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Outerplanar Graphs and Delaunay Triangulations

Ashraful Alam
University of Massachusetts Amherst

Igor Rivin
Temple University

Ileana Streinu
Smith College, istreinu@smith.edu

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Ashraful Alam * Igor Rivin† Ileana Streinu‡

Abstract

Over 20 years ago, Dillencourt [1] showed that all outerplanar graphs can be realized as Delaunay triangulations of points in convex position. His proof is elementary, constructive and leads to a simple algorithm for obtaining a concrete Delaunay realization. In this note, we provide two new, alternate, also quite elementary proofs.

1 Introduction

The Delaunay triangulation of a point set in convex position is, combinatorially, an outerplanar graph. Dillencourt [1] has shown, constructively, that the other direction is also true: any graph which arises from a triangulation of the interior of a simple polygon can be realized as a Delaunay triangulation. Dillencourt's proof uses a simple and natural criterion on the angles of triangles in a Delaunay triangulation, and gives an $O(n^2)$ time incremental algorithm to calculate their angles and infer a realization. Lambert [2] adapted this method into a linear time algorithm, whose implementation in Java is available at http://www.cse.unsw.edu.au/̃lambert/java/realize/

The general question, of characterizing and reconstructing arbitrary Delaunay triangulations (in two- or higher dimensions), is substantially more difficult. A closely related problem, going back to Steiner (see Grünbaum [6], page 284), asks for a characterization of the graphs of inscribable or circumscribable polyhedra: those whose vertices lie on a sphere, resp. whose faces are tangent to a sphere. Such graphs are said to be of inscribable or circumscribable type. The best result to date is due to Rivin [4], who proved necessary and sufficient conditions for a polyhedral graph to be of inscribable or circumscribable type. Dillencourt and Smith [3] linked inscribability of a graph to its realizability as a Delaunay triangulation, and gave a criterion relating Hamiltonicity to inscribability.

*Department of Computer Science, University of Massachusetts Amherst, MA, ashraful@cs.umass.edu. Research supported by NSF CCF-1016988 and DARPA HR0011-09-0003 grants.

†Department of Mathematics, Temple University, Philadelphia, rivin@math.temple.edu

‡Department of Computer Science, Smith College, Northampton, MA, streinu@smith.edu. Research supported by NSF CCF-1016988 and DARPA HR0011-09-0003 grants.

In this paper, we present two new simple and elementary proofs of the Delaunay realizability of outerplanar graphs. We are not aware of them previously appearing in the literature. The first one is an easy consequence of Dillencourt and Smith's [3] criterion relating Hamiltonianicity and inscribability. The second one, which occupies most of this note, uses Rivin's [4] inscribability criterion and constructs an explicit "witness" of this inscribability, in the form of certain weights assigned to the edges of the graph.

Preliminaries

A *graph* $G = (V, E)$ is a collection of vertices $V = \{1, \cdots, n\}$ and edges E, where an edge $e = ij \in E$ is a pair of vertices $i, j \in V$. A graph is *planar* if it can be drawn in the plane in a way such that no two edges cross, except perhaps at endpoints. A planar drawing of a planar graph is called a *plane graph*. It subdivides the plane into regions called faces. A plane graph has one unbounded face, called the *outer face*. A plane graph is denoted by its vertices, edges, faces and the outer face: $G = (V, E, F, f)$. A plane graph where *all* vertices lie on the outer face is called an *outerplanar graph*. A *stellated outerplanar graph* is obtained from an outerplanar graph by adding one vertex and connecting it to all the original vertices. We call this the *stellating* vertex, and all edges emanating from this vertex the *stellating* edges.

Two paths between two vertices are *independent* if they do not share any vertices except the end-points. A graph is connected if there is a path between any two vertices and it is *k-connected* if there are k independent paths between any two vertices. A *cutset* of a graph is the minimal set of edges whose removal makes the graph disconnected. A cutset is *coterminous* if all the edges emanate from a single point.

A graph is *polyhedral* if it is planar and 3-connected. In this case, the faces of a plane realization are uniquely determined up to the choice of the outer face. By Steinitz theorem (see Grünbaum [6]), any polyhedral graph can be realized as a convex polyhedron. A polyhedral graph is *inscribable* if its corresponding convex polyhedron is combinatorially equivalent to the edges and vertices of the convex hull of a set of noncoplanar points on the surface of the sphere.

