periodic framework* $\mathcal{F} = (G, \Gamma, p, \pi)$ is a periodic graph (G, Γ) together with a placement (p, π). Edges are represented as straight segments between their endpoints. Two frameworks are considered equivalent when one is obtained from the other by a Euclidean isometry. A *one-parameter deformation of the periodic framework* $\mathcal{F}$ is a (smooth) family of placements $p_\tau : V \rightarrow \mathbb{R}^d$ parametrized by time $\tau \in (-\epsilon, \epsilon)$ in a small neighborhood of the initial placement $p_0 = p$, which satisfies two conditions: (a) it maintains the lengths of all the edges $e \in E$, and (b) it maintains periodicity under Γ, via faithful representations $\pi_\tau : \Gamma \rightarrow \mathcal{T}(\mathbb{R}^d)$ which *may change with τ and give an associated variation of the periodicity lattice of translations* $\pi_\tau(\Gamma)$. Smoothness requires all functions $p_\tau(v)$ and $\pi_\tau(\gamma)$ to be (indefinitely) differentiable with respect to τ. (It is enough to have this condition satisfied for a complete set of vertex representatives modulo Γ and a set of generators for Γ.)

After choosing an independent set of d generators for the periodicity lattice Γ, the image $\pi(\Gamma)$ is completely described via the $d \times d$ matrix Λ with column vectors $(\lambda_i)_{i=1,\dots,d}$ given by the images of the generators under π. The Gram matrix for this basis will be $\omega = \Lambda^t \cdot \Lambda$.

Infinitesimal deformations in lattice coordinates. Infinitesimal deformations in Cartesian coordinates are discussed in [3], section 3(b). Let us fix now a complete set of vertex representatives $v_0, v_1, \dots, v_{n-1}$ for the n vertex orbits of (G, Γ). The framework $\mathcal{F}$ has them positioned at $p_i = p(v_i)$. When we pass from these Cartesian coordinates to lattice coordinates q_i, we consider v_0 to be the origin, that is $q_0 = 0$, and then $\Lambda q_i = p_i - p_0$.

We recall the form of the equations expressing the constant (squared) length

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of edges, when using parameters $(q_1, ..., q_{n-1}, \omega)$, cf. [5], formula (4.1). Let us consider an edge (denoted here simply by e_{ij}) which goes from v_i to a vertex in the orbit of v_j. Then, in *Cartesian coordinates*, the edge vector is given by $p_j + \lambda_{ij} - p_i$, with some period $\lambda_{ij} = \Lambda n_{ij} \in \pi(\Gamma)$ and $n_{ij} \in \mathbb{Z}^d$. In *lattice coordinates*, the edge vector is given by $e_{ij} = q_j + n_{ij} - q_i$ and the squared-length equation is:

$$\ell(e_{ij})^2 = \langle \omega e_{ij}, e_{ij} \rangle \tag{1}$$

For infinitesimal deformations we use the notation $(\dot{q}_1, ..., \dot{q}_{n-1}, \dot{\omega})$. From (1) we obtain by differentiating:

$$\langle \dot{\omega} e_{ij}, e_{ij} \rangle + 2 \langle \omega e_{ij}, \dot{e}_{ij} \rangle = 0 \tag{2}$$

and since $\dot{e}_{ij} = \dot{q}_j - \dot{q}_i$, we have

$$\langle \dot{\omega} e_{ij}, e_{ij} \rangle + 2 \langle \omega (\dot{q}_j - \dot{q}_i), e_{ij} \rangle = 0 \tag{3}$$

Key observation. The definition of auxetic behavior from [7], reviewed in Section 1, implies that an infinitesimal deformation is *strictly auxetic* precisely when $\dot{\omega}$ is *positive definite*. In this case, the geometric meaning of equation (3) is that *all edge vectors e_{ij} from the orbit of v_i to the orbit of v_j*, when seen as fixed vectors, have all their endpoints on the **ellipsoid**:

$$\langle \dot{\omega} x, x \rangle + 2 \langle \omega (\dot{q}_j - \dot{q}_i), x \rangle = 0 \tag{4}$$

which passes through their common origin $x = 0$ and has its center c_{ij} at:

$$c_{ij} = -\dot{\omega}^{-1} \omega (\dot{q}_j - \dot{q}_i) = \dot{\omega}^{-1} \omega (\dot{q}_i - \dot{q}_j) \tag{5}$$