Given a set P of points in the Euclidean plane, a triangulation of these points is a planar graph where all faces, with the possible exception of the outer face, are triangles. A Delaunay triangulation of P is a triangulation where the circumscircle of any triangle does not contain any other points of P.

Our result

We give two new proofs of Dillencourt's theorem:

Theorem 1 *Any outerplanar graph can be realized as a Delaunay triangulation.*

2 The first proof

The first proof that an outerplanar graph can be realized as a Delaunay triangulation relies on two elegant results due to Dillencourt and Smith [3] and to Rivin [4, 5]. They relate inscribability, realization as Delauney triangulation and Hamiltonicity.

A *Hamiltonian cycle* in a graph is a simple spanning cycle. Any graph that has a Hamiltonian cycle is called *Hamiltonian.* A graph is *1-Hamiltonian* if removing *any* vertex from the graph makes it Hamiltonian.

Dillencourt and Smith [3] proved:

Theorem 2 *A 1-Hamiltonian planar graph is of inscribable type.*

Next, we need this result of Rivin [4, 5].

Theorem 3 *A plane graph $G = (V, E, F, f)$, with f as the unbounded face, is realizable as a Delaunay triangulation for some point set P if and only if the graph G' obtained from G by stellating f is of inscribable type.*

To complete our first proof, we just need to show that:

Lemma 4 *A stellated outerplanar graph G' is 1-Hamiltonian.*

Proof. Let $\{1, 2, \cdots, n\}$ be the vertices of the underlying outerplanar graph G of G', in counterclockwise order on the outer face, with modulo n indices, and let s be the stellating vertex. If we remove a vertex i, $1 \leq i \leq n$, then we find a Hamiltonian cycle $i+1, i+2, \cdots, i-1, s, i+1$. If we remove vertex s, we get the original outerplanar graph G which is Hamiltonian. □

In the next section, we present our main result, which is a more technical proof for the same result. It is based on a very general criterion of Rivin, and has the advantage of illustrating specific properties (besides Hamiltonicity) of Delaunay triangulation realizations for outerplanar graphs.

3 The main proof

Rivin [4, 5] gave this very general criterion for the Steiner's problem:

Theorem 5 *A planar graph $G' = (V, E)$ is of inscribable type if and only if it satisfies the following conditions:*

1. *G' is a 3-connected planar graph.*
2. *A set of weights W can be assigned to the edges of G' such that:*
   (a) *For each edge e, $0 < w(e) \leq 1/2$.*
   (b) *For each vertex v, the sum of all weights of edges incident to v is 1.*
   (c) *For each non-coterminous cutset $C \subseteq E$, the sum of all the weights of edges of C must exceed 1.*

Combining this with Theorem 3, we have to prove that if G' is a stellated outerplanar graph, then a weight assignment as in Theorem 5 exists. But first, let us verify that any stellated outerplanar graph is 3-connected. The following lemma is straightforward:

Lemma 6 *Any outerplanar graph is 2-connected.*

Proof. In an outerplanar graph, all the vertices lie on the unbounded face f. If we label the vertices as $1, 2, \cdots, n$ in the order in which they appear on the outer face, there are two independent paths between any pair of vertices i and j: one from $i, i+1, \cdots, j$ and another is $i, i-1, \cdots, j$. □

This leads immediately to the verification of the first condition in Theorem 5:

Lemma 7 *Any stellated outerplanar graph G' is planar and 3-connected.*

Proof. Planarity is straightforward, since G' was obtained from an outerplanar graph G by stellating (with a new vertex s) its unbounded face. We show now that there exists three independent paths between any two vertices i and j of G'. Let i and j be two vertices of the outerplanar graph G. Lemma 6 showed that there are two independent paths between i and j. A third independent path is i, s, j where s is the stellating vertex. To complete the proof we show the existence of three independent paths between s and any other vertex i of G: they are (s, i), $(s, i-1, i)$ and $(s, i+1, i)$, where index arithmetic is done modulo n in the range $1, \cdots, n$. Notice that we implicitly assume that G has at least three vertices, otherwise the theorem is trivial. □

In the rest of the paper, we describe a weight assignment for G' which satisfies Rivin's criteria. In section 3.1 we give an inductive scheme to compute the weights, and prove that they satisfy the first two properties (2a and 2b) in Theorem 5. The proof is completed in section 3.2, where we verify the third property (2c) in Theorem 5.

3.1 Weight assignment

Instead of assigning weights on the edges of a stellated outerplanar graph G', we will assign them on the edges of the dual graph G'_D of G', so first we look closer at the structure of the dual of a stellated outerplanar graph.

The duals of the stellating edges of G' form a cycle, and the remaining ones form a tree (denoted by T_D) whose leaves lie on the cycle (see Fig 1). Furthermore, the tree is partitioned into a path whose vertices are not leaves (the *backbone*) and edges incident to the tree leaves, called leaf-edges. We thus partition the edges of G' into three classes, colored blue (cycle), green (backbone) and black (leaf edges). Primal and dual edges get the same color, see Fig 1(c). One end of the leaf edge is connected to a backbone edge and another end is connected to a cycle edge.

The edges of a face of G'_D are duals of edges incident to a vertex in G'. If there are n cycle edges, there will be n faces in G'_D. Let these faces be $f_1, f_2, \cdots, f_n$ in counter-clockwise order. Clearly, it suffices to make sure that the sum of all weights of cycle edges and sum of all weights of the edges of each face of G'_D are separately equal to 1. In addition, we have to make sure that the remaining two conditions 2a and 2b of Theorem 5 are also satisfied.

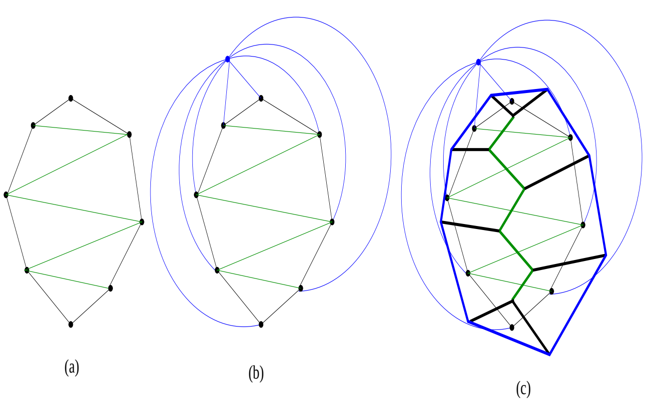

Figure 1: (a) An outerplanar graph (b) A stellated outerplanar graph obtained from (a). Blue edges are stellating edges and blue vertex is the stellating vertex. (c) Dual graph of the stellated outerplanar graph. Dual edges are colored according to their primal edges. Dark blue, green and black edges are cycle, backbone and leaf edges respectively.

The weight assignment is carried out in two steps, the *contraction step* and the *expansion step*. During contraction, all the backbone edges are contracted to obtain a very specific type of dual graph, on which a simple weight assignment is possible. The edges are then expanded back, and adjustments to the initial weights are locally performed, while maintaining Rivin's conditions.

Contraction:

In this step we contract all the backbone edges of T_D. Then all the cycle edges and leaf edges of G'_D remain unchanged, but all the faces become triangular (see Fig 2). Next, we assign a weight of $1/n$ to each cycle edge. Here n is the number of cycle edges of G'_D. Next we assign each leaf edge (one of the remaining two edges of a triangular face) a weight of $\frac{n-1}{2n}$. This weight assignment satisfies Rivin's conditions 2a and 2b for $n > 1$.

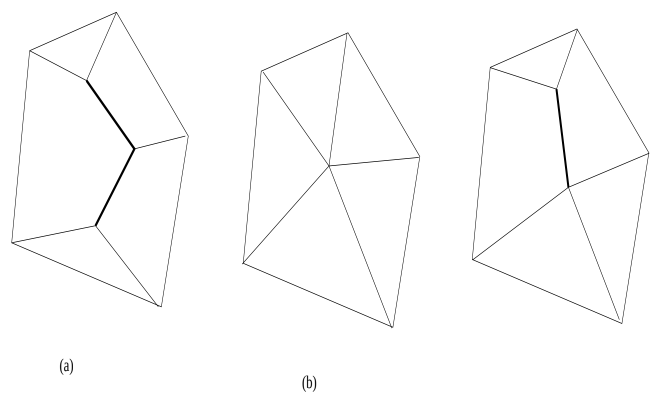

Figure 2: (a)A dual of a stellated outerplanar graph. Bold edges are backbone edges. (b) Dual graph after contraction of backbone edges. (c) Expansion of a single backbone edge.