Homothetic ellipsoids on the reduced quotient graph. Since all the ellipsoid formulae in (4) have the same quadratic part $\langle \dot{\omega} x, x \rangle$, they are all *homothetic* and can be labeled by the edges ij of the reduced quotient graph of the given framework. Under the assumption of a strictly auxetic infinitesimal deformation, it follows immediately that the quotient graph has no loops.

Conditions on cycles. Whenever we have a *cycle* in the reduced quotient graph, the sum of the vectors c_{ij} is zero (with the ordering ij corresponding to traversing the cycle). In other words, we obtain (up to translation) a *placement of the reduced quotient graph*, with edge vectors c_{ij} (or $2c_{ij}$).

It will be observed that, while these properties have been discovered in lattice coordinates, they hold true in Cartesian coordinates as well, since the two representations are related by an affine transformation, which takes homothetic ellipsoids to homothetic ellipsoids and zero-sum vector relations to zero-sum vector relations.

Illustration with Kagome frameworks. The structural features disclosed above will be necessarily present in planar frameworks of Kagome type. Such frameworks have $n = 3$ vertex orbits, $m = 6$ edge orbits and their deformation space is one dimensional. The quotient graph is a triangle with doubled edges. The auxetic character of Kagome periodic mechanisms with congruent equilateral triangles has been noticed earlier [8, 19, 20, 34]. In Figure 2, the three vertex orbits are colored in red, blue and green. The three ellipses are actually circles. The red circle passes through a green vertex and the two blue vertices connected with it. The centers of the circles are not shown, but the cycle condition is clearly satisfied by symmetry.
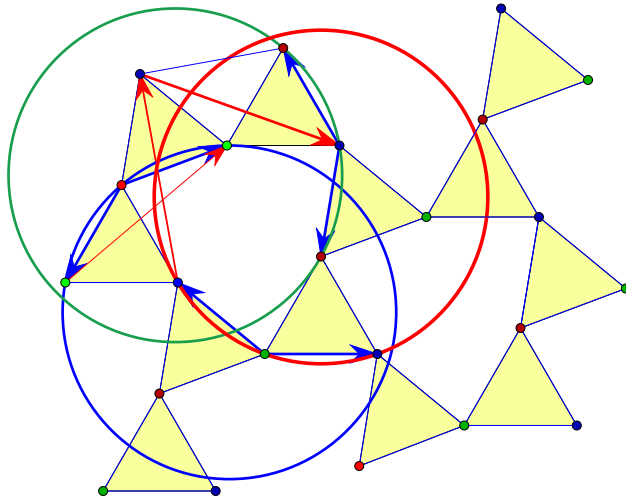

Figure 2: A Kagome framework has one degree of freedom and its deformation is strictly auxetic until the unit cell reaches maximal area. The geometric manifestation of this fact is the existence of three homothetic ellipses (which are circles, as a result of symmetry) satisfying the cycle condition.

Remarks. The return from lattice coordinates $(q_1, ..., q_n, \omega)$ to Cartesian coordinates requires finding a matrix Λ with $\omega = \Lambda^t \cdot \Lambda$. This equation

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is solved by diagonalizing the symmetric positive definite matrix ω via an orthogonal change of coordinates $\omega = M \cdot D \cdot M^t$, where D is diagonal, with positive diagonal entries and $M^t = M^{-1}$. Then one takes the positive square root $D^{1/2}$ of D and obtains the solution $\Lambda = M \cdot D^{1/2} \cdot M^t$. The corresponding framework will have the vertex orbit representatives at $p_0 = 0$ and $p_i = \Lambda q_i$, $i = 1, ..., n$ and periodicity matrix Λ. Since any other solution $\tilde{\Lambda}$ differs from Λ by some orthogonal transformation, $\tilde{\Lambda} = T\Lambda$, with $T^t = T^{-1}$, the framework is uniquely defined, up to equivalence under isometries. The fact that solving for Λ_τ in a smooth (local) family ω_τ can be done smoothly in τ follows from the holomorphic functional calculus for the operators with real positive spectra ω_τ.

Theorem. The reduced quotient graph (with its placement) and the family of homothetic ellipsoids (4), with all the edge vector representatives in them allows the reconstruction of the periodic framework and (up to a scalar) the corresponding strictly auxetic infinitesimal deformation.

Proof: The periodic framework is determined by the complete set of edge vector representatives. The strictly auxetic infinitesimal vector $(\dot{q}_1, ..., \dot{q}_{n-1}, \dot{\omega})$ can be retrieved, up to a non-zero scalar, since we have $\dot{\omega}$, up to a non-zero scalar, from any of the ellipsoids, and then the linear system (5) for $\dot{q}_i$ is compatible due to the vanishing of sums over cycles observed above.

Simplifying observation. Since affine transformations of a periodic framework preserve infinitesimal characteristics (such as strict auxeticity), we may adopt coordinates which turn all our homothetic ellipsoids into spheres.

Generating periodic frameworks with auxetic capabilities. We can construct (generate) such frameworks by following *in reverse* the steps of the previously described procedure. Particular attention must be paid to the vectors between endpoints of edge vector representatives in any given ellipsoid and to sums of edge vectors over cycles in the reduced quotient graph: these vectors must be periods and must generate, by linear combinations with *integer coefficients*, a (periodicity) lattice.

This *rational dependence* requirement is always fulfilled if we operate only with rational points (i.e. points with rational coordinates). More precisely, we have the following scenario (in arbitrary dimension $d \geq 2$).

Construction procedure. We start with a (connected) finite multi-graph without loops (which plays the role of a given quotient graph). We want to

construct periodic frameworks with strictly auxetic (infinitesimal) capabilities and with the given quotient graph.

(1) We take the reduced graph of the given multi-graph.

(2) We choose a (general, non-degenerate) placement of its vertices in $\mathbb{Q}^d$ i.e. at rational points in $\mathbb{R}^d$ and represent the edges by segments between adjacent vertices.

(3) We draw all spheres with these edges as diameters (hence centers are also rational points).

Lemma. Rational points are dense on all these spheres.

Proof: The standard stereographic projection (from a point e_1 on the unit sphere S^{d-1} to the tangent space $T_{-e_1}(S^{d-1})$ at the opposite point $-e_1$) gives a bijection between the rational points on the unit sphere and rational points in the tangent space $T_{-e_1}(S^{d-1})$. Thus, rational points are dense in the unit sphere. It follows that rational points are dense on any sphere with rational center and at least one rational point.

(4) When ij is an edge, we choose as many rational points on the corresponding sphere as dictated by the multiplicity in the given multi-graph. (Again, the choice in supposed to be non-degenerate.) The vectors from vertex v_i to the chosen points will represent all edge vectors connecting the orbit of v_i to the orbit of v_j.

(5) If degenerate choices are avoided, the span of all vectors between chosen points in any given sphere and all sums of edge vectors over cycles in the reduced quotient graph will be a (rank d) lattice, the periodicity lattice of the framework generated by articulating the chosen edge representatives.