Expansion:

Now we incrementally expand back the backbone edges of T_D. Consider a backbone edge e_b which is shared by face f_i and face f_j. We assign a weight of $0 < \epsilon < 1/2$ to e_b, for some positive ϵ to be determined later. This creates an imbalance into the sum of weights for the edges of faces f_i and f_j. We remove this imbalance by subtracting $\frac{\epsilon}{2}$ from the cycle edges of f_i and f_j, and subtracting $\frac{\epsilon}{4}$ from each of the two leaf edges of f_i and f_j, respectively. Although this restores the balance of weights for faces f_i and f_j, it creates an imbalance for faces $f_{i-1}, f_{i+1}, f_{j-1}, f_{j+1}$ and cycle edges of G'_D. To fully balance the weights, we add $\frac{\epsilon}{4}$ to the cycle edges of these four faces. This assignment of weights meets Rivin's conditions. The process is repeated for each expanded backbone edge. See Figure 3.

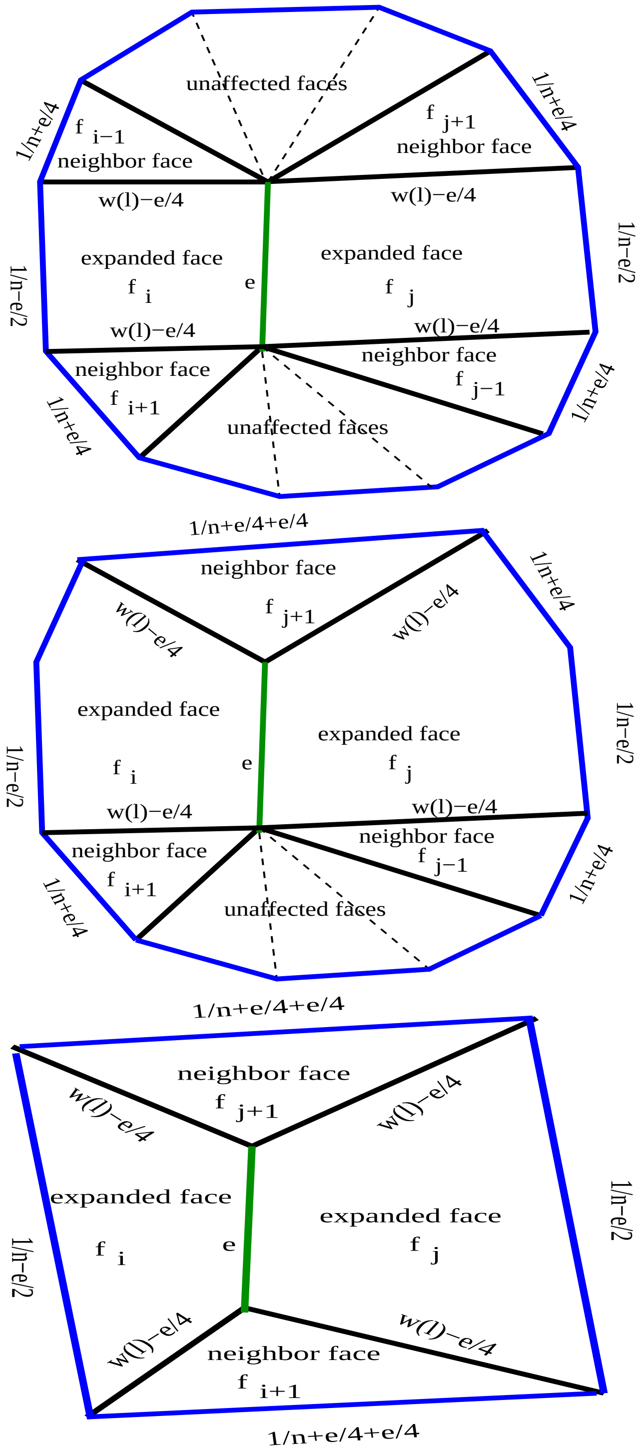

Figure 3: Three possible cases when a backbone edge is expanded. Expanded face is shared by (a) four distinct faces, (b) two distinct faces and one common face and (c) two common faces. When a backbone edge is expanded, only weights of these neighbor faces have to be adjusted; weights of other faces remain unchanged. Here $w(l) = \frac{n-1}{2n}$