An unfolding example. To illustrate the construction, we use the planar case shown in Figure 3. The quotient multi-graph for the example (not shown) has three vertices and double edges between any pair of vertices. The reduced graph is placed as an equilateral triangle in the plane, and three circles are drawn with diameters given by the three edges. On each circle, we have to position two edge vectors, for the two edge orbits in the multi-graph corresponding to its diameter edge. Choosing a counter-clockwise orientation for the triangle edges, the edge vectors emanate from the source vertex of the corresponding reduced oriented edge. The left hand side of the figure shows this stage of the construction, with the added particularity that in each circle, the red vector between the endpoints of the two edges is the same free vector. This particular choice avoids the need to maneuver with
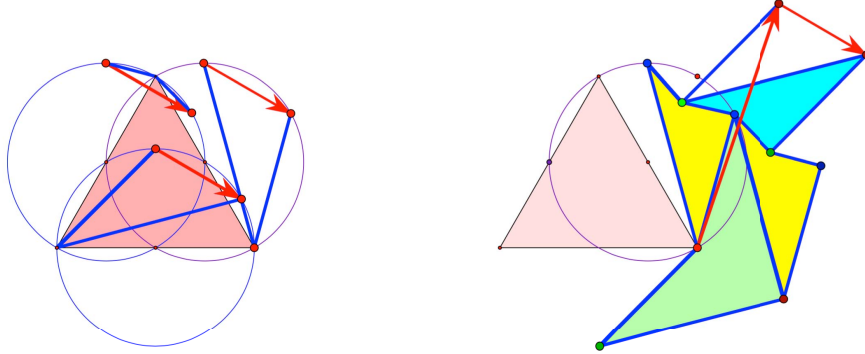
Figure 3: A planar example starting with a multigraph with $n = 3$ vertices and double edges between any pair of vertices ($m = 6$). The reduced graph is represented by the highlighted equilateral triangle. The period in each circle is the same. The framework generating process is shown on the right. In this case, the result is a periodic pseudotriangulation.

rational points, since this periodicity vector, together with one resulting from the single cycle in the reduced graph, will span the periodicity lattice.

The periodic framework on the right is now generated from the diagram constructed on the left, as follows. We start at one vertex and continue to place (all allowed) edge vectors emanating from the endpoints of the initial two edges. Indefinite repetitions of this type of edge addition will generate the entire (connected) periodic framework. The right hand side of the figure shows the stage reached after adding enough edges to see (in green, yellow and blue) representatives of the three type of pseudotriangular faces of the resulting planar framework. Obviously, the initial triangle and the circle are not part of the framework. As a periodic pseudotriangulation, the framework has one degree of freedom and its local deformation is not only auxetic, but actually expansive [8].

An example with higher symmetry. We discuss now a three-dimensional example based on a multi-graph with four vertices and double edges between any pair of vertices. The reduced graph is represented as a regular tetrahedron with vertices at: $(-1, -1, -1)$, $(1, 1, -1)$, $(1, -1, 1)$ and $(-1, 1, 1)$. We trace the six spheres with diameters given by the six edges of this tetrahedron. Our choices of edge vector representatives for the framework will use the symmetry group of the regular tetrahedron, represented in our setting by
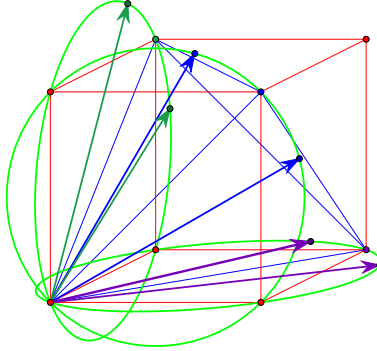
Figure 4: The essentials of the diagram used for a three-dimensional framework with $n = 4$ vertex orbits and $m = 12$ edge orbits. The reduced (quotient) graph is represented by the regular tetrahedron inscribed in the cube. The six edge vectors emanating from one vertex are shown as arrows. Only three great circles of the six spheres are traced. Tetrahedral symmetry completes the information recorded by the diagram.

permutations of the three coordinates and an even number of sign changes in front of them. Thus, it will be enough to indicate one of the two edge vectors emanating from $(-1, -1, -1)$ and ending at vertices in the orbit represented by $(1, 1, -1)$. We choose it to be of the form $(\alpha, \beta, 0)$. The condition that, when emanating from $(-1, -1, -1)$, the endpoint of the vector must be on the sphere over the tetrahedral edge from $(-1, -1, -1)$ to $(1, 1, -1)$ amounts to: $(\alpha - 1)^2 + (\beta - 1)^2 = 2$.

The twenty-four transforms of this edge vector give representatives for the twelve edge orbits in twelve pairs with opposite signs. The six edge vectors shown in Figure 4 are:

$$12A = (\alpha, \beta, 0), \quad 12B = (\beta, \alpha, 0),$$
$$13A = (0, \alpha, \beta), \quad 13B = (0, \beta, \alpha),$$
$$14A = (\beta, 0, \alpha), \quad 14B = (\alpha, 0, \beta).$$

With $IJA = -JIA$ and $IJB = -JIB$, the remaining six edge vector representatives are:

$$23A = (-\alpha, 0, \beta), \quad 23B = (-\beta, 0, \alpha),$$
$$24A = (0, -\beta, \alpha), \quad 24B = (0, -\alpha, \beta),$$

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$$34A = (\beta, -\alpha, 0), \quad 34B = (\alpha, -\beta, 0).$$

The periodicity lattice is spanned by the tetrahedral group transforms of vectors of the form $12A - 12B = (\alpha - \beta, \beta - \alpha, 0)$ and $12A + 23A + 31A = (0, \beta - \alpha, 0)$, which illustrates a period associated to a cycle in the reduced graph. It follows that, regardless of the rationality or irrationality of $\alpha \neq \beta$, the resulting periodicity lattice is $(\alpha - \beta)Z^3$.

The periodic framework itself is obtained by indefinite articulation of new edge vectors, proceeding from the endpoints of the six depicted arrows and always respecting the vertex orbits. A partial extension, with all twelve edge representatives, is shown, from a different perspective, in Figure 5. With the vertex marked 1 placed at the origin (and seen "in depth"), one may observe three skew quadrilaterals, resulting from the following relations.

$$12A + 23A = 14B + 43B = (0, \beta, \beta) = 33$$
$$13A + 34A = 12B + 24B = (\beta, 0, \beta) = 44$$
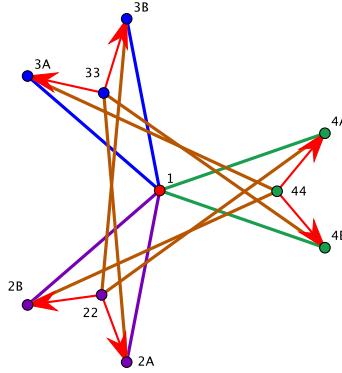$$14A + 42A = 13B + 32B = (\beta, \beta, 0) = 22$$

Figure 5: The essentials of the resulting framework. The periodicity lattice is generated by red vectors, with (22)(2A)=(44)(4B), (33)(3A)=(22)(2B), (44)(4A)=(33)(3B).

As a consequence of tetrahedral symmetry for the initial diagram, the resulting framework has a crystallographic symmetry group which is transitive on vertices and transitive on edges. The one-parameter auxetic deformation

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which maintains this crystallographic symmetry and extends the strictly auxetic infinitesimal deformation provided by our construction can be described as follows. Dilate the (cube and inscribed) tetrahedron of the initial diagram, but maintain the squared length of the new edge vector representative $(\tilde{\alpha}, \tilde{\beta}, 0)$, that is $\tilde{\alpha}^2 + \tilde{\beta}^2 = \alpha^2 + \beta^2$. Thus, at the vertex marked 1, the edge pairs $(12A, 12B)$, $(13A, 13B)$, $(14A, 14B)$ increase their angle in their respective planes (and by symmetry, this type of local motion is replicated at any other vertex).

Remarks. This family of periodic frameworks has (geometrically allowed) self-intersections. It can be shown that these equal edge-length frameworks have four degrees of freedom i.e. a smooth four-dimensional local deformation space.

This design invites comparison with a structure possessing cubical symmetry, introduced in [33] and studied in [12] from the perspective of cubic materials [29].

4 Perusal of the virtual auxetic design catalog

We have described above a procedure which associates to an *initial diagram* in $\mathbb{R}^d$, a d-periodic framework with a strictly auxetic infinitesimal deformation. The initial diagram consists of the following elements:

(i) a finite connected multi-graph without loops,
(ii) a placement of the (simple) reduced graph in $\mathbb{R}^d$,
(iii) spheres with diameters given by all edges of the placed reduced graph,
(iv) in a sphere, say, over the edge from v_i to v_j, a depiction of (framework edge) vectors e_{ij}^k, $k = 1, ..., m_{ij}$, where m_{ij} is the number of edges in the multigraph over the specified edge in the reduced graph; all vectors e_{ij}^k emanate from v_i and have their endpoints on the sphere,
(v) a rank d lattice generation condition for the span of the periodicity vectors; a periodicity vector is either a vector between the endpoints of two (framework edge) vectors in the same sphere or a (cycle) vector of the form $e_{i_1 i_2}^* + e_{i_2 i_3}^* + ... + e_{i_j i_1}^*$ for a cycle of edges in the reduced graph.