The weight assignment scheme leads to a maximum possible weight of $1/n$ for each edge. When $n > 2$, this is always smaller than $1/2$. When we expand a backbone edge, we subtract a value from *some* of the edges of G'_D, and *always* add $\frac{\epsilon}{4}$ to two cycle edges of the adjacent cells. The maximum amount that we subtract is $\epsilon/2$ from a cycle edge of G'_D. Therefore, the only case when the weight on any edge becomes negative is when we subtract more than the initial value of the edge.

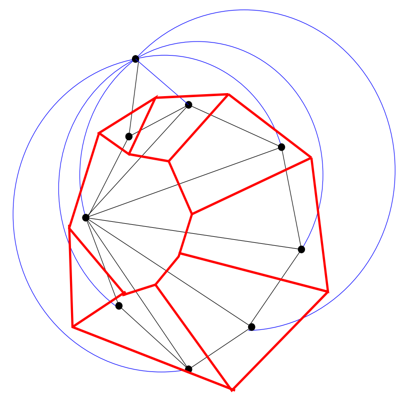

Figure 4: A stellated outerplanar graph and its dual (shown in red edges) where the backbone is a single vertex.

An extreme case occurs when, for each expansion of a backbone edge, $\epsilon/2$ is subtracted from the same edge. This is only possible when our original outerplanar graph G has all chords emanating from a single vertex. Consider a face in G'_D corresponding to such vertex in G. For each expansion of the backbone edge, the weight of the cycle edge of that face is decremented by $\epsilon/2$. Similarly, for each expansion, the weight of each of the two cycle edges is increased by $\frac{\epsilon}{4}$. It remains to prove that the weight of each edge of G'_D satisfies condition 2a of Theorem 5, if ϵ lies within a certain range.

Lemma 8 *The weight of each edge e lies within the range* $0 < w(e) \leq 1/2$.

Proof. We prove this first for the extreme case where a face f of G'_D shares all the backbone edge, as in Fig 4. First we show the upper bound of $w(e)$. Each time a backbone edge is expanded, the weight of each of the two cycle edges which are the edges of two adjacent faces of f is increased by $\frac{\epsilon}{4}$. Therefore, each such edge takes extra at most $\frac{\epsilon}{4}(k+1)$ from the weights, where k is the number of backbone edges in G'_D or chords in G'. In an outerplanar graph, k is exactly $n-3$. To maintain the upper bound of $w(e)$, we need $\frac{1}{n} + \frac{\epsilon(n-2)}{4} \leq \frac{1}{2}$ or $\epsilon \leq \frac{2}{n}$.

Now we prove the lower bound of $w(e)$. Each time a backbone edge is expanded, $\frac{\epsilon}{2}$ and $\frac{\epsilon}{4}$ are subtracted from the cycle and leaf edges of the face f respectively. Therefore, it suffices to show that the final weight of cycle edge remains positive. After adjusting weights for all backbone edges, the final weight of a boundary edge is $w(e) = \frac{1}{n} - \frac{(n-3)\epsilon}{2}$, as there are $(n-3)$ chords in the outerplanar graph. Since the base weight $1/n$ is always less than $1/2$, $w(e) < 1/2$. To make $w(e)$ positive, we have to choose ϵ such that $\frac{(n-3)\epsilon}{2} < \frac{1}{n}$ or $\epsilon < \frac{2}{n(n-3)}$. This completes the proof, with weights assignments $w(e)$ lying within the required range when $\epsilon < min\{\frac{2}{n(n-3)}, \frac{2}{n}\}$. □

3.2 Proof of non-coterminous cutset condition

A non-coterminous cutset is a cutset where all the edges of the cutset do not emanate from a single vertex. A non-coterminous cutset, like any other cutset, divides a connected graph into two components, where each component consists of at least two vertices. Since each vertex in the primal graph is represented by a face in the dual, the non-coterminous cutset in the primal is represented by a non-facial cycle in the dual. To prove the non-coterminous condition 2c of Theorem 5, we show now that for any non-facial cycle in the dual, the sum of weights of the edges of the cycle is strictly greater than 1.