The associated periodic framework may turn out to be a singular point in its deformation space and in this case additional investigations would be

required for deciding if the strictly auxetic infinitesimal deformation belongs to a local auxetic path or not. If the periodic framework has a smooth local deformation space, there are always strictly auxetic path extensions.

Thus, for a systematic and unencumbered generation of auxetic designs, we need a direct guarantee of property (v) for initial diagrams and a simple smoothness criterion for the local deformation space of the resulting periodic framework. We have already shown that, when operating with *rational initial diagrams*, that is, diagrams which have the vertices of the reduced graph at rational points and the endpoints of the (edge framework) vectors at rational points of the corresponding spheres, property (v) is guaranteed to hold. For the smoothness issue, a sensible restriction is to use as multi-graphs only *quotient graphs of minimally rigid periodic frameworks, with one or more edges removed*. The class of quotient graphs of minimally rigid periodic frameworks has been characterized in combinatorial terms in [4]. With this restriction, smoothness is guaranteed by a single (maximal rank) linear test.

Classification outlook for two vertex orbits. It is natural to organize the virtual catalog according to increasing values of the number n of vertex orbits and the number f of degrees of freedom. For $n = 2$, the reduced (quotient) graph is a single edge. Thus, we have a single ellipsoid and no cycles. We consider here the case of periodic frameworks with $f = 1$, i.e. one degree of freedom, for dimensions $d = 2$ and $d = 3$. For independent edge constraints, we have $f = dn + \binom{d}{2} - m$, with m denoting the number of edge orbits. This follows from formula (3.16) of [3], with readapted notation.

In the planar case, we have $m = 4$. A periodic framework (with four independent edge orbit constraints) is immediately recognized as a *strictly auxetic mechanism* by tracing the unique conic through a vertex and the four endpoints of the bars emanating from it: this conic must be an *ellipse*. Conversely, planar auxetic periodic mechanisms can be generated by adopting as periodicity lattice $\mathbb{Z}^2 \subset \mathbb{R}^2$. We choose four points in $\mathbb{Z}^2$, forming a (non-degenerate) convex quadrilateral. In general, the $\mathbb{Z}$-span of the edge vectors will be a sublattice of $\mathbb{Z}^2$, but we shall retain only choices with equality (without any loss of design patterns). For classification purposes, one must identify equivalent choices of this nature. There is a one parameter family of ellipses running through four such points. We may select any ellipse and place the vertex representative of the second vertex orbit on it (away from the four points). The four (framework) edge vectors from this new point to the previous four points provide the data for articulating the associated
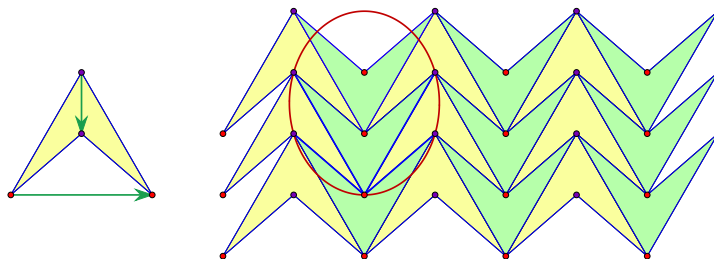
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auxetic periodic mechanism.

Figure 6: The double arrowhead framework can be generated from a single quadrilateral by adopting the diagonals as periods. The auxetic character is confirmed by the ellipse through one vertex and the endpoints of the four edges emanating from it.

In Figure 6, we illustrate a design obtained by topology optimization in [25]. It simplifies a structure proposed in [27]. The double arrowhead framework has $n = 2$ vertex orbits and $m = 4$ edge orbits. It can be obtained from a single quadrilateral (in a pseudotriangle configuration) by a construction principle described in [11]: the two diagonal vectors generate a periodicity lattice and the periodic framework is made of the joint translates of the quadrilateral. The result is a periodic pseudotriangulation, hence the local one-degree-of-freedom motion is expansive and therefore auxetic [8]. From the perspective of the present paper, the auxetic capability is demonstrated by the ellipse running through one vertex and the endpoints of the four edges emanating from it.

In dimension three, we have $m = 8$. A periodic framework (with eight independent edge orbit constraints) is immediately recognized as a *strictly auxetic mechanism* by tracing the unique quadric surface through a vertex and the eight endpoints of the bars emanating from it: this quadric must be an *ellipsoid*. The design procedure is slightly more complicated than in the planar case because eight points in strictly convex position need not lie on any ellipsoid. We start with eight points in $\mathbb{Z}^3$ in strictly convex position and check that the $\mathbb{Z}$-span of the vectors between all point pairs equals $\mathbb{Z}^3$. Then we check that the family of quadrics passing through the eight points is one dimensional (i.e. a pencil) and contains ellipsoids. If this is the case, the choice is valid. Again, for classification purposes, one must identify equivalent

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choices of this nature. Auxetic mechanisms will be obtained, as above, by selecting an ellipsoid through the eight points and a vertex representative of the second vertex orbit on it (away from the quartic curve through the eight points which is the axis of the pencil, that is, the common vanishing locus of the pencil's quadrics). Up to an affine transformation, Figure 1 illustrates one such blueprint.

5 Conclusion

Although auxetic behavior is typically non-linear, we have found a structural characterization of periodic auxetics from infinitesimal i.e. linear considerations. This necessary and sufficient condition for periodic bar-and-joint frameworks with strictly auxetic infinitesimal deformations is expressed in a diagram with homothetic ellipsoids over the reduced quotient graph and leads to endless, yet systematic possibilities for generating auxetic periodic designs.

References

[1] K. Bertoldi. Harnessing instabilities to design tunable architected cellular materials. *Annu. Rev. Mater. Res.*, 47:51–61, 2017.

[2] K. Bertoldi, V. Vitelli, J. Christensen, and M. van Hecke. Flexible mechanical metamaterials. *Nature Reviews Materials*, 2:17066, 2017.

[3] C. S. Borcea and I. Streinu. Periodic frameworks and flexibility. *Proc R Soc Lond A Math Phys Sci*, 466:2633–2649, 2010.

[4] C. S. Borcea and I. Streinu. Minimally rigid periodic graphs. *Bulletin of the London Mathematical Society*, 43:1093–1103, 2011.

[5] C. S. Borcea and I. Streinu. Frameworks with crystallographic symmetry. *Philos Trans A Math Phys Eng Sci*, 372:20120143, 2014.

[6] C. S. Borcea and I. Streinu. Kinematics of expansive planar periodic mechanisms. In Jadran Lenarcic and Oussama Khatib, editors, *Advances in Robot Kinematics (ARK'14)*, pages 395–407. Springer Verlag, Ljubljana, Slovenia, June 2014.

[7] C. S. Borcea and I. Streinu. Geometric auxetics. *Proc R Soc Lond A Math Phys Sci*, 471:20150033, 2015.

[8] C. S. Borcea and I. Streinu. Liftings and stresses for planar periodic frameworks. *Discrete Comput Geom*, 53:747–782, 2015.

[9] C. S. Borcea and I. Streinu. Auxetic deformations and elliptic curves. *arXiv:1612.02100*, preprint, 2016.

[10] C. S. Borcea and I. Streinu. Deforming diamond. *IMA J Appl Math*, 82(2):371–383, April 2017.

[11] C. S. Borcea and I. Streinu. New principles for auxetic periodic design. *SIAM Journal on Applied Algebra and Geometry*, 1:442–458, 2017.

[12] L. Cabras and M. Brun. A class of auxetic three-dimensional lattices. *J. Mech. Phys. Solids*, 91:56–72, 2016.