For simplicity, let us consider a non-coterminous cutset where the non-facial cycle contains two adjacent faces in the dual, as in Fig 5. Let e be the edge shared by this two faces. According to condition 2b, the sum of the weights of the edges of each of these two faces is 1. Therefore, the sum of the weights of edges of the non-facial cycle is $2 - 2w(e)$. Since the weight of any edge is less than $\frac{1}{2}$, the weight of the cutset is strictly greater than 1. It is important to note that this cutset divides the primal graph into two components where one component has only two vertices joined by an edge. The two faces of the non-facial cycle represents these two vertices in the dual and edge e is the dual of the connecting edge in the primal.

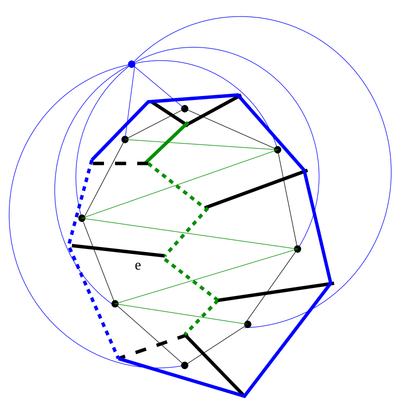

Figure 5: The dotted lines show the non-facial cycle in the dual graph. Edge e is the extra edge in the cycle.

Now consider a non-facial cycle C which contains n faces $f_1, f_2, \cdots, f_n$. An edge e_x is called an *extra edge*, if it is shared by two faces f_i and f_j, where $1 \leq i \neq j \leq n$. Denote by k the extra edges in C and by w_{max} the largest weight possible on any of these k edges (which is essentially less than $\frac{1}{2}$). In order to satisfy the condition 2c, we need $n - 2kw_{max} > 1$. We need an upper bound of w_{max} (lower bound is trivially greater than 0). In order to find that, we need an upper bound of k too.

Recall that the extra edges in any non-facial cycle represent the edges of one of the two components. Since a non-coterminous cycle divides the primal graph into exactly two components, one of the components has at least as many vertices than the other one. Therefore, the number of vertices of the smaller component is $v \leq \frac{n}{2}$ and the number of edges in that component satisfy $e \leq 2v - 3 = n - 3$. Therefore the upper bound of k is $n-3$. Substituting this value in the equation above, we get $w_{max} < \frac{n-1}{2(n-3)}$, where w_{max} is clearly less than $\frac{1}{2}$.

Finally, we need to convert w_{max} in terms of ϵ. Initially, in the contracted form, the cycle edges are assigned $\frac{1}{n}$ each and leaf edges are assigned $\frac{n-1}{2n}$ each. Both of these initial weights are within the bound of w_{max}. Whenever a backbone edge of face f_i is expanded, the weights of cycle and leaf edges of f_i are decreased. The only edges whose weights are increased are the cycle edges of face f_{i-1} and f_{i+1}. Hence it is possible for these edges to violate only the w_{max} condition. We get the bound of w_{max} based on these edges. The maximum weights of such cycle edges are encountered when the outerplanar graph is similar to the extreme case stated in lemma 8. In that case, only one face f_i shares all the backbone edges and the cycle edges of faces f_{i-1} and f_{i+1} incur maximum additional weights. Since there can be at most $n-3$ possible chords in the primal or backbone edges in the dual, the weight of each of these two cycle edge is $\frac{1}{n} + \frac{(n-3)\epsilon}{4}$. Therefore, the weights of these edges has to satisfy the condition that $\frac{1}{n} + \frac{(n-3)\epsilon}{4} \leq w_{max}$ or $\frac{1}{n} + \frac{(n-3)\epsilon}{4} < \frac{n-1}{2(n-3)}$. Solving this equation, we get $\epsilon < \frac{2(n^2-3n+6)}{n(n-3)^2}$.

To satisfy both conditions 2a and 2c, we will choose an ϵ such that $0 < \epsilon < min\{\frac{2}{n(n-3)}, \frac{2}{n}, \frac{2(n^2-3n+6)}{n(n-3)^2}\}$

This concludes our main proof.

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