[13] M. T. Dove. Theory of displacive phase transitions in minerals. *American Mineralogist*, 82:213–244, 1997.

[14] J. C. A. Elipe and A. D. Lantada. Comparative study of auxetic geometries by means of computer-aided design and engineering. *Smart Materials and Structures*, 21:105004, 2012.

[15] K. E. Evans, M. A. Nkansah, and I. J. Hutchinson. Auxetic foams: Modeling negative Poisson's ratios. *Acta Metall. Mater.*, 42(4):1289–1294, 1994.

[16] K. E. Evans, M. A. Nkansah, I. J. Hutchinson, and S. C. Rogers. Molecular network design. *Nature*, 353:124–125, 1991.

[17] G. N. Greaves, A. L. Greer, R. Lakes, and T. Rouxel. Poisson's ratio and modern materials. *Nature Materials*, 10:823–837, 2011.

[18] J. N. Grima, R. Caruana-Gauci, D. Attard, and R. Gatt. Three-dimensional cellular structures with negative Poisson's ratio and negative compressibility properties. *Proc R Soc Lond A Math Phys Sci*, 468:3121–3138, 2012.

[19] J. N. Grima, E. Chetcuti, E. Manicaro, D. Attard, M. Camilleri, R. Gatt, and K.E. Evans. On the auxetic properties of generic rotating rigid triangles. *Proc R Soc Lond A Math Phys Sci*, 468:810–830, 2012.

[20] J. N. Grima and K.E. Evans. Self expanding molecular networks. *Chemical Communications*, pages 1531–1532, 2000.

[21] C. Huang and L. Chen. Negative Poisson's ratio in modern functional materials. *Advanced Materials*, 28:8079–8096, 2016.

[22] H. M. A. Kolken and A. A. Zadpoor. Auxetic mechanical metamaterials. *RSC Advances*, 7:5111–5129, 2017.

[23] R. Lakes. Foam structures with a negative Poisson's ratio. *Science*, 235:1038–1040, 1987.

[24] R. S. Lakes. Negative-Poisson's-ratio materials: Auxetic solids. *Annu. Rev. Mater. Res.*, 47:63–81, 2017.

[25] U. D. Larsen, O. Sigmund, and S. Bouwstra. Design and fabrication of compliant mechanisms and material structures with negative Poisson's ratio. *J. of MicroElectroMechanical Systems,*, 6:99–106, 1997.

[26] J-H. Lee, J. P. Singer, and E. L. Thomas. Micro-/nanostructured mechanical metamaterials. *Advanced Materials*, 24:4782–4810, 2012.

[27] G. W. Milton. Composite materials with Poisson's ratio close to -1. *J. Mech. Phys. Solids*, 40:1105–1137, 1992.

[28] H. Mitschke, F. Schury, K. Mecke, F. Wein, M. Stingl, and G. E. Schroeder-Turk. Geometry: The leading parameter for the Poisson's ratio of bending-dominated cellular solids. *International Journal of Solids and Structures*, 100–101:1–10, 2016.

[29] A. N. Norris. Poisson's ratio in cubic materials. *Proc R Soc Lond A Math Phys Sci*, 462:3385–3405, 2006.

[30] H. S. Park, S. Y. Kim, and J-W. Jiang. Auxetic nanomaterials: Recent progress and future development. *Applied Physics Reviews*, 3:041101, 2016.

[31] P. M. Reis, H. M. Jaeger, and M. van Hecke. Designer matter: A perspective. *Extreme Mechanics Letters*, 5:25–29, 2015.

[32] K. K. Saxena, R. Das, and E. P. Calius. Three decades of auxetics research: Materials with negative Poisson's ratio: A review. *Advanced Engineering Materials*, 18(11):1847–1870, 2016.

[33] O. Sigmund. Tailoring materials with prescribed elastic properties. *Mechanics of Materials*, 20:351–368, 1995.

[34] K. Sun, A. Souslov, X. Mao, and T. C. Lubensky. Surface phonons, elastic response, and conformal invariance in twisted kagome lattices. *Proc. Nat. Acad. Sci.*, 109:12369–12374, 2012.